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Module-Amenability on Module Extension Banach Algebras

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Abstract. Let A be a Banach algebra and E be a Banach A-bimodule then $S = A \oplus E$, the l^1-direct sum of A and E becomes a module extension Banach algebra when equipped with the algebras product $(a, x).(a', x') = (aa', a.x' + x.a')$. In this paper, we investigate $\triangle$-amenability for these Banach algebras and we show that for discrete inverse semigroup S with the set of idempotents E_S, the module extension Banach algebra $S = l^1(E_S) \oplus l^1(S)$ is $\triangle$-amenable as a $l^1(E_S)$-module if and only if $l^1(E_S)$ is amenable as Banach algebra.

Keywords: Module-amenability, module extension, Banach algebras

1. Introduction

The concept of amenability for Banach algebras was introduced by Johnson in [8]. The main Theorem in [8] asserts that the group algebra $L^1(G)$ of a locally compact group G is amenable if and only if G is amenable. This is far from true for semigroups. If S is a discrete inverse semigroup, $l^1(S)$ is amenable if and only if E_S is finite and all the maximal subgroups of S are amenable [6]. This failure is due to the fact that $l^1(S)$, for a discrete inverse semigroup S with the set of idempotents E_S, is equipped with two algebraic structures. It is a Banach algebra and a Banach module over $l^1(E_S)$.

The concept of module amenability for Banach algebras was introduced by M.Amini in [1]. The main theorem in [1] asserts that for an inverse semigroup S, with the set of idempotents E_s , $l^1(S)$ is module amenable as a Banach module over $l^1(E_s)$ if and only if S is amenable. Also the second named author study the concept of weak module amenability in [2] and showed that for a commutative inverse semigroup S, $l^1(S)$ is always weak module amenable as a Banach module over $l^1(E_s)$. There are many examples of Banach modules which do not have any natural algebra structure One example is $L^p(G)$ which is a left Banach $L^1(G)$-module, for a locally compact group G [4]. The theory of amenability in [8] and module amenability developed in [1] does not cover these examples. There is one thing in common in these examples and that is the existence of a module homomorphism from the Banach module to the underlying Banach algebra. For instance if G is a compact group and $f \in L^q(G)$, then on has the module homomorphism $\triangle_f : L^p(G) \to L^1(G)$ which sends g to $f * g$. The concept of $\triangle$-amenability in [7] is defined for a Banach module E over a Banach algebra A with a given mod-

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ule homomorphism $\triangle : E \to A$. The authors in [7] gives the basic properties of $\triangle$-amenability and in particular establishes the equivalence of this concept with the existence of module virtual (approximate) diagonals in an appropriate sense. Also the main example in [7] asserts that for a discrete abelian group G, $L^p(G)$ is $\triangle$-amenable as an $L^1(G)$-module if and only if G is amenable. In this paper we shall focus on an especial kind of Banach algebras which are constructed from a Banach algebra A and a Banach A-bimodule E, called module extension Banach algebras and we verify the concept of $\triangle$-amenability for these Banach algebras.

2. Preliminaries

Let A be a Banach algebra and E be a Banach space with a left A-module structure such that, for some $M > 0, \| a.x \| \leqslant M \| x \|$ $(a \in A, x \in E)$. Then E is called a left Banach A-module. Right and two-sided Banach A-modules are defined similarly. Throughout this section E is a Banach A-bimodule and $\triangle : E \to A$ is a bounded Banach A-biomodule homomorphism.

Definition 2.1 *let X be a Banach A-Bimodule. A bounded linear map $D : A \to X$ is called a module derivation (or more specifically $\triangle$-derivation) if*

$$D(\triangle(a.x)) = a.D(\triangle(x)) + D(a).\triangle(x)$$

$$D\triangle(x.a)) = D(\triangle(x)).a + \triangle(x).D(a)$$

For each $a \in A$ and $x \in E$. Also D is called inner (or $\triangle$-inner) if there is $f \in X$ such that

$$D(\triangle(x)) = f.\triangle(x) - \triangle(x).f \qquad (x \in E)$$

Definition 2.2 *A bimodule E is called module amenable (or more specifically $\triangle$-amenable as a A-bimodule) if for each Banach A-bimodule X, all $\triangle$-derivation from A to X^* are $\triangle$-inner.*

It is clear that A is A-module amenable (which $\triangle$ =id) if and only if it is amenable as a Banach algebra. A right bounded approximate identity of E is a bounded net a_α in A such that for each $x \in E$, $(\triangle(x).a_\alpha - \triangle(x)) \to 0$ As $\alpha \to 0$. The left and two sided approximate identities are defined similarly.

