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# Some results on graded $S$-strongly prime submodules

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**Abstract.** Let $G$ be a group with identity $e$ and $R$ be a commutative $G$-graded ring with nonzero identity, $S \subseteq h(R)$ a multiplicatively closed subset of $R$ and $M$ be a graded $R$-module. A graded submodule $N$ of $M$ with $(N :_R M) \cap S = \emptyset$ is said to be graded $S$-strongly prime if there exists $s \in S$ such that whenever $((N + Rx_g) :_R M)y_h \subseteq N$, then $sx_g \in N$ or $sy_h \in N$ for all $x_g, y_h \in h(M)$. The aim of this paper is to introduce and investigate some basic properties of the notion of graded $S$-strongly prime submodules, especially in graded multiplication modules. Moreover, we investigate the behaviour of this structure under graded module homomorphisms, localizations of graded modules, quotient graded modules, Cartesian product.

**Keywords:** Graded S-prime submodule, graded S-strongly prime submodule, graded multiplication module.

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## 1. Introduction

In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied in [1, 2, 4–8, 10, 12–15]. In this paper, first, we introduce and study the notions of graded $S$-strongly prime submodules and graded $S$-strongly semiprime submodules of a graded $R$-module $M$ as a generalization of graded prime submodules and we investigate some properties of such graded submodules. For example, we show that if $N$ is a graded $S$-strongly prime submodule of $M$, then $N$ is a graded $S$-strongly semiprime submodule and $(N :_R M)$ is a graded $S$-prime ideal of $R$. Also, we give some characterizations of graded $S$-strongly prime submodules in graded multiplication modules. Second, we

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investigate the behaviour of this structure under graded module homomorphisms, localizations, quotient graded modules, Cartesian product.

Let $G$ be a group with identity $e$ and $R$ be a ring. Then $R$ is said to be a $G$-graded if $R = \bigoplus_{g\in G} R_g$ such that $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$, where $R_g$ is an additive subgroup of $R$ for all $g \in G$ [13]. The elements of $R_g$ are homogeneous of degree $g$. An element $r$ of $R$ has a unique decomposition as $r = \sum_{g\in G} r_g$ with $r_g \in R_g$ for all $g \in G$, but the sum being a finite sum, i.e. almost all $r_g$ zero. Let $R = \bigoplus_{g\in G} R_g$ be a graded ring and $I$ be an ideal of a graded ring $R$. Then $I$ is said to be a graded ideal of $R$, if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g\in G} x_g$, where $x_g \in I$ for all $g \in G$. Moreover, $R/I$ becomes a $G$-graded ring with $g$-component $(R/I)_g = (R_g + I)/I$ for $g \in G$ [13]. A graded ring $R$ is called graded quasilocal ring if it has a unique graded maximal ideal [12]. We call $S \subseteq h(R)$ is a multiplicatively closed subset of $R$ if $0 \notin S$, $1 \in S$ and $s_g s'_{g'} \in S$ for all $s_g, s'_{g'} \in S$ [12]. Let $R$ be a graded ring and $M$ an $R$-module. We say that $M$ is a graded $R$-module if there exists a family of subgroups $\{M_g\}_{g\in G}$ of $M$ such that $M = \bigoplus_{g\in G} M_g$ and $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. The elements of $M_g$ are called homogeneous of degree $g$. It is clear that $M_g$ is an $R_e$-submodule of $M$ for all $g \in G$. Moreover, $h(M) = \bigcup_{g\in G} M_g$ [13]. Let $N$ be an $R$-submodule of a graded $R$-module $M$. Then $N$ is said to be a graded $R$-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i.e. for $m \in N$, $m = \sum_{g\in G} m_g$, where $m_g \in N$ for all $g \in G$. Moreover, $M/N$ becomes a $G$-graded module with $g$-component $(M/N)_g = (M_g + N)/N$ for $g \in G$ [13]. A proper graded submodule $N$ of a graded $R$-module $M$ is said to be graded prime if $r_gm_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then $m_h \in N$ or $r_g \in (N : M)$. A graded $R$-module $M$ is called graded prime if the zero graded submodule is graded prime in $M$ [2]. A proper graded submodules $N$ of a graded $R$-module $M$ is call graded semiprime if $r_g^k m_h \in N$ for some $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{N}$, then $r_gm_h \in N$ [9]. Let $S \subseteq h(R)$ be a multiplicatively closed subset of $R$ and $N$ be a graded submodule of a graded $R$-module $M$ with $(N :_R M) \cap S = \emptyset$. Then $N$ is said to be a graded $S$-prime submodule if there exists $s \in S$ such that whenever $r_gm_h \in N$, then $sm_h \in N$ or $sr_g \in (N : M)$ for each $r_g \in h(R)$ and $m_h \in h(M)$ [15]. A graded $R$-module $M$ is called graded finitely generated if $M = Rm_{g_1} + Rm_{g_2} + \cdots + Rm_{g_n}$ for some $m_{g_1}, \ldots, m_{g_n} \in h(M)$ [2]. Let $S \subseteq h(R)$ be a multiplicatively closed subset of $R$ and $M$ be a graded $R$-module. Then $S^{-1}M$ is a graded $S^{-1}R$-module with

