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# Product of normal edge-transitive Cayley graphs

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**Abstract.** For two normal edge-transitive Cayley graphs on groups $H$ and $K$ which have no common direct factor and $gcd(|H/H'|, |Z(K)|) = 1 = gcd(|K/K'|, |Z(H)|)$, we consider four standard products of them and it is proved that only tensor product of factors can be normal edge-transitive.



**Keywords:** Cayley graph, Normal edge-transitive, Product of graphs.

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## 1. Introduction

Let $\Gamma = (V, E)$ be a simple graph where $V$ is the set of vertices and $E$ is the set of edges of $\Gamma$. An edge joining the vertices $u$ and $v$ is denoted by $\{u, v\}$. The group of automorphisms of $\Gamma$ is denoted by $\mathrm{Aut}(\Gamma)$, which acts on vertices, edges and arcs of $\Gamma$. If $\mathrm{Aut}(\Gamma)$ acts transitively on vertices, edges or arcs of $\Gamma$, then $\Gamma$ is called *vertex-transitive, edge-transitive* or *arc-transitive*, respectively. If $\Gamma$ is vertex and edge-transitive but not arc-transitive, then $\Gamma$ is called $\frac{1}{2}$ *arc-transitive.*

There are four standard products of graphs. (see [5], [4], [16] and [17].)

**Definition 1.1** Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two simple graphs, then the Cartesian product, tensor product, strong product and lexicographic product of $\Gamma_1$ and

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$\Gamma_2$ denoted by $\Gamma_1 \square \Gamma_2$, $\Gamma_1 \times \Gamma_2$, $\Gamma_1 \boxtimes \Gamma_2$ and $\Gamma_1 \odot \Gamma_2$ respectively, is a graph with vertex set $V = V_1 \times V_2$ and two vertices $(v_1, v_2)$ and $(u_1, u_2)$ are adjacent if one of the relevant conditions happen.

- Cartesian product iff $(v_1 = u_1$ and $(v_2, u_2) \in E_2)$ or $((v_1, u_1) \in E_1$ and $v_2 = u_2)$.
- tensor product iff $(v_1, u_1) \in E_1$ and $(v_2, u_2) \in E_2$.
- strong product iff $(v_1 = u_1$ and $(v_2, u_2) \in E_2)$ or $((v_1, u_1) \in E_1$ and $v_2 = u_2)$ or $((v_1, u_1) \in E_1$ and $(v_2, u_2) \in E_2)$.
- lexicographic product iff $((v_1, u_1) \in E_1)$ or $(v_1 = u_1$ and $(v_2, u_2) \in E_2)$.

Let $G$ be a finite group and $S$ be an inverse closed subset of $G$ which does not contain the identity element of the group $G$, i.e. $S = S^{-1}$, such that $1 \notin S$. The *Cayley graph* $\Gamma = \mathrm{Cay}(G, S)$ on $G$ with respect to $S$ is a graph with vertex set $G$ and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. $\Gamma$ is connected if and only if $G =< S >$. For $g \in G$ define the mapping $\rho_g : G \longrightarrow G$ by $\rho_g(x) = xg, x \in G$. Clearly, $\rho_g \in \mathrm{Aut}(\Gamma)$ for every $g \in G$, thus $R(G) = \{\rho_g \mid g \in G\}$ is a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to $G$, forcing $\Gamma$ to be a vertex-transitive graph.

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph of a finite group $G$ on $S$. Let $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ and $A = \mathrm{Aut}(\Gamma)$. Then the normalizer of $R(G)$ in $A$ is equal to

$$N_A(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S),$$

where $\rtimes$ denotes the semi-direct product of two groups ([7]).

A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is called *normal* if $R(G)$ is a normal subgroup of $\mathrm{Aut}(\Gamma)$. This concept was first introduced with Xu [15].

