

G -Frames, g -orthonormal bases and g -Riesz bases

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Abstract. G -Frames in Hilbert spaces are a redundant set of operators which yield a representation for each vector in the space. In this paper we investigate the connection between g -frames, g -orthonormal bases and g -Riesz bases. We show that a family of bounded operators is a g -Bessel sequences if and only if the Gram matrix associated to its defines a bounded operator.

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1. Introduction

Let \mathcal{H}, \mathcal{K} be separable Hilbert spaces, let I, J and every J_i denote the countable (or finite) index sets. Let $\{W_j\}_{j \in J}$ is a sequence of closed subspaces of \mathcal{K} and let $B(\mathcal{H}, W_j)$ denote the algebra of all bounded linear operators from \mathcal{H} to W_j . Recall that a family $\mathcal{F} = \{f_j\}_{j \in J}$ is called a frame for \mathcal{H} if there exist constants $0 < A \leq B < \infty$ such that,

$$A\|f\|^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}. \quad (1)$$

Frames have many nice properties which make them very useful in the characterization of function spaces, signal processing and many other fields. Gabor [5], in 1946 introduced a technique for signal processing which eventually led to wavelet theory. Later in 1952, Duffin and Schaeffer [3] in the context of nonharmonic Fourier series introduced

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frame theory for Hilbert spaces. In 1986, Daubechies, Grassman and Meyer [2] showed that Duffin and Schaeffers definition was an abstraction of Gabors concept. Sun in [6] introduced g -frames and g -Riesz bases in a complex Hilbert space and discussed some properties of them. We refer to [1, 4, 7] for an introduction to the frames and g -frames and its applications.

Definition 1.1 A family $\Lambda = \{\Lambda_j \in B(\mathcal{H}, W_j) \mid j \in J\}$ is called a generalized frame, or simply a g -frame for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ if there are two positive constant C and D such that

$$C\|f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq D\|f\|^2 \quad \forall f \in \mathcal{H}. \quad (2)$$

The real numbers $0 < C \leq D < \infty$ are called the lower and upper g -frame bounds, respectively. We call this family a g -frame for \mathcal{H} with respect to \mathcal{K} whenever $\mathcal{K} = W_j$ for all $j \in J$. The family Λ is called a C -tight g -frame if $C = D$ and if $C = D = 1$, it's called a Parseval g -frame, the $\sup\{rank(\Lambda_j) : j \in J\}$ is called the multiplicity of the g -frame. If $\|\Lambda_i\| = \|\Lambda_j\| = \lambda$ for all $i, j \in J$, then the g -frame is called λ -uniform. If we only have the upper bound, we call Λ a g -Bessel sequence for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ with g -Bessel bound D . The family Λ is called

- (i) A g -complete set for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ if $\mathcal{H} = \overline{\text{span}\{\Lambda_j^*(W_j)\}_{j \in J}}$.
- (ii) A g -orthonormal system for \mathcal{H} with respect to $\{W_j\}_{j \in J}$, if:

$$\langle \Lambda_i^* g, \Lambda_j^* g' \rangle = \delta_{ij} \langle g, g' \rangle \quad \forall i, j \in J, g \in W_i, g' \in W_j.$$

- (iii) A g -orthonormal basis for \mathcal{H} with respect to $\{W_j\}_{j \in J}$, if it is a g -orthonormal system for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ and $\{\Lambda_j^*(e_{ij})\}_{j \in J, i \in J_j}$ is a basis for \mathcal{H} , where $\{e_{ij}\}_{i \in J_j}$ is an orthonormal basis for W_j for all $j \in J$.