$$(S^{-1}M)_g = \{\frac{m}{s} : (\deg m)(\deg s)^{-1} = g\}$$

and $(S^{-1}R)_g = \{\frac{r}{s} : (\deg r)(\deg s)^{-1} = g\}$ [13] Let $M = \bigoplus_{g\in G} M_g$ and $M' = \bigoplus_{g\in G} M'_g$ be two graded $R$-modules. A mapping $f$ from $M$ into $M'$ is said to be a graded homomorphism, if for all $m, n \in M$;

(1) $f(m + n) = f(m) + f(n)$,
(2) $f(rm) = rf(m)$, for any $r \in R$ and $m \in M$,
(3) For any $g \in G$; $f(M_g) \subseteq M'_g$ [12].

Let $R_1$ and $R_2$ be $G$-graded rings. Then $R = R_1 \times R_2$ is a $G$-graded ring with $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Let $M_1$ be a $G$-graded $R_1$-module, $M_2$ be a $G$-graded $R_2$-module and $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a $G$-graded $R$-module with $M_g = (M_1)_g \times (M_2)_g$ for all $g \in G$. Also, if $S_1 \subseteq h(R_1)$ is a multiplicatively closed subset of $R_1$ and $S_2 \subseteq h(R_2)$ is a multiplicatively closed subset of $R_2$, then $S = S_1 \times S_2$ is a multiplicatively closed subset of $R$. Furthermore, each graded submodule of $M$ is of the form $N = N_1 \times N_2$ where $N_i$ is a graded submodule of $M_i$ for $i = 1, 2$ [12]. A graded

$R$-module $M$ is called graded multiplication module, if every graded submodule $N$ of $M$, $N = IM$ for some graded ideal $I$ of $R$ [3].

Throughout this work, $R$ is a commutative graded rings with identity and $M$ is a graded $R$-module. Also, $S \subseteq h(R)$ is a multiplicatively closed subset of $R$.

## 2. Characterizations of graded $S$-strongly prime submodules

**Definition 2.1** ($a$) A proper graded submodule $N$ of $M$ is said to be a graded strongly prime submodule if $((N + Rx_g) :_R M)y_h \subseteq N$, then $x_g \in N$ or $y_h \in N$ for each $x_g, y_h \in h(M)$.

($b$) A graded submodule $N$ of $M$ with $(N :_R M) \cap S = \emptyset$ is said to be graded $S$-strongly prime if there exists $s \in S$ such that whenever $((N + Rx_g) :_R M)y_h \subseteq N$, then $sx_g \in N$ or $sy_h \in N$ for each $x_g, y_h \in h(M)$.

Note that if we consider $R$ as a graded $R$-module, then graded $S$-strongly prime submodules are exactly graded $S$-prime ideals of $R$.

The following Lemma is known, but we write it here for the sake of references.

**Lemma 2.2** Let $M$ be a graded module over a graded ring $R$. Then the following hold:

(i) If $I$ and $J$ are graded ideals of $R$, then $I + J$ and $I \bigcap J$ are graded ideals of $R$.

(ii) If $I$ is a graded ideal of $R$, $N$ is a graded submodule of $M$, $r_g \in h(R)$ and $x_h \in h(M)$, then $Rx_h$, $IN$, $r_g N$ and $(0 :_M I)$ are graded submodules of $M$.

(iii) If $N$ and $K$ are graded submodules of $M$, then $N + K$ and $N \bigcap K$ are also graded submodules of $M$ and $(N :_R M)$ is a graded ideal of $R$. Also, $Ann_R(M) = (0 :_R M)$ is a graded ideal of $R$.