Therefore according to [7], $\Gamma = \mathrm{Cay}(G, S)$ is normal if and only if $A := \mathrm{Aut}(\Gamma) = R(G) \rtimes \mathrm{Aut}(G, S)$, and in this case $A_1 = \mathrm{Aut}(G, S)$ where $A_1$ is the stabilizer of the identity element of $G$ under $A$. The normality of Cayley graphs has been extensively studied from different points of views by many authors. Wang et.al [14] obtained all disconnected normal Cayley graphs. Therefore, it suffices to study the connected Cayley graphs when one investigates the normality of Cayley graphs, which we use in this paper.

Therefore throughout the paper a Cayley graph is $\Gamma = \mathrm{Cay}(G, S)$, where $G$ is a finite group and $S$ is a non-empty generating subset of $G$ such that $1 \notin S$ and $S = S^{-1}$, and 1 denotes the identity element of the relevant group. We also denote $\mathrm{Aut}(\Gamma)$ by $A$.

**Definition 1.2** A Cayley graph $\Gamma$ is called normal edge-transitive or normal arc-transitive if $N_A(R(G))$ acts transitively on the set of edges or arcs of $\Gamma$ respectively. If $\Gamma$ is normal edge-transitive, but not normal arc-transitive, then it is called normal $\frac{1}{2}$ arc-transitive Cayley graph.

Edge-transitivity of Cayley graphs of small valency have received attention in the literature. A relation between regular maps and edge-transitive Cayley graphs of valency 4 is studied in [12], and Li et.al [11] characterized edge-transitive Cayley graphs of valency 4 and odd order. Houlis [9] classified normal edge-transitive Cayley graphs of groups $\mathbb{Z}_{pq}$ where $p$ and $q$ are distinct primes. Normal edge-transitive Cayley graphs on some abelian groups of valency at most 5 have been studied by Alaeiyan [1] . Edge-transitive Cayley graphs of valency four on non-abelian simple groups are studied in [6]. Besides, Darafsheh et.al in [2] classified all normal edge-transitive Cayley graphs of non-abelian groups of order $4p$, for prime $p$.

In this paper, we consider the standard products of normal edge-transitive graphs. We prove that only tensor product of two normal edge-transitive Cayley graphs is normal edge-transitive under some conditions.

## 2. Preliminary Results

Keeping fixed terminologies used in section 1, we mention a few results whose proofs can be found in the literature.

The following result is proved in [15] and [7].

Result 2.1 Let $\Gamma = \mathrm{Cay}(G, S)$, then the followings hold:

(1) $N_A(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S)$
(2) $R(G) \trianglelefteq A$ if and only if $A = R(G) \rtimes \mathrm{Aut}(G, S)$
(3) $\Gamma$ is normal if and only if $A_1 = \mathrm{Aut}(G, S)$

The result that we will use in our investigation of normal edge-transitive Cayley graph is the following that makes it possible to characterize normal edge-transitivity in terms of the action of $\mathbb{A}ut(G, S)$ on $S$ (see [13]).

Result 2.2 Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley graph (undirected) on $S$. Then $\Gamma$ is normal edge-transitive if and only if $\mathrm{Aut}(G, S)$ is either transitive on $S$, or has two orbits in $S$ in the form of $T$ and $T^{-1}$ where $T$ is a non-empty subset of $S$ such that $S = T \cup T^{-1}$.

For a general graph $\Gamma = (V, E)$, if $v$ is a vertex in $\Gamma$, then $\Gamma(v)$ denotes the set of the so called neighbors of $v$, i.e. $\Gamma(v) = \{u \in V \mid \{u, v\} \in E\}$. The following result which can be deduced from a result in [8] characterizes normal arc-transitive Cayley graphs in terms of the action of $\mathrm{Aut}(G, S)$ on $S$.

Result 2.3 Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley graph (undirected) on $S$. Then $\Gamma$ is normal arc-transitive if and only if $\mathrm{Aut}(G, S)$ acts transitively on $S$.

The next theorem is proved in [3]

Result 2.4 Let $G = H \times K$, where $H$ and $K$ be two groups with no common direct factor and $gcd(|H/H'|, |Z(K)|) = 1 = gcd(|K/K'|, |Z(H)|)$, then $\mathrm{Aut}G = \mathrm{Aut}H \times \mathrm{Aut}K$.

The following result shows that all four kinds of product of two Cayley graphs are also a Cayley graph [10].