Notation 1.2 Let $\Lambda = \{\Lambda_j\}_{j \in J}$ be a g -frame for \mathcal{H} with respect to $\{W_j\}_{j \in J}$. The representation space associated to Λ denotes by

$$\left(\sum_{j \in J} \oplus W_j \right)_{\ell^2} = \left\{ \{g_j\}_{j \in J} \mid g_j \in W_j \text{ and } \sum_{j \in J} \|g_j\|^2 < \infty \right\}. \quad (3)$$

which is a Hilbert space with inner product as follows:

$$\langle \{g_j\}_{j \in J}, \{g'_k\}_{k \in J} \rangle = \sum_{j \in J} \langle g_j, g'_j \rangle \quad \forall \{g_j\}_{j \in J}, \{g'_j\}_{j \in J} \in \left(\sum_{j \in J} \oplus W_j \right)_{\ell^2}.$$

Moreover, if $\{e_{ij}\}_{i \in J_j}$ is an orthonormal basis for W_j for all $j \in J$, then $\{u_{ij}\}_{j \in J, i \in J_j}$ is called the standard orthonormal basis of $\left(\sum_{j \in J} \oplus W_j \right)_{\ell^2}$ where $u_{ij} = \{\delta_{kj} e_{ij}\}_{k \in J}$ and δ_{kj} is the Kronecker delta.

Definition 1.3 The synthesis operator of a g -frame $\Lambda = \{\Lambda_j\}_{j \in J}$ is defined by

$$\Theta_\Lambda : \left(\sum_{j \in J} \oplus W_j \right)_{\ell^2} \longrightarrow \mathcal{H} \quad \text{with} \quad \Theta_\Lambda(\{g_j\}_{j \in J}) = \sum_{j \in J} \Lambda_j^* g_j \quad (4)$$

The associated adjoint operator given by

$$\Theta_{\Lambda}^* : \mathcal{H} \longrightarrow \left(\sum_{j \in J} \oplus W_j \right)_{\ell^2} \quad \text{with} \quad \Theta_{\Lambda}^*(f) = \{\Lambda_j f\}_{j \in J}. \quad (5)$$

is called the analysis operator. By composing Θ_{Λ} and Θ_{Λ}^* we obtain the g -frame operator

$$S_{\Lambda} : \mathcal{H} \longrightarrow \mathcal{H} \quad \text{with} \quad S_{\Lambda}(f) = \Theta_{\Lambda} \Theta_{\Lambda}^*(f) = \sum_{j \in J} \Lambda_j^* \Lambda_j(f) \quad (6)$$

which is a bounded, invertible, and positive operator. This provides the reconstruction formula

$$f = \sum_{j \in J} \Lambda_j^* \tilde{\Lambda}_j f = \sum_{j \in J} \tilde{\Lambda}_j^* \Lambda_j f \quad (7)$$

where $\tilde{\Lambda}_j = \Lambda_j S_{\Lambda}^{-1}$. The family $\tilde{\Lambda} = \{\tilde{\Lambda}_j\}_{j \in J}$ is also a g -frame for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ with g -frame bounds $\frac{1}{D}$ and $\frac{1}{C}$ respectively.

The well-known relations between a frame and associated analysis and synthesis operator also holds in g -frames situation.

Theorem 1.4 Let $\Lambda_j \in B(\mathcal{H}, W_j)$ for all $j \in J$. Then the following are equivalent:

- (i) $\Lambda = \{\Lambda_j\}_{j \in J}$ is a g -frame for \mathcal{H} with respect to $\{W_j\}_{j \in J}$.
- (ii) The synthesis operator Θ_{Λ} is bounded, linear and onto.
- (iii) The analysis operator Θ_{Λ}^* is injective with closed range.

Proof. This claim holds in an analogous way as in frame theory. ■

A family $\Lambda = \{\Lambda_j\}_{j \in J}$ is called a g -frame sequence for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ if Λ is a g -frame for $\overline{\text{span}}\{\Lambda_j^*(W_j)\}_{j \in J}$ with respect to $\{W_j\}_{j \in J}$. The definition shows that if Λ is a g -frame for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ then Λ is g -complete set for \mathcal{H} with respect to $\{W_j\}_{j \in J}$. Theorem 1.4 leads to a statement about g -frame sequence.