(iv) Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a collection of graded submodules of $M$. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of $M$.

**Proposition 2.3**

(i) Every graded strongly prime submodule $N$ of $M$ with $(N :_R M) \cap S = \emptyset$ is also a graded $S$-strongly prime submodule of $M$.

(ii) Let $S \subseteq h(R)$ be a multiplicatively closed subset of $R$ consisting of units in $R$. Then a graded submodule $N$ of $M$ is graded strongly prime if and only if $N$ is graded $S$-strongly prime.

**Proof.** The proof is completely straightforward. ■

By setting $S = \{1\}$, we conclude that every graded strongly prime submodule is a graded $S$-strongly prime submodule by Proposition 2.3. The following example shows that the converse is not true in general.

***Example* 2.4**

(i) Let us observe $R = \mathbb{Z}$ as a trivially $\mathbb{Z}_2$-graded ring and $M = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ be a $\mathbb{Z}_2$-graded $R$-module with $M_0 = \mathbb{Z}/n\mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}/n\mathbb{Z}$ where $n$ is a positive integer with $M_0$. Let $p$ be a prime factor of $n$ and $S = \mathbb{Z} - p\mathbb{Z}$. Then the submodule $p\mathbb{Z}/n\mathbb{Z} \times \{0\}$ is a graded $S$-strongly prime submodule of $M$.

(ii) Let $R = \mathbb{Z}[i]$ be $\mathbb{Z}_2$-graded $R$-module with $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$ and $S = \{2^n \mid n \in \mathbb{N} \cup \{0\}\}$. Consider the graded submodule $N = \langle 4i \rangle$ of graded $R$-module $R$. Put $s = 4$. It is easy to see that $N$ is a graded $S$-strongly prime submodule. But $N$ is not a graded strongly prime submodule.

**Definition 2.5** $(a)$ Let $N$ be a graded submodule of $M$ such that $(N :_R M) \cap S = \emptyset$. Then $N$ is said to be a graded $S$-semiprime submodule if there exists $s \in S$ such that whenever $r_g^2 m_h \in N$, then $sr_g m_h \in N$ for all $r_g \in h(R)$ and $m_h \in h(M)$.

$(b)$ Let $N$ be a graded submodule of $M$ such that $(N :_R M) \cap S = \emptyset$. Then $N$ is said to be a graded $S$-strongly semiprime submodule if there exists $s \in S$ such that whenever $((N + Rx_g) :_R M)x_g \subseteq N$, then $sx_g \in N$ for all $x_g \in h(M)$.

$(c)$ A graded ideal $I$ of $R$ is called graded $S$-semiprime if $I \cap S = \emptyset$ and there exists $s \in S$ such that whenever $a_g^2 \in I$, then $sa_g \in I$ for all $a_g \in h(R)$.

**Lemma 2.6** Every graded $S$-strongly semiprime submodule is a graded $S$-semiprime submodule.

**Proof.** Let $N$ be a graded $S$-strongly semiprime submodule of $M$ and suppose $r_g^2 m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$. Thus, $((N + R(r_g m_h)) :_R M)(r_g m_h) = r_g((N + R(r_g m_h)) :_R M)m_h \subseteq r_g(N + R(r_g m_h)) \subseteq N$. Since $N$ is a graded $S$-strongly semiprime submodule, there exists $s \in S$ such that $sr_g m_h \in N$. Therefore, $N$ is a graded $S$-semiprime submodule. ■

**Proposition 2.7** If $N$ is a graded $S$-strongly semiprime submodule of $M$, then $(N :_R M)$ is a graded $S$-semiprime ideal of $R$.

**Proof.** Let $a_g^2 \in (N :_R M)$ where $a_g \in h(R)$. Let $m \in M$. Hence $m = \sum_{h \in G} m_h$ where $m_h \in M_h$ for all $h \in G$. Suppose $m_h \in M_h$. Thus $((N + R(a_g m_h)) :_R M)(a_g m_h) = a_g((N + R(a_g m_h)) :_R M)m_h \subseteq a_g(N + R(a_g m_h)) \subseteq N$. Since $N$ is graded $S$-strongly semiprime, there exists $s \in S$ such that $sa_g m_h \in N$, so $sa_g m \in N$ and $sa_g \in (N :_R M)$. Therefore, $(N :_R M)$ is a graded $S$-semiprime ideal of $R$. ■

The following example shows that the converse of Proposition 2.7 is not hold.

