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Cubic spline Numerov type approach for solution of Helmholtz equation

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Abstract. We have developed a three level implicit method for solution of the Helmholtz equation. Using the cubic spline in space and finite difference in time directions. The approach has been modified to drive Numerov type finite difference method. The method yield the tri-diagonal linear system of algebraic equations which can be solved by using a tri-diagonal solver. Stability and error estimation of the presented method are analyzed.The obtained results satisfied the ability and efficiency of the method.

Keywords: Cubic spline, Finite difference, Numerov type, Stability, Helmholtz equation.

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1. Introduction

The Helmholtz equation arises in a variety of engineering and scientific applications such as acoustic radiation, scattering, electromagnetic field, wave propagation and heat conduction [1,3,8,20]. Consider the linear elliptic differential operator $L_\tau = \Delta - t^2$, where Δ is the Laplacian operator and $\tau \in C$ is a given parameter. Let Ω be a bounded connected domain in the real d-dimensional Euclidean spaceR^d, with sufficiently regular boundary$\partial\Omega$. The Helmholtz equation is given by

$$L_\tau u(x) = f(x) \quad x \in \Omega$$
$$\beta u(x) = g(x) \quad x \in \partial\Omega,$$

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whereβ defines a boundary linear operator,f and g are known functions on Ω and $\partial\Omega$, respectively. The boundary conditions could be of Dirichlet-type or Neumann-type. Finite element method (FEM) used for solving Helmholtz equation [9,10,11]. This method has high-order accurate and requires a generation of a mesh. Both coding and mesh generation for FEM become increasingly difficult when the dimension of the space increases. In the last decade, meshless methods using radial basis functions have been extensively developed for numerical approximation of partial differential equations. Meshless methods based on thin plate splines radial basis functions is proposed for solving numerically the modified Helmholtz equation.The collocation method based on RBFs have been used in [4,5,13,14,15]. We assume that the boundary Ω is sufficiently smooth to ensure the existence of a solution to the boundary problem(1). Paige and Saunders [18] used least-square (LSQR) method. Apart from LSQR on wave equations using splines [6,7] and cubic spline, Rashidinia et al. [19]. Mohanty et al. [17] have studied the cubic spline and compact finite difference method for the numerical solution of hyperbolic problems. In this paper we consider the one-space dimensional second-order quasi-linear hyperbolic equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial t^2}+\rho u=0 \quad 0<x<1, t>0 \tag{1}$$

with the following initial conditions:

$$u(x,0)=\phi(x), \quad u_t(0,t)=\psi(t), \quad 0<x<1 \tag{2}$$

And the boundary equations:

$$u(0,t)=p_0(t), \quad u(1,t)=p_1(t), \quad t>0 \tag{3}$$

2. Formulation and numerical technique

In this section formulations of this method are described. It is supposed that, the domain of equation (1) $[0,1]\times[0,t]$ is divided to an $(n+1)\times m$ such that: $h=\frac{1}{n+1}$ in x-direction and $k=\frac{t}{m}$ in t-direction,where n and m are positive integers. The mesh ratio parameter is given by $\lambda=\frac{k}{h}$. and the notation u_l^j be a grid function for discrete approximation at the grid point $(x_l,t_j)=(lh,jk)$.
In the mesh point (x_l,t_j) we can write the helmholtz differential equation as

$$u_{xxl}^j+u_{ttl}^j+\rho u_l^j=0 \tag{4}$$

A polynomial spline function $s_j(x)$ of class $C^2[x_0, x_n]$ which interpolate $u(x,t)$ in mesh point $x_0...x_n$ in each segment $[x_l, x_{l+1}]$ has the following form [2]

$$s_j(x) = \frac{(x_{l+1}-x)^3 M_l^j + (x-x_l)^3 M_{l+1}^j}{6h} + \frac{(x_{l+1}-x)u_l^j + (x-x_l)u_{l+1}^j}{h} \tag{5}$$
$$- \frac{h}{6}[(x_{l+1}-x)M_l^j + (x-x_l)M_{l+1}^j]$$
$$x_l \le x \le x_{l+1} \quad j = 1(1)m \quad l = 1(1)n$$