Corollary 1.5 A sequence $\Lambda = \{\Lambda_j\}_{j \in J}$ is a g -frame sequence for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ if and only if

$$\Theta_{\Lambda} : \left(\sum_{j \in J} \oplus W_j \right)_{\ell^2} \longrightarrow \mathcal{H} \quad , \quad \Theta_{\Lambda}(\{g_j\}_{j \in J}) = \sum_{j \in J} \Lambda_j^* g_j,$$

is a well-defined bounded operator with closed range.

Proof. This follows immediately from Theorem 1.4. ■

2. G -Frames, g -orthonormal bases and g -Riesz bases

If $\Lambda = \{\Lambda_j\}_{j \in J}$ is a g -Bessel sequence for \mathcal{H} with respect to $\{W_j\}_{j \in J}$, then the Gram matrix associated to Λ is defined by

$$\Theta_{\Lambda}^* \Theta_{\Lambda} = \{ \langle \Lambda_j^* e_{ij}, \Lambda_n^* e_{mn} \rangle \}_{j, n \in J, i \in J_j, m \in J_n}. \quad (8)$$

Theorem 2.1 Let $\Lambda_j \in B(\mathcal{H}, W_j)$ for all $j \in J$, then the following are equivalent:

- (i) $\Lambda = \{\Lambda_j\}_{j \in J}$ is g -Bessel sequence with bound B for \mathcal{H} with respect to $\{W_j\}_{j \in J}$.
(ii) The Gram matrix associated to Λ defines a bounded operator on $(\sum_{j \in J} \oplus W_j)_{\ell^2}$, with norm at most B .

Proof. (i) \Rightarrow (ii) Let $\{g_j\}_{j \in J} \in (\sum_{j \in J} \oplus W_j)_{\ell^2}$, we show that $\Theta_\Lambda(\{g_j\}_{j \in J})$ is well-defined. Fix $I \subseteq J$ with $|I| < \infty$, we have

$$\begin{aligned} \left\| \sum_{i \in I} \Lambda_i^* g_i \right\|^2 &= \sup_{\|f\|=1} \left| \langle f, \sum_{i \in I} \Lambda_i^* g_i \rangle \right|^2 \\ &= \sup_{\|f\|=1} \left| \sum_{i \in I} \langle \Lambda_i f, g_i \rangle \right|^2 \\ &\leq \sup_{\|f\|=1} \left(\sum_{i \in I} \|\Lambda_i f\|^2 \right) \left(\sum_{i \in I} \|g_i\|^2 \right) \leq B \sum_{i \in I} \|g_i\|^2. \end{aligned}$$

It follows that $\sum_{j \in J} \Lambda_j^* g_j$ is weakly unconditionally Cauchy and hence unconditionally convergent in \mathcal{H} . Thus $\Theta_\Lambda(\{g_j\}_{j \in J})$ is well-defined. Clearly $\Theta_\Lambda^* \Theta_\Lambda$ is bounded and $\|\Theta_\Lambda^* \Theta_\Lambda\| \leq B$.

(ii) \Rightarrow (i) suppose that $\{g_j\}_{j \in J} \in (\sum_{j \in J} \oplus W_j)_{\ell^2}$, then we have

$$\sum_{j \in J} \left\| \sum_{k \in J} \Lambda_j \Lambda_k^* g_k \right\|^2 \leq B^2 \sum_{j \in J} \|g_j\|^2.$$

Given arbitrary $I \subseteq J$ with $|I| < \infty$, we have

$$\begin{aligned} \left\| \sum_{i \in I} \Lambda_i^* g_i \right\|^4 &= \left| \langle \sum_{i \in I} \Lambda_i^* g_i, \sum_{k \in I} \Lambda_k^* g_k \rangle \right|^2 \\ &= \left| \sum_{i \in I} \langle g_i, \sum_{k \in I} \Lambda_i \Lambda_k^* g_k \rangle \right|^2 \\ &\leq \left(\sum_{i \in I} \|g_i\|^2 \right) \left(\sum_{i \in I} \left\| \sum_{k \in I} \Lambda_i \Lambda_k^* g_k \right\|^2 \right) \\ &\leq B^2 \left(\sum_{i \in I} \|g_i\|^2 \right)^2. \end{aligned}$$