Result 2.5 Let $\Gamma_1 = \mathrm{Cay}(H, S)$ and $\Gamma_2 = \mathrm{Cay}(K, T)$ be two Cayley graphs, $\Gamma_\square = \Gamma_1 \square \Gamma_2$, $\Gamma_\times = \Gamma_1 \times \Gamma_2$, $\Gamma_\boxtimes = \Gamma_1 \boxtimes \Gamma_2$ and $\Gamma_\odot = \Gamma_1 \odot \Gamma_2$, then $\Gamma_\square$, $\Gamma_\times$, $\Gamma_\boxtimes$ and $\Gamma_\odot$ all are Cayley graphs on the group $G = H \times K$ relative to the sets $S_\square$, $S_\times$, $S_\boxtimes$ and $S_\odot$ respectively, where

(1) $S_\square = (\{1_H\} \times T) \cup (S \times \{1_K\})$
(2) $S_\times = S \times T$
(3) $S_\boxtimes = (\{1_H\} \times T) \cup (S \times \{1_K\}) \cup (S \times T)$
(4) $S_\odot = (S \times K) \cup (\{1_H\} \times T)$

## 3. Products of normal edge-transitive Cayley graphs

Now we can focus on the Cayley graphs which arise from product of normal edge-transitive Cayley graphs. But from result 2.3, it is convenient if we can describe the automorphism group of the group $\mathrm{Aut}(G, S_*)$ with the automorphism groups $\mathrm{Aut}(H, S)$ and $\mathrm{Aut}(K, T)$, where * can be replaced by $\square, \boxtimes, \times$ and $\odot$.

**Lemma 3.1** Let $G$, $H$ and $K$ be groups which satisfy the assumption in Result 2.4, $S$ and $T$ two closed inverse subset of $H$ and $K$, respectively which does not contain identity of the correspoding group, and $G = H \times K$. Then

$$\mathrm{Aut}(G, S_*) = \mathrm{Aut}(H, S) \times \mathrm{Aut}(K, T),$$

where $* \in \{\square, \boxtimes, \times, \odot\}$.

**Proof.** With the assumption in the Result 2.5 and Result 2.4, we can deduce that $\mathrm{Aut}G = \mathrm{Aut}H \times \mathrm{Aut}K$. i.e.

$$\mathrm{Aut}(G) = \{\sigma = (\alpha, \delta) | \alpha \in \mathrm{Aut}(H), \delta \in \mathrm{Aut}(K)\}.$$

Since $\mathrm{Aut}(G, S_*) = \{\sigma \in \mathrm{Aut}(G) | \sigma(S_*) = S_*\}$ and from Result 2.5, if $\sigma \in \mathrm{Aut}(G, S_*)$ we can distinguish four cases of $*$ in the following:

(1) *Cartesian product:* For $(1_H, t) \in \{1_H\} \times T \subset S_\square$ we have

$$\sigma((1, t)) = (\alpha, \delta)(1, t) = (1, \delta(t)) \in S_\square = (\{1_H\} \times T) \cup (S \times \{1_K\}).$$

$1 \notin S$ implies $(1, \delta(t)) \in \{1_H\} \times T$, thus $\delta(t) \in T$, .i.e. $\delta \in \mathrm{Aut}(K, T)$ and $\sigma \in \mathrm{Aut}(H) \times \mathrm{Aut}(K, T)$. But for $s \in S$ we have $(s, 1_T) \in S \times \{1_K\} \subset S_\square$ which include similarly that $\sigma \in \mathrm{Aut}(H, S) \times \mathrm{Aut}(K)$. Therefore

$$\sigma \in (\mathrm{Aut}(H, S) \times \mathrm{Aut}(K)) \cap (\mathrm{Aut}(H) \times \mathrm{Aut}(K, T)),$$

which yields $\sigma \in \mathrm{Aut}(H, S) \times \mathrm{Aut}(K, T)$.

