

## A generalized cost Malmquist index to compare the productivities of units with negative data in DEA

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**Abstract.** In some data envelopment analysis (DEA) applications, some inputs of DMUs have negative values with positive cost. This paper generalizes the global cost Malmquist productivity index to compare the productivity of different DMUs with negative inputs in any two periods of times under variable returns to scale (VRS) technology, and then the generalized index is decomposed to several components. The obtained components are computed using the nonparametric linear programming models, known as DEA. To illustrate the generalized index and its components, a numerical example at three successive periods of time is given.

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### 1. Introduction

Data Envelopment Analysis (DEA) is a linear programming for evaluating the relative efficiency of each member of a set of organizational units (Charnes et al. [2]). The units, called Decision Making Units (DMUs), consume various levels of each specified inputs and produce various levels of each specified output. DEA evaluates the efficiency of a DMU relative to an empirical production possibility frontier determined by all DMUs under assumptions regarding returns to scale and orientation. In recent years the measurement and analysis of productivity change has enjoyed a great deal of interest among researchers studying firm performance and behavior. Research effort has focused on the investigation

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of the causes of productivity change and on its decomposition. In this framework, the Malmquist index was first introduced in productivity literature by Caves et al. [1]. Fare et al. [4] decomposed productivity change into a part attributable to change of technical efficiency and technical change and used non-parametric mathematical programming models for its computation. The cost Malmquist index proposed by Maniadakis and Thanassoulis [5] was extended by Tohidi et al. [11] into the Profit Malmquist index (PM) that can be used when input and output prices are available and producers want to maximize total profit of DMUs. To compare the cost efficiency of DMUs in the different periods of time, Tohidi et al. [10] used the convex combination (weighted average) of inputs costs in different periods of time and obtained a common cost for inputs. They used common cost and obtained a global cost efficient frontier, as single base, and proposed a global cost Malmquist index. This index is circular and it gives a single measure of productivity change, and its models are always feasible. Then Tohidi and Razavyan [12] extended the global cost index for the case when the prices of outputs are also known and named it the circular global profit Malmquist index. They applied Multi-Objective Programming (MOP) and proposed a method without decision maker preferences to obtain the common costs and prices as the coefficients of the base profit efficient frontier. Traditional DEA model that is applied for computing Malmquist indexes cannot deal with negative data. The approach named range directional model (RDM) was developed by Portela et al. [6] to measure efficiency under negative data. They introduced a model based on an ideal point and a directional vector that will be described later. Other approaches are also introduced to measure efficiency of a DMU in Sharp et al. [9] and Emrouznejad et al. [3] for the case where data are negative. For example Sharp et al. [9] presented a modified slacks-based measure (MSBM) model that can deal with both negative inputs and negative outputs. The directional vector is named the SP range in the MSBM model and the efficiency score of a DMU that is calculated by this approach cannot be greater than the RDM efficiency of Portela et al. [6]. Portela and Thanassoulis [8] calculated the meta-Malmquist index using the RDM model. Furthermore, Tohidi et al. [13] defined new profit efficiency and presented the profit Malmquist index and its global form in the presence of the negative data. They decomposed the index and used the range directional model (RDM) to compute the proposed index and its component. This paper presents a global cost Malmquist index for comparing the productivity of a DMU in two time periods when some inputs have negative values. Decomposition of the proposed index and their interpretations is presented. For computing the index and its components the RDM is used. This paper is organized as follows. Section 2 presents a background. In Section 3 the global cost Malmquist index and its components are presented under negative data. Section 4 provides a numerical example. Section 5 concludes.

## 2. Background

The RDM introduced by Portela et al. [6] can be used for comparing DMUs when some inputs and/or outputs are negative. Assume that in time period  $t, t = 1, \dots, T$ ,  $j$ th unit,  $j = 1, \dots, n$ , uses an input vector  $x_j^t = (x_{1j}^t, x_{2j}^t, \dots, x_{mj}^t)$ , to produce an output vector  $y_j^t = (y_{1j}^t, y_{2j}^t, \dots, y_{sj}^t)$ . Consider a point with maximum outputs and minimum inputs observed in period as an ideal point of time period  $t$  (for each input  $i, i = 1, \dots, m$ ,  $IP$  is as  $\min_j \{x_{ij}^t\}$  and for each output  $r, r = 1, \dots, s$ ,  $IP$  is as  $\max_j \{y_{rj}^t\}$  and the directional vector as  $(g_x, g_y) = (g_{x_1^t}, \dots, g_{x_m^t}, g_{y_1^t}, \dots, g_{y_s^t})$ . In the RDM model this directional vector for  $DMU_k$  in a given time period  $t$  is as,  $g_{x_{ik}^t} = R_{x_{ik}^t} = x_{ik}^t - \min_j \{x_{ij}^t\}; i = 1, \dots, m$  and