It follows that $\Theta_\Lambda(\{g_j\}_{j \in J}) = \sum_{j \in J} \Lambda_j^* g_j$ is convergent and $\|\Theta_\Lambda\| \leq \sqrt{B}$. Hence for all $f \in \mathcal{H}$ we obtain

$$\sum_{j \in J} \|\Lambda_j f\|^2 = \|\Theta_\Lambda^* f\|^2 \leq B \|f\|^2.$$

■

Theorem 2.2 Let $\{\Lambda_j\}_{j \in J}$ and $\{\Gamma_j\}_{j \in J}$ be g -Bessel sequences for \mathcal{H} with respect to $\{W_j\}_{j \in J}$, $\{V_j\}_{j \in J}$ then the series $\sum_{j \in J} \Gamma_j^* \Lambda_j f$ converges unconditionally for all $f \in \mathcal{H}$.

Proof. Since $\{\Lambda_j\}_{j \in J}$ is a g -Bessel sequence for \mathcal{H} with respect to $\{W_j\}_{j \in J}$, hence $\{\Lambda_j f\}_{j \in J} \in \ell^2(\mathcal{K}, J)$ for all $f \in \mathcal{H}$. Fix $I \subset J$ with $|I| < \infty$, and let B be the g -Bessel

bound of $\{\Gamma_j\}_{j \in J}$. Then for all $f \in \mathcal{H}$ we have

$$\begin{aligned} \left\| \sum_{i \in I} \Gamma_i^* \Lambda_i f \right\|^2 &= \sup_{\|g\|=1} \left| \left\langle \sum_{i \in I} \Gamma_i^* \Lambda_i f, g \right\rangle \right|^2 \\ &= \sup_{\|g\|=1} \left| \sum_{i \in I} \langle \Lambda_i f, \Gamma_i g \rangle \right|^2 \\ &\leq \sup_{\|g\|=1} \left(\sum_{i \in I} \|\Lambda_i f\|^2 \right) \left(\sum_{i \in I} \|\Gamma_i g\|^2 \right) \\ &\leq B \sum_{i \in I} \|\Lambda_i f\|^2. \end{aligned}$$

It follows that $\sum_{j \in J} \Gamma_j^* \Lambda_j f$ is weakly unconditionally Cauchy and hence unconditionally convergent in \mathcal{H} . ■

Proposition 2.3 Let $\{\Xi_j\}_{j \in J}, \{\Xi'_j\}_{j \in J}$ be g -orthonormal bases for \mathcal{H}, \mathcal{U} with respect to $\{W_j\}_{j \in J}$ respectively, and let $T : \mathcal{H} \rightarrow \mathcal{U}$ be a bounded linear operator such that $\Xi'_j T = \Xi_j$ for all $j \in J$. Then T is a unitary operator.

Proof. For each $f \in \mathcal{H}$ we have

$$\|Tf\|^2 = \sum_{j \in J} \|\Xi'_j Tf\|^2 = \sum_{j \in J} \|\Xi_j f\|^2 = \|f\|^2,$$

which implies that T is an isometry operator. Further for every $g \in \mathcal{U}$ we compute

$$T^*g = \sum_{j \in J} T^* \Xi'_j \Xi'_j g = \sum_{j \in J} \Xi_j^* \Xi'_j g.$$

This yields

$$\begin{aligned} \|T^*g\|^2 &= \left\langle \sum_{j \in J} \Xi_j^* \Xi'_j g, \sum_{k \in J} \Xi_k^* \Xi'_k g \right\rangle \\ &= \sum_{j \in J} \sum_{k \in J} \delta_{jk} \langle \Xi'_j g, \Xi'_k g \rangle \\ &= \sum_{j \in J} \|\Xi'_j g\|^2 = \|g\|^2. \end{aligned}$$