***Example* 2.8** Let $R = \mathbb{Z}$ be a trivially $\mathbb{Z}_2$-graded ring and $M = \mathbb{Q} \times \mathbb{Q}$ where $\mathbb{Q}$ is the field of rational numbers be a $\mathbb{Z}_2$-graded module with $M_0 = \mathbb{Q} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Q}$. Take the graded submodule $N = \mathbb{Z} \times \{0\}$ and the multiplicatively closed subset $S = \mathbb{Z} - \{0\}$ of $\mathbb{Z}$. Then the graded ideal $(N :_{\mathbb{Z}} M) = 0$ is a graded $S$-semiprime, but $N$ is not a graded $S$-strongly semiprime submodule of $M$. Let $s$ be an arbitrary element of $S$. Choose a prime number $p$ with $gcd(p, s) = 1$. Note that $((N + R(\frac{1}{p}, 0)) :_R M)(\frac{1}{p}, 0) \subseteq N$, but $(\frac{s}{p}, 0) \notin N$.

**Proposition 2.9**

(i) Every graded $S$-strongly prime submodule is a graded $S$-prime submodule.
(ii) Every graded $S$-strongly prime submodule is a graded $S$-strongly semiprime submodule.
(iii) Every graded maximal submodule $N$ of $M$ with $(N :_R M) \cap S = \emptyset$ is a graded $S$-strongly prime submodule.

**Proof.** $(i)$ Let $N$ be a graded $S$-strongly prime submodule of $M$. Thus there exists $s \in S$ such that whenever $((N + Rx_g) : M)y_h \subseteq N$ for all $x_g, y_h \in h(M)$, implies that $sx_g \in N$ or $sy_h \in N$. Let $r_g m_h \in N$ and $sm_h \notin N$ for some $r_g \in h(R)$ and $m_h \in h(M)$. We show that $sr_g \in (N :_R M)$. Let $x = \sum_{k \in G} x_k \in M$. Thus we have $((N + Rm_h) :_R M)(r_g x_k) = r_g((N + Rm_h) :_R M)x_k \subseteq r_g(N + Rm_h) \subseteq N$ for any $x_k \in M_k$, since $sm_h \notin N$ and $N$ is a graded $S$-strongly prime submodule of $M$, we conclude $sr_g x_k \in N$ for any $x_k \in M_k$. Hence $sr_g x \in N$. Therefore $sr_g M \subseteq N$ and so $sr_g \in (N :_R M)$.

$(ii)$ It is clear.

$(iii)$ Let $N$ be a graded maximal submodule of $M$ such that $(N :_R M) \cap S = \emptyset$. Let $x_g, y_h \in h(M)$ and $((N + Rx_g) : M)y_h \subseteq N$. Let $x_g \notin N$. Thus $N + Rx_g = M$, hence $(N + Rx_g :_R M) = R$ and we conclude $y_h \in N$. Therefore $N$ is a graded strongly prime submodule, and since $(N :_R M) \cap S = \emptyset$, then by Proposition 2.3, $N$ is a graded $S$-strongly prime submodule of $M$. ■

The following example shows that the concept of graded $S$-strongly prime submodules is different from the concept of graded $S$-prime submodules.

***Example* 2.10** Let $R$ be a $G$-graded ring, $P$ be a graded prime ideal of $R$ and $S = h(R) - P$. Then $P \times P$ is a graded $S$-prime submodule of graded $R$-module $R \times R$, because $P \times P$ is a graded prime submodule of $R \times R$ and $(P \times P :_R R \times R) \cap S = P \cap S = \emptyset$. But it is not a graded $S$-strongly prime submodule of $M$. Let $s$ be an arbitrary element of $S$. Then $((P \times P + R(1, 0)) :_R R \times R)(0, 1) \subseteq P \times P$, but $s(1, 0) \notin P \times P$ and $s(0, 1) \notin P \times P$.

**Proposition 2.11** Let $M$ be a graded module over a graded field $R$ and $N$ be a proper graded submodule of $M$. Then $N$ is a graded maximal submodule of $M$ if and only if $N$ is a graded $S$-strongly prime submodule of $M$.

