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E-Clean Matrices and Unit-Regular Matrices

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Abstract. Let $a, b, k \in K$ and $u\ , v \in U(K)$. We show for any idempotent $e \in K$, $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is e-clean iff $\begin{pmatrix} a & 0 \\ u(vb+ka) & 0 \end{pmatrix}$ is e-clean and if $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is 0-clean, $\begin{pmatrix} ua & 0 \\ u(vb+ka) & 0 \end{pmatrix}$ is too.

Keywords: matrix ring, unimodular column, unit-regular, clean , e-clean.

1. Introduction

Throughout this paper, all rings are associative with identity and we let $U(K)$ be the units group of ring K and $(a', b')^t$ as a column of 2×2 matrix. An element in a ring K is said to be clean (respectively, unit-regular) if it is the sum (respectively, product) of an idempotent element and an invertible element. Answering a question of Nicholson, Yu, Camilo and Khurana proved that a unit-regular ring is clean [1] and [2]. Lam and Khurana have shown that a single unit-regular need not to be clean. More generally, they gave a criterion for a matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ to be clean in a matrix ring $M_2(K)$ over any commutative ring K [3]. In this note, similarly, we gave a criterion for a matrix $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ to be clean in a matrix ring $M_2(K)$ over any commutative ring K. Finally, we show that for $a, b, k \in K$ and $u, v \in U(K)$ and for any idempotent $e \in K$, $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is e-clean iff $\begin{pmatrix} a & 0 \\ u(vb+ka) & 0 \end{pmatrix}$ is e-clean and if matrix $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is 0-clean, $\begin{pmatrix} ua & 0 \\ u(vb+ka) & 0 \end{pmatrix}$ is too.

2. Main results

Throughout this work, an unimodular column means a column whose entries generate the unit ideal.

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Proposition 2.1 A matrix $B = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is unit-regular in $M_2(K)$ iff there exists an idempotent $e \in K$ and a unimodular column $(a', b')^t \in K^2$ such that $(a,\ b)^t = (a',\ b')^t e$.

Proof. If $(a\ , b)^t$ has the form $(a', b')^t e$ as above, since $(a', b')^t$ is unimodular, $a'y - b'x = 1$ for some $x,\ y \in K$ and the equation

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a' & x \\ b' & y \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$$

shows that B is unit-regular. Conversely, assume that B is unit-regular; say $B = UE$, where $E = E^2$ and $U \in GL_2(K)$. $E = \begin{pmatrix} e & r \\ s & t \end{pmatrix}$ and $U = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Then $(r,\ t\ )^t = U^{-1}.(0,0)^t = (0,0)^t$, and so

$$E = E^2 = \begin{pmatrix} e^2 & 0 \\ se & 0 \end{pmatrix}$$

shows that $e = e^2$, and $s = se$ .Therefore, $(a, b)^t = (a', b')^t\ e$, where

$$(a', b')^t = (w,\ y\ )^t + (x, z)^t s$$

is an unimodular column since (w, x) and (x, z) are the columns of a matrix in $GL_2(K)$. □

We shall now introduce a matrix that will be crucial for the work in the rest of this paper. For any three elements $e,\ x,\ k \in K$ such that $e = e^2$ and $ex = 0$, we define the matrix

$$E(e, x, k) := \begin{pmatrix} e - kx & x \\ ke - k(kx+1) & kx+1 \end{pmatrix} \in M_2(K) \tag{1}$$

The basic properties of $E(e,\ x\ ,\ k)$ are summarized as follows [3].

Proposition 2.2 $E := E(e,\ x\ ,\ k)$ is an idempotent matrix over K with $tr(E) = e + 1$ and $det(E) = e$.

Proposition 2.3 Let $E = (a_{ij})$ be any 2×2 idempotent matrix over K with determinant e. Then $e^2 = e$, and $ea_{ij} = \delta_{ij}\ e$ (where $\{\delta_{ij}\}$ are the Kronecker deltas). If the last column of E happens to be unimodular, then $a_{22} \equiv 1(mod a_{12}K)$.

Proof. First, we have $e = det(E) = det(E^2) = (\det(E)\ )^2 = e^2$. Let $f = 1 - e$ be the complementary idempotent of e. Over the factor ring K/fK , $\overline{E}$ has determinant $\overline{1}$, and is thus invertible. But then $\overline{E} = \overline{E}^2$ implies that $\overline{E}$ is the identity matrix. This means that $a_{ii} \in 1 + fK$ for all i, and $a_{ij} \in fK$ for $i \neq j$. Multiplying these by e, we see that $ea_{ii} = e$ for all i , and $ea_{ij} = 0$ for $i \neq j$. If, in addition, the last column of E is unimodular, then, over the factor ring $K/a_{12}K$, a_{22} becomes a unit, and E becomes an (idempotent) block-upper triangular matrix. The latter implies that the image of a_{22} in $K/a_{12}K$ is an idempotent, and thus we must have $a_{22} \equiv 1(mod a_{12}K)$.□

It is known [[3],definition(3.1)] that if e be a given idempotent in K, and matrix $M \in M_n(K)$ can be written in the form $E + U$, for some $E = E^2$ of determinant e and some $U \in GL_n(K)$, M is e-clean.