Thus T is a co-isometry, which finishes the proof. ■

Theorem 2.4 Let $\Xi = \{\Xi_j\}_{j \in J}$ be a g -orthonormal system for \mathcal{H} with respect to $\{W_j\}_{j \in J}$. Then the following conditions are equivalent:

- (i) Ξ is a g -orthonormal basis for \mathcal{H} with respect to $\{W_j\}_{j \in J}$.
- (ii) $f = \sum_{j \in J} \Xi_j^* \Xi_j f \quad \forall f \in \mathcal{H}$.
- (iii) $\|f\|^2 = \sum_{j \in J} \|\Xi_j^* \Xi_j f\|^2 \quad \forall f \in \mathcal{H}$.
- (iv) $\|f\|^2 = \sum_{j \in J} \|\Xi_j f\|^2 \quad \forall f \in \mathcal{H}$.
- (v) $\langle f, g \rangle = \sum_{j \in J} \langle \Xi_j f, \Xi_j g \rangle \quad \forall f, g \in \mathcal{H}$.
- (vi) If $\Xi_j f = 0$ for all $j \in J$ then $f = 0$.

- (vii) $\mathcal{H} = \overline{\text{span}}\{\Xi_j^* \Xi_j(\mathcal{H})\}_{j \in J}$.
- (viii) $\mathcal{H} = \overline{\text{span}}\{\Xi_j^*(W_j)\}_{j \in J}$.

Proof. (i) \Rightarrow (ii) By the assumptions $\{\Xi_j^* e_{ij}\}_{j \in J, i \in J_j}$ is an orthonormal basis for \mathcal{H} . Therefore for all $f \in \mathcal{H}$ we have

$$\langle \sum_{j \in J} \Xi_j^* \Xi_j f, f \rangle = \sum_{j \in J} \|\Xi_j f\|^2 = \sum_{j \in J} \sum_{i \in J_j} |\langle f, \Xi_j^* e_{ij} \rangle|^2 = \|f\|^2 = \langle f, f \rangle .$$

From this (ii) follows.

(ii) \Rightarrow (iii) Since Ξ is a g -orthonormal system for \mathcal{H} with respect to $\{W_j\}_{j \in J}$, hence for all $f \in \mathcal{H}$ and $j \in J$ we have $(\Xi_j^* \Xi_j)^2 f = \Xi_j^* \Xi_j f$. This yields

$$\|f\|^2 = \langle \sum_{j \in J} \Xi_j^* \Xi_j f, f \rangle = \sum_{j \in J} \|\Xi_j^* \Xi_j f\|^2,$$

which implies (iii). The implications (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) are clear. To prove (vi) \Rightarrow (vii) assume that $f \perp \overline{\text{span}}\{\Xi_j^* \Xi_j(\mathcal{H})\}_{j \in J}$, hence $\|\Xi_j f\|^2 = \langle f, \Xi_j^* \Xi_j f \rangle = 0$ and so $\Xi_j f = 0$ for all $j \in J$, it shows that $f = 0$ and thus (vii) follows. Also the implication (vii) \Rightarrow (viii) is obvious. To prove (viii) \Rightarrow (i) suppose that

$$\mathcal{A} = \left\{ f \in \mathcal{H} : \sum_{j \in J} \Xi_j^* \Xi_j f = f \right\}.$$