**Proof.** Let $N$ be a graded maximal submodule of $M$. We have $(N :_R M) \cap S = \emptyset$, because if $s \in (N :_R M) \cap S$, then $1 = s^{-1}s \in (N :_R M)$, a contradiction. Thus by Proposition 2.9, $N$ is a graded $S$-strongly prime submodule of $M$. Conversely, let $N$ be a graded $S$-strongly prime submodule of $M$ which is not a graded maximal submodule of $M$. Then there exists $x_g \in h(M) \setminus N$ such that $Rx_g + N \neq M$. Let $y = \sum_{h \in G} y_h \in M$. Hence for any $y_h \in M_h$, we have $((N + Rx_g) :_R M)y_h = \{0\}y_h = \{0\} \subseteq N$. Thus there exists $s \in S$ such that $sx_g \in N$ or $sy_h \in N$ since $N$ is a graded $S$-strongly prime submodule of $M$. Since $x_g \notin N$, so $sx_g \notin N$. We conclude that $sy_h \in N$ and $y_h \in N$, so $y \in N$. Thus, $N = M$, which is a contradiction. ■

**Corollary 2.12** Let $N$ a graded submodule of $M$ with $(N : M) = P$ and $S = h(R) - P$. If $P$ is a graded maximal ideal of $R$, then there exists a graded $S$-strongly prime submodule $\mathcal{M}$ of $M$ with $(\mathcal{M} : M) = P$.

**Proof.** Note that $M/N$ is a graded module over the graded field $R/P$, so it has a graded maximal submodule, say $\mathcal{M}/N$. Then $\mathcal{M}$ is a graded maximal submodule of $M$ containing of $N$ and hence $P = (N : M) \subseteq (\mathcal{M} : M)$, we have $(\mathcal{M} : M) = P$. Since $(\mathcal{M} : M) \cap S = \emptyset$, then by Proposition 2.9, $\mathcal{M}$ is a graded $S$-strongly prime submodule of $M$. ■

**Definition 2.13** A graded submodule $N$ of a graded $R$-module $M$ is called graded $S$-$I$-maximal if $(N :_R M) = I$ and there exists $s \in S$ such that whenever $K$ is a graded submodule of $M$ containing of $N$ with $(K :_R M) = I$, then $sK \subseteq N$.

**Theorem 2.14** Let $M$ be a graded $R$-module and $N$ be a graded submodule of $M$.

(i) $N$ is a graded $S$-strongly prime submodule of $M$.
(ii) $N$ is a graded $S$-strongly semiprime submodule of $M$ and $N$ is a graded $S$-prime submodule of $M$.
(iii) $N$ is a graded $S$-strongly semiprime submodule of $M$ and $(N :_R M)$ is a graded $S$-prime ideal of $R$.
(iv) $N$ is a graded $S$-$(N :_R M)$-maximal submodule of $M$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.

**Proof.** $(i) \Rightarrow (ii)$ Apply Proposition 2.9.

$(ii) \Rightarrow (iii)$ Note that for every graded $S$-prime submodule $N$ of $M$, the graded ideal $(N :_R M)$ is a graded $S$-prime ideal of $R$ (see [15, Proposition 2.6]).

$(iii) \Rightarrow (iv)$ Since $N$ is a graded $S$-strongly semiprime submodule of $M$, there exists $s \in S$ such that whenever $((N + Rx_g) : M)x_g \subseteq N$, then $sx_g \in N$ for all $x_g \in M$. Let $K$ be a graded submodule of $M$ containing of $N$ with $(K :_R M) = (N :_R M)$. We show that $sK \subseteq N$. Let $x = \sum_{g \in G} x_g$ be an arbitrary element of $K$. Since $N \subseteq N + Rx_g \subseteq K$ for any $g \in G$, then $(N :_R M) \subseteq ((N + Rx_g) :_R M) \subseteq (K :_R M) = (N :_R M)$ and so $((N + Rx_g) :_R M) = (N :_R M)$. Thus, $((N + Rx_g) :_R M)x_g = (N :_R M)x_g \subseteq N$, since $N$ is a graded $S$-strongly semiprime submodule of $M$, $sx_g \in N$ and hence $sx \in N$. Therefore, $sK \subseteq N$ as required. ∎

**Proposition 2.15** Let $\{N_i\}_{i \in I}$ be a family of graded $S$-strongly prime submodules of $M$ such that $(N_i :_R M) = P$ for all $i \in I$. If $\bigcap_{i \in I} N_i$ is a graded $S$-strongly prime submodule of $M$, then there exists $s \in S$ such that $sN_i \subseteq N_j$ for all $i, j \in I$.