THEOREM 2.4 *Let e be a given idempotent in K. Then $B = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is e-clean iff there exits $x\ ,y \in K$ with $ex = 0$ and $y \equiv 1(mod xK)$ such that $ay - bx$ is e-clean.*

Proof. For the "if " part, write y in the form $kx+1$ with $ex = 0$, and let $ay-bx = e+u$, where $u \in U(K)$.We can then form the idempotent matrix $E := E(e,\ x\ ,\ k)$ in (1), with $det\,(E) = e$ by Proposition (2.2). Letting $U := B - E$, we have

$$det\,(U) = -(ay\ + bx) + det(E) = -(e + u) + e = -u \in U(K).$$

Thus, $U \in GL_2(K)$, and $B = E + U$ shows that B is e-clean. For the "only if" part, assume that $B = E + U$ with $E = E^2 = \begin{pmatrix} p & x \\ q & y \end{pmatrix}$ of determinant e, and $U \in GL_2\,(K)$. Since $U = \begin{pmatrix} a-p & -x \\ b-q & -y \end{pmatrix}$ and $U \in GL_2\,(K)$, we have $(-x, -y)^t$ is an unimodular column so the last column of E is unimodular. Thus, by Proposition (2.3), we have $ex = 0$, and $y \equiv 1(mod\ xK)$. Now let $u\ := -\ det(U) \in U(K)$. Then

$$-u = det\begin{pmatrix} a-p & -x \\ b-q & -y \end{pmatrix} = -ay + bx + e$$

so $ay - bx = e + u \in K$ is e-clean, as desired.□

Corollary 2.5 Let $k \in K$ and $u, v \in U(K)$. For any idempotent $e \in K$,

(1) $B = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is e-clean iff $B' = \begin{pmatrix} a & 0 \\ vb + ka & 0 \end{pmatrix}$ is e-clean.

(2) $B = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is e-clean iff $B'' = \begin{pmatrix} a & 0 \\ u(vb + ka) & 0 \end{pmatrix}$ is e-clean.

Proof.

(1) It's proof is similar to [[3], Corollary (3.3)].
(2) Let B is e-clean, there exist (by Theorem (2.4))
$ex = 0$, $y \equiv 1\,(mod\ xK)$ and $ay - bx$ is e-clean.

Let $b' = u(vb + ka)$ .Then $ay - bx = ay - v^{-1}\left(u^{-1}b' - ka\right)x = ay_1 - b'x_1$ is e-clean for $x_1 = v^{-1}u^{-1}x$ and $y_1 = y + v^{-1}kx$. Since $ex_1 = eu^{-1}v^{-1}x = 0$ and $y_1 \equiv y \equiv 1$ modulo the ideal $x_1K = xK$, B'' is e-clean again by theorem(2.4). Conversely, Assuming B'' is e-clean, there exist $x_1, y_1 \in K$ such that $ex_1 = 0$, $y_1 \equiv 1\,(mod\ x_1K)$ and $ay_1 - b'x_1$ is e-clean.

Let $b' := u(vb+ka)$.Then $ay_1-b'x_1 = ay_1-u(vb+ka)x_1 = ay_1-uvbx_1-ukax_1 = ay - bx$ for $x = uvx_1$ and $y = y_1 - ukx_1$ .Since $ex = euvx_1 = 0$ and $y \equiv y_1 \equiv 1$ modulo the ideal $x_1K = xK$, so B is e-clean.

If $e = 0$ then B is 0-clean. The following theorem gives some properties for B to be 0-clean. It's proof is analogue of [[3], Corollary (2.5.2)].□

Corollary 2.6 $B = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is 0-clean iff there exsist $x_0, y_0 \in K$ such that $ay_0-bx_0 = 1$ and y_0+x_0K contains a unit of K . (In this case, B is also unit-regular, according to Proposition (2.1).)

Corollary 2.7 Let $a, b \in K$ and $u, v \in U(K)$. If $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is 0-clean, so $\begin{pmatrix} ua & 0 \\ u(vb + ka) & 0 \end{pmatrix}$ is.

Proof. If $u = 1$, this is covered by corollary (2.5) (even in the e-clean case). Thus, it only remains to make the passage from B to $\begin{pmatrix} ua & 0 \\ ub & 0 \end{pmatrix}$ .This can be done (albeit only in the 0-clean case) by rewriting the equation $ay_0 - bx_0 = 1$ for some $x_0, y_0 \in K$ in Corollary (2.6) in the form $au\left(u^{-1}y_0\right) - bu\left(u^{-1}x_0\right) = 1$, and noting that

$$u^{-1}y_0 + u^{-1}x_0K = u^{-1}(y_0 + x_0K).$$

□

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