It is obvious that \mathcal{A} is a closed subspace of \mathcal{H} . It follows by assumption that for every $i, j \in J$ and $f \in \mathcal{H}$ we have $\langle \Xi_j^* \Xi_j f, \Xi_i^* \Xi_i f \rangle = \delta_{ij} \langle \Xi_j f, \Xi_i f \rangle$, which implies that $\Xi_j^* \Xi_j \Xi_i^* \Xi_i = \delta_{ij} \Xi_j^* \Xi_i$. Thus $\Xi_j^* \Xi_j f \in \mathcal{A}$. Now suppose $f \in \mathcal{A}^\perp$, then for all $j \in J$ we compute $\|\Xi_j f\|^2 = \langle \Xi_j^* \Xi_j f, f \rangle = 0$, hence $\Xi_j f = 0$. Let $j \in J$ and $g \in W_j$ then $\langle f, \Xi_j^* g \rangle = \langle \Xi_j f, g \rangle = 0$. It follows that $f \perp \overline{\text{span}}\{\Xi_j^*(W_j)\}_{j \in J} = \mathcal{H}$ and so $f = 0$, therefore $\mathcal{H} = \mathcal{A}$. For every $f \in \mathcal{H}$ we further have

$$f = \sum_{j \in J} \Xi_j^* \Xi_j f = \sum_{j \in J} \sum_{i \in J_j} \langle f, \Xi_j^* e_{ij} \rangle \Xi_j^* e_{ij}.$$

From this the result follows. ■

Let $\{Z_j\}_{j \in J}$ be a family of closed subspaces in \mathcal{H} , then $\{Z_j\}_{j \in J}$ is called an orthonormal fusion basis for \mathcal{H} if $\mathcal{H} = \bigoplus_{j \in J} Z_j$.

Corollary 2.5 Let $\Xi = \{\Xi_j\}_{j \in J}$ be a g -orthonormal basis for \mathcal{H} with respect to $\{W_j\}_{j \in J}$, and let $V_j = \Xi_j^* \Xi_j(\mathcal{H})$ for all $j \in J$. Then $\{V_j\}_{j \in J}$ is an orthonormal fusion basis for \mathcal{H} .

Proof. This claim follows immediately from the fact that for each $i, j \in J$ we have

$$\Xi_j^* \Xi_j \Xi_i^* \Xi_i = \delta_{ij} \Xi_j^* \Xi_i.$$

■

Corollary 2.6 Let $\Xi = \{\Xi_j\}_{j \in J}$ be a Parseval g -frame of co-isometries for \mathcal{H} with respect to $\{W_j\}_{j \in J}$. Then Ξ is a g -orthonormal basis for \mathcal{H} with respect to $\{W_j\}_{j \in J}$.

Proof. Fix $i \in J$, since Ξ is a Parseval g -frame for \mathcal{H} with respect to $\{W_j\}_{j \in J}$. Thus for every $g \in W_i$, we have

$$\|\Xi_i^* g\|^2 = \sum_{j \in J} \|\Xi_j \Xi_i^* g\|^2 = \|\Xi_i^* g\|^2 + \sum_{\substack{j \in J \\ j \neq i}} \|\Xi_j \Xi_i^* g\|^2.$$

Hence $\sum_{\substack{j \in J \\ j \neq i}} \|\Xi_j \Xi_i^* g\|^2 = 0$. So $\Xi_j \Xi_i^* g = 0$ for all $j \neq i$. This shows that $\Xi = \{\Xi_j\}_{j \in J}$ is a g -orthonormal system for \mathcal{H} with respect to $\{W_j\}_{j \in J}$. Now the result follows from the Theorem 2.4. ■

Definition 2.7 Let $\{\Lambda_j\}_{j \in J}$ and $\{\Gamma_j\}_{j \in J}$ be sequences for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ and $\{V_j\}_{j \in J}$ respectively. Then

(i) $\{\Lambda_j\}_{j \in J}$ and $\{\Gamma_j\}_{j \in J}$ are said to be biorthogonal for \mathcal{H} with respect to $\{W_j\}_{j \in J}, \{V_j\}_{j \in J}$ if

$$\langle \Lambda_i^* g, \Gamma_j^* g' \rangle = \delta_{ij} \langle g, g' \rangle \quad \forall i, j \in J, g \in W_i, g' \in V_j.$$