(2) *tensor product:* For $g = (s, t) \in S_\times = S \times T$ we have

$$\sigma(g) = (\alpha, \delta)(s, t) = (\alpha(s), \delta(t)) \in S \times K,$$

implies $\alpha(h) \in H$ and $\delta(t) \in T$, i.e. $\sigma \in \mathrm{Aut}(H, S) \times \mathrm{Aut}(K, T)$.

(3) *strong product:* Comes from the cases Cartesian product and strong product, since $S_\boxtimes = S_\square \cup S_\times$ and the fact that neither $S$ nor $T$ contains the identity element.

(4) *lexicographic product:* For $s \in S$ and $k \in K$ we have

$$\sigma(s, k) = (\alpha, \delta)(s, k) = (\alpha(s), \delta(k)),$$

but $\alpha \in \mathrm{Aut}(H)$ and $1_H \notin S$, implies $\alpha(s) \neq 1_H$, i.e. $\sigma(s,k) \in S \times K$, hence $\alpha \in \mathrm{Aut}(H,S)$.

For $(1_H,t) \in \{1_H\} \times T$, we have $\sigma(1,t) = (1,\delta(t)) \in \{1_H\} \times T$, thus $\delta(t) \in T$ and therefore $\delta \in \mathrm{Aut}(K,T)$.

Conversely, if $\sigma \in \mathrm{Aut}(H,S) \times \mathrm{Aut}(K,T)$, it is easy to see in any cases $\sigma \in \mathrm{Aut}(G,S_*)$ ■

Now, we want to find out when the product of two normal edge-transitive Cayley graphs is also normal edge-transitive.

**Theorem 3.2** Let $G$, $H$, $K$, $S$ and $T$ be the ones mentioned in Lemma 3.1, $\Gamma_1 = \mathrm{Cay}(H,S)$ and $\Gamma_2 = \mathrm{Cay}(K,T)$. Then the Cartesian product, strong product and lexicographic product of $\Gamma_1$ and $\Gamma_2$ all are non normal edge-transitive Cayley graphs.

**Proof.** Let $\Gamma_\square$, $\Gamma_\boxtimes$ and $\Gamma_\odot$ be the Cartesian product, strong product and lexicographic product of the Cayley graphs $\Gamma_1$ and $\Gamma_2$, respectively. By result 2.2, $\Gamma_*$ is normal edge-transitive if and only if $\mathrm{Aut}(G,S_*)$ acts transitively on $S_*$ or $S_* = T_* \cup T_*^{-1}$ where $\mathrm{Aut}(G,S_*)$ acts transitively on $T_*$, where $* \in \{\square, \boxtimes, \odot\}$. Now in each product we prove there is a contradiction to normal edge-transitivity of $\Gamma_*$. Assume $\Gamma_*$ is normal edge-transitive, then we have the followings:

(1) *Cartesian product:* For $(1_H,t),(s,1_K) \in S_\square$, since $(1_H,t)^{-1} = (1_H,t^{-1}) \neq (s,1_K)$, thus some $\sigma \in \mathrm{Aut}(G,S_\square)$ should sends $(1_H,t)$ to $(s,1_K)$ which is impossible since by Lemma 3.1, $\sigma = (\alpha,\delta)$, thus $\sigma(1_H,t) = (1_H,\delta(t)) \neq (s,1_H)$.
(2) *strong product:* Similar to Cartesian product.
(3) *lexicographic product:* Similar argument of Cartesian product can occur for $(s,k),(1_H,t) \in S_\odot$

■

By Theorem 3.2, three kinds of product of two normal edge-transitive Cayley graph can not be normal edge-transitive. But tensor product of them can be normal edge-transitive. In the next Theorem, we find out the conditions under which this can happen.

**Theorem 3.3** Let $\Gamma_1$ and $\Gamma_2$ be two Cayley graphs with the assumptions in Theorem 3.2. Then

(I) $\Gamma_1 \times \Gamma_2$ is normal arc-transitive iff $\Gamma_1$ and $\Gamma_2$ are normal arc-transitive .
(II) $\Gamma_1 \times \Gamma_2$ is normal half arc-transitive iff one of $\Gamma_1$ and $\Gamma_2$ is normal arc-transitive and the other one is half arc-transitive.
(III) If $\Gamma_1$ and $\Gamma_2$ both are normal half arc-transitive, then $\Gamma_1 \times \Gamma_2$ is not normal edge-transitive.