$gy_{rk}^t = Ry_{rk}^t = \max_j \{y_{rj}^t\} - y_{rk}^t; r = 1, \dots, s$ , that reflects ranges of possible improvement for this DMU. The RDM model for  $DMU_k$  in time period  $t$  is as follows (Portela et al. [6]):

$$\overline{DR}^t(x_k^t, y_k^t, R_{x_k^t}, R_{y_k^t}) = \sup\{\beta | (x_k^t - \beta R_{x_k^t}, y_k^t + \beta R_{y_k^t}) \in T^t\}, \tag{1}$$

where  $T^t = \{(x^t, y^t) | x^t \text{ can produce } y^t\}$  is the production technology of period  $t$  and exhibits variable returns to scale (VRS). This paper applies the input-oriented RDM model that is a particular type of model (1). In this case  $R_{y_k^t}$  is set to the zero vector. For  $DMU_k$ , in time period  $t$ , the input oriented RDM model is as follows (Portela et al. [6]):

$$\begin{aligned} \beta_k^* &= \max \beta_k \\ \text{s.t. } &\sum_{j=1}^n \lambda_j x_{ij}^t \leq x_{ik}^t - \beta_k R_{x_{ik}^t}, i = 1, \dots, m \\ &\sum_{j=1}^n \lambda_j y_{rj}^t \geq y_{rk}^t, r = 1, \dots, s \\ &\sum_{j=1}^n \lambda_j = 1 \\ &\lambda_j \geq 0, j = 1, \dots, n. \end{aligned} \tag{2}$$

The value of  $\beta_k^*$  in model (2) is an inefficiency measure and is equal to  $\beta_k^* = \overline{DR}^t(x_k^t, y_k^t, R_{x_k^t}, 0)$ . Therefore the measure of input efficiency of unit  $k$  is  $RDM^t(x_k^t, y_k^t, R_{x_k^t}, 0) = 1 - \beta_k^*$ . Also for each input  $i$  we have  $1 - \beta_k^* = \frac{x_{ik}^t - \min_j x_{ij}^t}{x_{ik}^t - \min_j x_{ij}^t}$  and this input efficiency measure reflects the distance between observed and target input levels ( $x_{ik}^{t*}$ ). In fact the observed and target levels are compared by the ideal unobserved DMU.

### 3. A Malmquist index to compare the productivities of units with negative data in DEA

Assume  $DMU_j, j = 1, \dots, n$ , uses a vector  $x_j^t = (x_{Pj}^t, x_{Nj}^t), x_{Pj}^t \in R^q, x_{Nj}^t \in R^l, q+l = m$ , that variables  $x_{Pj}^t$  indicate the positive inputs and variables  $x_{Nj}^t$  indicate the negative inputs of  $DMU_k$  for producing an output vector  $y_j^t \in R^s$  in time period  $t$ . Also assume  $w^t \in R^m, w^t = (w_P^t, w_N^t)$  is the input price vector of period  $t$ , where  $w_P^t$  is the prices of positive and  $w_N^t$  is the prices of negative inputs of DMUs. We compute the cost function  $C_k^t(y^t, w_P^t, w_N^t)$  using the following model:

$$\begin{aligned}
 C_k^t(y^t, w_P^t, w_N^t) = \min & \sum_{i=1}^q w_{ip}^t(x_{ik}^t - \beta R_{x_{ik}^t}^{GCF}) - \sum_{i=q+1}^{q+l} w_{iN}^t(x_{ik}^t - \beta R_{x_{ik}^t}^{GCF}) \\
 \text{s.t.} & \sum_{j=1}^n \lambda_j y_{rj}^t \geq y_{rk}^t, r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j x_{ij}^t \leq x_{ik}^t - \beta R_{x_{ik}^t}^{GCF}, i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, j = 1, \dots, n.
 \end{aligned} \tag{3}$$

Similarly, we have

$$\begin{aligned}
 C_k^G(y^t, w_P^G, w_N^G) = \min & \sum_{i=1}^q w_{ip}^G(x_{ik}^t - \beta R_{x_{ik}^t}^{GCF}) - \sum_{i=q+1}^{q+l} w_{iN}^G(x_{ik}^t - \beta R_{x_{ik}^t}^{GCF}) \\
 \text{s.t.} & \sum_{t=1}^T \sum_{j=1}^n \lambda_j y_{rj}^t \geq y_{rk}^t, r = 1, \dots, s \\
 & \sum_{t=1}^T \sum_{j=1}^n \lambda_j x_{ij}^t \leq x_{ik}^t - \beta R_{x_{ik}^t}^{GCF}, i = 1, \dots, m \\
 & \sum_{t=1}^T \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j^t \geq 0,
 \end{aligned} \tag{4}$$