(ii) $\{\Lambda_j\}_{j \in J}$ is called a g -Riesz basis for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ if it's g -complete set for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ and there exist constants $0 < A \leq B < \infty$ such that for any finite subset $I \subset J$ and $g_i \in W_i, (i \in I)$ we have

$$A \sum_{i \in I} \|g_i\|^2 \leq \left\| \sum_{i \in I} \Lambda_i^* g_i \right\|^2 \leq B \sum_{i \in I} \|g_i\|^2. \tag{9}$$

Theorem 2.8 Let $\Lambda = \{\Lambda_j\}_{j \in J}$ be a g -Riesz basis for \mathcal{H} with respect to $\{W_j\}_{j \in J}$, then there exists a sequence $\Gamma = \{\Gamma_j\}_{j \in J}$ for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ such that

$$f = \sum_{j \in J} \Gamma_j^* \Lambda_j f \quad \forall f \in \mathcal{H}. \tag{10}$$

Γ is also a g -Riesz basis for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ and Λ, Γ are biorthogonal for \mathcal{H} with respect to $\{W_j\}_{j \in J}$. Moreover the series (10) converges unconditionally for all $f \in \mathcal{H}$.

Proof. By [6, Corollary 3.4] there is a g -orthonormal basis $\{\Xi_j\}_{j \in J}$ for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ and a bounded invertible operator T on \mathcal{H} such that $\Lambda = \{\Lambda_j\}_{j \in J} = \{\Xi_j T\}_{j \in J}$. Put $\Gamma_j = \Xi_j (T^{-1})^*$ for all $j \in J$. Obviously $\Gamma = \{\Gamma_j\}_{j \in J}$ is a g -Riesz basis for \mathcal{H} with respect to $\{W_j\}_{j \in J}$ and we have

$$\langle \Lambda_i^* g, \Gamma_j^* g' \rangle = \langle T^* \Xi_j^* g, T^{-1} \Xi_j^* g' \rangle = \delta_{ij} \langle g, g' \rangle \quad \forall i, j \in J, g \in W_i, g' \in W_j$$

which implies that Λ, Γ are biorthogonal for \mathcal{H} with respect to $\{W_j\}_{j \in J}$. Moreover, for all $f \in \mathcal{H}$ we observe that

$$\sum_{j \in J} \Gamma_j^* \Lambda_j f = \sum_{j \in J} T^{-1} \Xi_j^* \Xi_j T f = T^{-1} T f = f.$$

Since every g -Riesz basis is a g -Bessel sequence thus, convergent unconditionally of the above series follows by Theorem 2.2. ■

Let $\Lambda = \{\Lambda_j\}_{j \in J}$ and $\Gamma = \{\Gamma_j\}_{j \in J}$ be g -Bessel sequences for \mathcal{H} with respect to $\{W_j\}_{j \in J}, \{V_j\}_{j \in J}$ respectively. Then Γ is called a dual g -frame of Λ for \mathcal{H} with respect to $\{V_j\}_{j \in J}, \{W_j\}_{j \in J}$ if

$$f = \sum_{j \in J} \Gamma_j^* \Lambda_j f \quad \forall f \in \mathcal{H}.$$

It is easy to check that Γ is a dual g -frame of Λ for \mathcal{H} with respect to $\{V_j\}_{j \in J}, \{W_j\}_{j \in J}$ if and only if $\Theta_\Gamma \Theta_\Lambda^* = Id_{\mathcal{H}}$, in this case Λ and Γ are also g -frames for \mathcal{H} with respect to $\{W_j\}_{j \in J}, \{V_j\}_{j \in J}$. Since $\Theta_\Lambda \Theta_\Gamma^* = Id_{\mathcal{H}}$, hence Λ is also a dual g -frame of Γ for \mathcal{H} with respect to $\{W_j\}_{j \in J}, \{V_j\}_{j \in J}$.

The following example shows that the dual g -frame of a g -orthonormal basis is not unique.