**Proof.** We consider three cases

**Case 1.** Let $\Gamma_1$ and $\Gamma_2$ be normal arc-transitve and $(s,t),(s',t') \in S_\times = S \times T$. By Result 2.3 there exist $\alpha \in \mathrm{Aut}(H,S)$ and $\delta \in \mathrm{Aut}(K,T)$ such that $\alpha(s) = s'$ and $\delta(t) = t'$.

By Lemma 3.1, $\sigma = (\alpha, \delta) \in \mathrm{Aut}(G, S_\times)$ satisfies the condition $\sigma(s, t) = (s', t')$, i.e. $\Gamma_\times = \Gamma_1 \times \Gamma_2$ is normal arc-transitive.

Conversely, suppose $\Gamma_\times$ is normal arc-transitive, $s, s' \in S$ and $t, t' \in T$ . By Result 2.3 there exists $\sigma \in \mathrm{Aut}(G, S_\times)$ which sends $(s, t)$ to $(s', t')$. By Result 3.1, $\alpha \in \mathrm{Aut}(H, S)$ and $\delta \in \mathrm{Aut}(K, T)$ exists which send $s$ to $s'$ and $t$ to $t'$, respectively. By implying Lemma 2.3, it is easy to verify $\Gamma_1$ and $\Gamma_2$ be two normal arc-transitive .

**Case 2.** Without lost of generality let $\Gamma_1$ is normal arc-transitive and $\Gamma_2$ is normal half arc-transitive. Then By Result 2.3 and Result 2.2 we conclude that $T = W \cup W^{-1}$ and $\mathrm{Aut}(H, S)$ acts transitively on $S$ as well as $\mathrm{Aut}(K, T)$ on $W$. Now, we can deduce that $S \times T$ is also the distinct union of $S \times W$ and $S \times W^{-1}$. Similarly, in the case (1) one can verify that $\mathrm{Aut}(\Gamma_\times, S_\times)$ acts transitively on $S \times W$ as well as $S \times W^{-1}$ implying that $\Gamma_1 \times \Gamma_2$ is normal half arc-transitive.

Conversely, if $\Gamma_\times$ is normal half arc-transitive, by Results 2.2 and 2.3 we can write $S \times T$ in the form of $X \cup X^{-1}$ where $X$ and $X^{-1}$ are subsets of $S \times T$ and orbits of $\mathrm{Aut}(G, S_\times)$ as well. Set $V := \pi_1(X)$ and $W := \pi_2(X)$ where $\pi_i$ is the projective function form $S \times T$ into $S$ or $T$, resprectively. For $s_1, s_2 \in V$ , there are $t_1, t_2 \in W$ such that $(s_1, t_1), (s_2, t_2) \in X$ and hence for some $\sigma = (\alpha, \delta)$, we have $\sigma(s_1, t_1) = (s_2, t_2)$. Now if we define $\eta = (\alpha, id) \in \mathrm{Aut}(G, S_\times)$, then we observe that $\eta(s_1, t_1) = (s_2, t_2)$, i.e. $(s_2, t_1) \in X$. Hence for all $t \in W$ we have $V \times \{t\} \in X$ and we can similarly proof that for all $s \in V$ we have $\{s\} \times W \in X$, i.e. $V \times W \in X$, and therefore $X = V \times W$.

$S \times T$ is the disjoint union of $X$ and $X^{-1}$ yields that $V = V^{-1}$ and $W \cap W^{-1} = \emptyset$ or vise versa. In the first case, $\mathrm{Aut}(G, S_\times)$ acts transitively on $S \times T$. By the Lemma 3.1 we conclude that $\Gamma_1$ is arc-transitive and $\Gamma_2$ is half transitive Cayley graph. The proof of latter case is similar.

**Case 3.** If $\Gamma$ is normal edge-transitive, then $\Gamma$ is normal arc-transitive or normal half arc-transitive. By case (1) and case (2) the proof is obvious. ■

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