$$s_j(x) = \frac{(x_l-x)^3 M_{l-1}^j + (x-x_{l-1})^3 M_l^j}{6h} + \frac{(x_l-x)u_{l-1}^j + (x-x_{l-1})u_l^j}{h} \tag{6}$$
$$- \frac{h}{6}[(x_l-x)M_{l-1}^j + (x-x_{l-1})M_l^j]$$
$$x_{l-1} \le x \le x_l \quad j = 1(1)m \quad l = 1(1)n$$

$$s'_j(x) = \frac{-(x_l-x)^2}{2h}M_{l-1}^j + \frac{(x-x_{l-1})}{2h}M_l^j + \frac{u_l^j - u_{l-1}^j}{h} - \frac{h}{6}[M_l^j - M_{l-1}^j] \tag{7}$$
$$s''_j(x) = \frac{(x_l-x)}{h}M_{l-1} + \frac{(x-x_{l-1})}{h}M_l^j, \tag{8}$$

where

$$M_l^j = s''_j(x_l) = u_{xxl}^j = -(u_{ttl}^j + \rho u_l^j) \tag{9}$$

Using continuity of first derivative at (x_l, t_j) , that is $s'_j(x_l^+) = s'_j(x_l^-)$ the following relation of spline is obtained for $l = 1, 2, ..., n$.[2]

$$u_{l+1}^j - 2u_l^j + u_{l-1}^j = \frac{h^2}{6}[M_{l+1}^j + 2M_l^j + M_{l-1}^j], \tag{10}$$

Now by using the relations (9) and (10) we have

$$u_{l+1}^j - 2u_l^j + u_{l-1}^j = -\frac{h^2}{6}[u_{ttl+1}^j + \rho u_{l+1}^j + 2u_{ttl}^j + 2\rho u_l^j + u_{ttl-1}^j + \rho u_{l-1}^j] \tag{11}$$

We have following approximations for partial derivative

$$u_{ttl}^j \simeq \frac{u_l^{j+1} - 2u_l^j + u_l^{j-1}}{k^2}, \tag{12}$$

$$u^j_{ttl+1} \simeq \frac{u^{j+1}_{l+1} - 2u^j_{l+1} + u^{j-1}_{l+1}}{k^2}, \tag{13}$$

$$u^j_{ttl-1} \simeq \frac{u^{j+1}_{l-1} - 2u^j_{l-1} + u^{j-1}_{l-1}}{k^2}. \tag{14}$$

By substitution (12)-(14) in equation (11) we obtain the cubic spline method.

$$u^{j+1}_{l+1} + 4u^{j+1}_l + u^{j+1}_{l-1} = -(u^{j-1}_{l+1} + 4u^{j-1}_l + u^{j-1}_{l-1}) + a_0(u^j_{l+1} + u^j_{l-1}) + a_1 u^j_l \tag{15}$$

$$\lambda = \frac{k}{h}$$

$$a_0 = 2 - 6\lambda^2 - k^2\rho$$

$$a_1 = 8 - 4k^2\rho + 12\lambda^2$$

The cubic spline local truncation error is

$$\begin{aligned}\bar{T}^j_l &= u^j_{l+1} - 2u^j_l + u^j_{l-1} + \frac{h^2}{6}[u^j_{ttl+1} + \rho u^j_{l+1} + 2u^j_{ttl} \\ &\quad +2\rho u^j_l + u^j_{ttl-1} + \rho u^j_{l-1}] + O(k^4 + h^2k^2)\end{aligned} \tag{16}$$

Following by using the cubic spline finite difference method of Numerov type [17],we can obtain the modified method as follow

$$\begin{aligned}6\lambda^2(u^j_{l+1} - 2u^j_l + u^j_{l-1}) &= -\frac{k^2}{2}(\bar{u}^j_{tt_{l+1}} + \bar{u}^j_{tt_{l-1}} + 10\bar{u}^j_{tt_l}) \\ &-\frac{k^2}{2}\rho(\bar{u}^j_{l+1} + \bar{u}^j_{l-1} + 10\bar{u}^j_l) + \bar{T}^j_l \qquad l = 1(1)n,\end{aligned} \tag{17}$$

where

$$\bar{u}^j_{ttl} = \frac{u^{j+1}_l - 2u^j_l + u^{j-1}_l}{k^2} + O(k^2)$$

$$\bar{u}^j_{ttl+1} = \frac{u^{j+1}_{l+1} - 2u^j_{l+1} + u^{j-1}_{l+1}}{k^2} + O(k^2 + k^2h^2)$$

$$\bar{u}^j_{ttl-1} = \frac{u^{j+1}_{l-1} - 2u^j_{l-1} + u^{j-1}_{l-1}}{k^2} O(k^2 - k^2h^2).$$