Example 2.9 Fix some $n \in \mathbb{N}, 1 \leq j \leq n$ and define $W_j \subset \mathbb{C}^{n+1}$, by $W_j = \text{span}\{\sum_{k=1}^{j+1} e_k\}$, where $\{e_i\}_{i=1}^{n+1}$ is the standard orthonormal basis for \mathbb{C}^{n+1} . Also define

$$\Xi_j : \mathbb{C}^n \rightarrow W_j \quad \text{with} \quad \Xi_j(\{z_i\}_{i=1}^n) = \frac{z_j}{\sqrt{j+1}} \sum_{k=1}^{j+1} e_k.$$

Then $\Xi = \{\Xi_j\}_{j=1}^n$ is a g -orthonormal basis for \mathbb{C}^n with respect to $\{W_j\}_{j=1}^n$. Therefore by Theorem 2.4, Ξ is a dual g -frame of itself for \mathbb{C}^n with respect to $\{W_j\}_{j=1}^n$. Now if for each $1 \leq j \leq n$, we define $V_j = \text{span}\{e_j\}$ and

$$\Gamma_j : \mathbb{C}^n \rightarrow V_j \quad \text{and} \quad \Gamma_j(\{z_i\}_{i=1}^n) = \sqrt{j+1} z_j e_j.$$

Then for all $z \in \mathbb{C}^n$ we have $z = \sum_{j=1}^n \Gamma_j^* \Xi_j z$, that is $\Gamma = \{\Gamma_j\}_{j=1}^n$ is a dual g -frame of $\Xi = \{\Xi_j\}_{j=1}^n$ for \mathbb{C}^n with respect to $\{V_j\}_{j=1}^n, \{W_j\}_{j=1}^n$ respectively.

Proposition 2.10 Let $\Lambda = \{\Lambda_j\}_{j \in J}$ be a g -frame for \mathcal{H} with respect to $\{W_j\}_{j \in J}$, then there exists a g -orthonormal basis $\{\Xi_j\}_{j \in J}$ for $(\sum_{j \in J} \oplus W_j)_{\ell^2}$ with respect to $\{W_j\}_{j \in J}$ such that $\Xi_j \Theta_\Lambda^* = \Lambda_j$ for all $j \in J$.

Proof. For all $j \in J$ define $\Xi_j : (\sum_{j \in J} \oplus W_j)_{\ell^2} \rightarrow W_j$ by $\Xi_j(\{g_k\}_{k \in J}) = g_j$, then $\Xi_j^* g = \{\delta_{kj} \pi_{W_k} g\}_{k \in J}$ for all $g \in \mathcal{K}$, where δ_{kj} is the Kronecker delta. First of all, $\{\Xi_j\}_{j \in J}$ is a g -orthonormal system for \mathcal{H} with respect to $\{W_j\}_{j \in J}$. To see this, let $g \in W_j, g' \in W_i$ and $i, j \in J$. Then we have

$$\begin{aligned} \langle \Xi_j^* g, \Xi_i^* g' \rangle &= \sum_{k \in J} \delta_{kj} \delta_{ki} \langle \pi_{W_k} g, \pi_{W_k} g' \rangle \\ &= \delta_{ji} \langle \pi_{W_j} g, \pi_{W_j} g' \rangle = \delta_{ji} \langle g, g' \rangle. \end{aligned}$$

On the other hand for any $g = \{g_k\}_{k \in J} \in (\sum_{j \in J} \oplus W_j)_{\ell^2}$ we compute

$$\sum_{j \in J} \|\Xi_j g\|^2 = \sum_{j \in J} \|g_j\|^2 = \|g\|^2.$$

By Theorem 2.4 $\{\Xi_j\}_{j \in J}$ is a g -orthonormal basis for $(\sum_{j \in J} \oplus W_j)_{\ell^2}$ with respect to $\{W_j\}_{j \in J}$. It is easy to check that $\Xi_j \Theta_\Lambda^* = \Lambda_j$ for all $j \in J$. Now the conclusion follows. ■

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