where  $R_{x_{ik}^t}^{GCF} = x_{ik}^t - \min_t \{ \min_j \{ x_{ij}^t \} \}$ ,  $i = 1, \dots, m$ . Here, the ideal point is a global ideal point defined over the global technology. That is, for each input  $IP_i = \min_t \{ \min_j \{ x_{ij}^t \} \}$ ,  $i = 1, \dots, m$ . Now we can define the index  $CM_k^G$  under VRS technology in the presence of negative data to evaluate the productivity of  $DMU_k$  as

$$CM_k^G = \frac{C_k^G(y^t, w_P^G, w_N^G) / (w_P^G x_p^t - w_N^G x_N^t)}{C_k^G(y^{t+1}, w_P^G, w_N^G) / (w_P^G x_p^{t+1} - w_N^G x_N^{t+1})}. \tag{5}$$

The term  $w_P^G x_p^t - w_N^G x_N^t$  in (5) is the observed cost of  $DMU_k$  in time period  $t$ . The value greater than 1 for  $CM_k^G$  reflects the productivity regressed from period  $t$  to  $t + 1$ .

### 3.1 Decomposition of the proposed index

The proposed index can be decomposed as follows:

Table 1. Input-output data of period 1

DMU	Input 1 level	Input 2 level	Output level	Input 1 price	Input 2 price
DMU1	2.50	-0.02	3.00	1	2
DMU2	5.00	3.00	2.50	1	2
DMU3	0.75	-1.05	5.00	1	2
DMU4	1.50	-0.03	2.00	1	2
DMU5	2.00	2.00	1.00	1	2
DMU6	4.00	-5.00	4.00	1	2

Table 2. The proposed index and its components

$CM^G$	$OEK_k^G$	$CTC_k^G$
0.8356	1.0755	0.7769

$$CM_k^G = \frac{C_k^t(y^t, w_P^t, w_N^t) / (w_P^t x_P^t - w_N^t x_N^t)}{C_k^{t+1}(y^{t+1}, w_P^{t+1}, w_N^{t+1}) / (w_P^{t+1} x_P^{t+1} - w_N^{t+1} x_N^{t+1})} \times \left[ \frac{C_k^{t+1}(y^{t+1}, w_P^{t+1}, w_N^{t+1}) / (w_P^{t+1} x_P^{t+1} - w_N^{t+1} x_N^{t+1})}{C_k^G(y^{t+1}, w_P^G, w_N^G) / (w_P^G x_P^{t+1} - w_N^G x_N^{t+1})} \times \frac{C_k^G(y^t, w_P^G, w_N^G) / (w_P^G x_P^t - w_N^G x_N^t)}{C_k^t(y^t, w_P^t, w_N^t) / (w_P^t x_P^t - w_N^t x_N^t)} \right] \tag{6}$$

The term outside the brackets in (6) is the overall efficiency change component of unit  $k$  from  $t$  to  $t + 1$  ( $OEK_k^G$ ) and The term inside the brackets provides cost frontier shift between periods  $t$  and  $t + 1$  under VRS production technologies of two time  $t$  and  $t + 1$  ( $CTC_k^G$ ). The value greater than 1 of component  $OEK_k^G$  reflects the overall efficiency has regressed and the value under 1 for component  $CTC_k^G$  indicates there is a technical progress for DMU $k$  from period of time 1 to 2.

### 4. Numerical example

In Table 1 we have six DMUs with two inputs and one output available. In period 2 DMU1 improves its input levels by 10% and the other DMUs improve their input levels by 25%. The price of input 2 is reduced by 50% in period 2. In order to evaluate the productivity change of DMU 1 between two time periods 1 and 2 by computing the  $CM^G$  index and its components we assume that  $w^G = \frac{1}{2}w^1 + \frac{1}{2}w^2$ . Based on the data in Table 1 and the data of period 2, the values of the  $CM^G$  index and its components for DMU1 are shown in Table 2.

Because the value of  $CM^G$  index is less than 1, the productivity of DMU1 has improved from period 1 to period 2. The value of  $OEK_k^G$  is 1.0755, it means that the overall efficiency of DMU1 has regressed between two time 1 and 2. Similarly, the productivity of the other DMUs can be analyzed.

### 5. Conclusions

Tohidi et al.'s [10] cost Malmquist productivity index is a circular index for measuring the productivity change of a DMU in the case that the prices of inputs are known and the inputs are positive. But, this index cannot be used when negative inputs occur. To solve

this problem, this paper presented the  $CM^G$  index and decomposed the proposed index. The index and its components presented here satisfy circularity and are calculated by using the RDM model introduced by Portela et al. [6] and some modified DEA models.

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