The local truncation error of the method (17)

$$\begin{aligned}\bar{T}^j_l &= 6\lambda^2(u^j_{l+1} - 2u^j_l + u_{l-1}) + \frac{k^2}{2}(u^j_{tt_{l+1}} + u^j_{tt_{l-1}} + 10u^j_{tt_l}) \\ &+ \frac{k^2}{2}\rho(u^j_{l+1} + u^j_{l-1} + 10u^j_l) + O(k^4 + k^2h^4)\end{aligned}$$

By ignoring the truncation error in (17) and simplifying it,we have

$$u^{j+1}_{l+1} + u^{j+1}_{l-1} + 10u^{j+1}_l = a_1(u^j_{l+1} + u^j_{l-1}) + a_2 u^j_l - (u^{j-1}_{l+1} + u^{j-1}_{l-1} + 10u^{j-1}_l), \tag{18}$$

$$l = 1(1)n \quad j = 1(1)m-1,$$

Where

$$\gamma_1 = \frac{-1}{12\lambda^2}, \quad \gamma_2 = \frac{-h^2\rho}{12}, \quad a_1 = \frac{1}{\gamma_1} + 2 - \frac{\gamma_2}{\gamma_1}, \quad a_2 = \frac{-2}{\gamma_1} + 20 - 10\frac{\gamma_2}{\gamma_1}$$

The present method is an implicit three level scheme.By using boundary condition in (18) yield the three diagonal linear system.
To start any computation,it is necessary to know that the solution of u, at first time level is calculated by

$$u_l^1 = u_l^0 + ku_{t_l}^0 + \frac{k^2}{2}u_{tt_l}^0 + O(k^3) \tag{19}$$

From equation(1) we have:

$$(u_{tt})_l^0 = -((u_{xx})_l^0 + \rho u_l^0) \tag{20}$$

Thus using the initial conditions (2) we can obtain:

$$u_l^0 = \phi(lh) \tag{21}$$

$$(u_t)_l^0 = \psi(lh) \tag{22}$$

By substitute (20)-(22) in (19) we can compute the value of u at first level

$$u_l^1 = \phi(lh) + k\psi(lh) - \frac{k^2}{2}(\phi_{xx}(lh) + \rho\phi(lh)) + O(k^3) \tag{23}$$

3. Stability of the method

For stability of the method (18), we follow the technique used by [16].
let $\varepsilon_l^j = U_l^j - u_l^j$ be the discretization error(in absent of round-off error) at each interval grid point (x_l, t_j).
Now by neglecting truncation error we obtain an error equation

$$\varepsilon_{l+1}^{j+1} + \varepsilon_{l-1}^{j+1} + 10\varepsilon_l^{j+1} = a_1(\varepsilon_{l+1}^j + \varepsilon_{l-1}^j) + a_2\varepsilon_l^j - (\varepsilon_{l+1}^{j-1} + \varepsilon_{l+1}^{j-1} + \varepsilon_{l+1}^{j-1}) \tag{24}$$

To establish stability for the difference scheme(18),we substitute

$$\varepsilon_l^j = A\xi^j e^{i\beta lh}$$

into the equation (24) ,where A is a non-zero parameter to be determined,ξ is in general complex,β is an arbitrary real number and $i = \sqrt{-1}$.We obtain the characteristic equation

$$\xi^2(e^{i\beta h} + e^{-i\beta h} + 10) - a_1\xi(e^{i\beta h} + e^{-i\beta h}) - a_2\xi + (e^{i\beta h} + e^{-i\beta h} + 10) = 0, \tag{25}$$

Therefore

$$\xi^2(-4\sin^2\frac{\beta h}{2}+12)+\xi(4a_1\sin^2\frac{\beta h}{2}-12a_1-a_2)+(-4\sin^2\frac{\beta h}{2}+12)=0, \quad (26)$$