Proposition 2.3 *If E is module amenable, Then E has a bounded approximate identity.*

Proposition 2.4 *If I is a closed ideal of A which contains a bounded approximate identity, E is a Banach A-bimodule with module homomorphism $\triangle : E \to A$, and X is a essential Banach I-module, then X is a Banach A-module and each $\triangle_I$-derivation $D : I \to X$ uniquely extends to a $\triangle$-derivation $D : A \to X$ which is continuous with respect to the strict topology of A (induced by I) and W-topology of X^*.*

Proposition 2.5 *If $\triangle : E \to A$ has a dense range, then $\triangle$-amenability of E is equivalent to amenability of A.*

Definition 2.6 *let $\triangle : A\hat{\otimes}A \to A$ be the continuous lift of the multiplication map of A to the projective tensor product$A\hat{\otimes}A$. A module approximate diagonal of E is* *a bounded net e_α in $A\hat{\otimes}A$ such that*

$$\begin{aligned} &\|e_\alpha \triangle(x) - \triangle(x).e_\alpha\| \to 0 \\ &\|\pi(e_\alpha.\triangle(x) - \triangle(x)\| \to 0\,, \quad (x \in E) \end{aligned}$$

As $\alpha \to \infty$. A module virtual diagonal of E is an element M in $(A\hat{\otimes}A)^{**}$ such that

$$\begin{aligned} M.\triangle(x) - \triangle(x).M = 0 \\ \pi^{**}(M).\triangle(x) - \triangle(x) = 0\,, \quad (x \in E) \end{aligned}$$

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It is clear that if E has a module virtual diagonal, then A contains a bounded approximate identity.

Theorem 2.7 *Consider the following assertions*

i) E is module amenable,
ii) E has a module virtual diagonal,
iii) E has a module approximate diagonal.

We have (i) $\to$ (ii)$\to$(iii). If moreover $\triangle$ has a dense range, all the assertions are equivalent.

Example 2.8 let $1 \leqslant P < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ then l^1 is a Banach algebra and l^p is a Banach l^1 module, both with respect to pointwise multiplication. Also each $f \in l^q$ defines a module homomorphism $\triangle_f : l^p \to l^1$ by $\triangle_f(g) = g^* f$. If $f = \sum_{k=-\infty}^{\infty} \delta_k$, then $\triangle_f$ has dense range and l^p is $\triangle_f$-amenable.

3. $\triangle$-amenability of Module extension Banach algebras

The module extension Banach algebra corresponding to A and E is $S = A \oplus E$, the l^1-direct sum of A and E , with the algebra product defined a follows:

$$(a, x).(a', x') = (aa', a.x' + x.a') \qquad (a, a' \in A, x, x' \in E).$$

Some aspects of algebras of this form have been discussed in [3] and [5] also the amenability and n-weak amenability of module extension Banach algebras investigated by zhang in [**?**]. In this section we show that the amenability of Banach algebra A is equivalent to $\triangle$-amenability $A \oplus E$ as a Banach A-module.

By the following module actions the module extension Banach algebra $A \oplus E$ is a Banach A-module

$$a.(b, x) = (ab, x), \quad (b, x).a = (ba, x) \quad (a, b \in A, x \in E).$$

Also $\triangle : A \oplus E \to A$ by $(a, x) \to a(a \in A, x \in E)$ is a surjective A-module homomorphism ,so we have:q

Proposition 3.1 *The Banach algebra A is amenable if and only if the module extension Banach algebra $A \oplus E$ is $\triangle$-amenable as a A-module.*

Example 3.2 Let S is a discrete inverse semigroup with the set of idempotents E_S and $E = l^1(S)$,$A = l^1(E_S)$ and $l^1(E_S)$ act on $l^1(S)$ by multiplication in this case: the module extension Banach algebra $S = l^1(E_S) \oplus l^1(S)$ is $\triangle$-amenable as a $l^1(E_S)$-module if and only if $l^1(E_S)$ is amenable.

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