An alternative to this procedure consists of applying transformation to the characteristic equation (26) which maps the interior of the unit circle onto the half-plane .Then using the Routh-Hurwitz criterion [12] which gives the necessary and sufficient conditions for the characteristic equation to have negative real part.
The transformation is

$$\xi=\frac{1+z}{1-z},$$

so equation (26) is translated to the following equation

$$(-4\sin^2\frac{\beta h}{2}+24-4a_1\sin^2\frac{\beta h}{2}+12a_1+a_2-4\sin^2\frac{\beta h}{2})z^2$$
$$-8\sin^2\frac{\beta h}{2}+24+4a_1\sin^2\frac{\beta h}{2}-12a_1-a_2=0,$$

For stability by Routh-Hurwitz criterion it is required that

$$-8\sin^2\frac{\beta h}{2}+24-4a_1\sin^2\frac{\beta h}{2}+12a_1+a_2>0 \quad (27)$$

$$-8\sin^2\frac{\beta h}{2}+24+4a_1\sin^2\frac{\beta h}{2}-12a_1-a_2>0 \quad (28)$$

By (27) and(28) we obtain that

$$k^2<\frac{10}{\frac{36}{h^2}+9\rho} \quad (29)$$

Or the mesh ratio satisfied in

$$\lambda^2<\frac{10}{36+9\rho h^2} \quad (30)$$

Thus the presented method is conditionally stable.

4. Numerical illustrations

To illustrate accuracy and ability of the present method by applying (18) and (15) on Helmholtz equation(1) subjected to the following initial and boundary conditions

$$\begin{aligned}
u(x,0) &= C(A\cos(x\mu_1) + B\sin(x\mu_2)),\\
u_t(0,t) &= D(A\cos(x\mu_1) + B\sin(x\mu_1)\mu_2 \quad 0 < x < 1\\
u(0,t) &= A(C\cos(t\mu_2) + D\sin(t\mu_2)),\\
u(1,t) &= (A\cos(\mu_1) + B\sin(\mu_1))(C\cos(t\mu_2) + D\sin(t\mu_2)), \qquad t > 0
\end{aligned}$$

Whose exact solution are known to us.

$$\begin{aligned}
u(x,t) &= (A\cos\mu_1 x + B\sin\mu_1 x)(C\cos\mu_2 t + D\sin\mu_2 t),\\
\rho &= \mu_1^2 + \mu_2^2.
\end{aligned}$$

WhereA, B, C and$D$ are constant.

With various values of $h = 0.1$ and $k = 0.01$ .The maximum absolute error in the solution are tabulated in tables 1-3. We compare the modified method(18) by cubic spline method in (16) we obtain

The maximum norm error between the exact solution and approximation solution $u(x_l, t_j)$ at the mesh point is computed.

x	scheme(18)	scheme(15)
0.1	5.91541(−16)	5.89633(−15)
0.3	6.80012(−16)	5.78877(−15)
0.5	7.04298(−16)	4.13211(−15)
0.7	5.77663(−16)	3.53884(−15)
0.9	3.41741(−16)	3.48853(−15)

Table 1. Maximum absolute error for $t = 0.2$, $h = 0.1$, $k = 0.01$, $\mu_1 = \mu_2 = 0.01$.

x	scheme(18)	scheme(15)
0.1	3.23237(−12)	8.95593(−10)
0.3	7.16232(−12)	2.09046(−9)
0.5	7.59292(−12)	2.21997(−9)
0.7	5.63978(−12)	1.61717(−9)
0.9	2.15612(−12)	5.96565(−10)

Table 2. Maximum absolute error for $t = 0.6$, $h = 0.1$, $k = 0.01$, $\mu_1 = \mu_2 = 0.01$.

5. conclusion

In this article,we have outlined a new idea for solving Helmholtz equation by using Numerov type finite difference method based on cubic spline approximation.The method

x	scheme(18)	scheme(15)
0.1	4.09352(−8)	4.45937(−4)
0.3	1.0054(−7)	1.13972(−3)
0.5	1.14179(−7)	1.3571(−3)
0.7	8.66733(−8)	1.05986(−3)
0.9	3.21878(−8)	3.96548(−4)

Table 3. Maximum absolute error for $t = 0.8$, $h = 0.1$, $k = 0.01$, $\mu_1 = \mu_2 = 0.01$.

contain 9-grid points of order $O(k^2+h^4)$.This numerical method is conditionally stable.It has been found that the present algorithm gives considerable accurate numerical results and it is more efficient than the cubic spline method.

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