**Proof.** Let $N = \bigcap_{i \in I} N_i$. Thus ,$(N :_R M) = \bigcap_{i \in I}(N_i :_R M) = P = (N_j :_R M)$ for each $j \in I$. Since $N$ is graded $S$-strongly semiprime and by Proposition 2.14($(i) \Rightarrow (iv)$), $N$ is graded $S$-$(N :_R M)$-maximal and $N \subseteq N_j$ with $(N :_R M) = (N_j :_R M)$, so there exists $s \in S$ such that $sN_j \subseteq N$ for all $j \in I$. ∎

**Lemma 2.16** Let $N$ be a graded $S$-strongly prime submodule of a graded $R$-module $M$. Then the following statements hold for some $s \in S$.

(i) $(N :_M s') \subseteq (N :_M s)$ for all $s' \in S$.
(ii) $((N :_R M) :_R s') \subseteq ((N :_R M) :_R s)$ for all $s' \in S$.

**Proof.** $(i)$ Let $m = \sum_{g \in G} m_g \in (N :_M s')$ where $s' \in S$. Then $s'm_g \in N$ for any $m_g \in h(M)$. Since every graded $S$-strongly prime is graded $S$-prime, there exists $s \in S$ such that $sm_g \in N$ or $ss' \in (N :_R M)$. As $(N :_R M) \cap S = \emptyset$, we get $sm_g \in N$ so $sm \in N$, namely $m \in (N :_M s)$.

$(ii)$ It follows from $(i)$. ∎

**Theorem 2.17** Let $N$ be a graded submodule of a graded $R$-module $M$ provided $(N :_R M) \cap S = \emptyset$. Then $N$ is a graded $S$-strongly prime submodule of $M$ if and only if $(N :_M s)$ is a graded strongly prime submodule of $M$ for some $s \in S$.

**Proof.** Assume that $N$ is a graded $S$-strongly prime submodule of $M$. Then there exists $s \in S$ such that whenever $((N + Rx_g) :_R M)y_h \subseteq N$, then $sx_g \in N$ or $sy_h \in N$ for all $x_g, y_h \in h(M)$. We prove that $(N :_M s)$ is a graded strongly prime submodule. Taking $x_g, y_h \in M$ with $(((N :_M s) + Rx_g) :_R M)y_h \subseteq (N :_M s)$, we have $(((N :_M s) + Rx_g) :_R M)(sy_h) \subseteq s(N :_M s) \subseteq N$. Since $N \subseteq (N :_M s)$, $((N + Rx_g) :_R M)(sy_h) \subseteq N$. Thus, $sx_g \in N$ or $s^2y_h \in N$. If $sx_g \in N$, then $x_g \in (N :_M s)$. If $s^2y_h \in N$, then $y_h \in (N :_M s^2) \subseteq (N :_M s)$ by Lemma 2.16. Hence ,$(N :_M s)$ is a graded strongly prime submodule of $M$. Conversely, assume that $(N :_M s)$ is a graded strongly prime submodule of $M$. Let $((N + Rx_g) :_R M)y_h \subseteq N$ for some $x_g, y_h \in h(M)$. Since $N \subseteq (N :_M s)$, we have $x_g \in (N :_M s)$ or $y_h \in (N :_M s)$. Thus, $sx_g \in N$ or $sy_h \in N$, and so $N$ is a graded $S$-strongly prime submodule of $M$. ∎

**Theorem 2.18** Let $N$ be a graded submodule of $M$ provided $(N :_R M) \subseteq Jac^{gr}(R)$, where $Jac^{gr}(R)$ is the intersection of all graded maximal ideals of $R$. Then the following statements are equivalent:

(i) $N$ is a graded strongly prime submodule of $M$.
(ii) $N$ is a graded prime submodule of $M$ and $N$ is a graded $(h(R) - \mathfrak{m})$-strongly prime

submodule of $M$ for each graded maximal ideal $\mathfrak{m}$.

**Proof.** $(i) \Rightarrow (ii)$ Let $N$ be a graded strongly prime submodule of $M$. Then $N$ is a graded prime submodule of $M$. Since $(N :_R M) \subseteq Jac^{gr}(R)$, $(N :_R M) \subseteq \mathfrak{m}$ for each graded maximal ideal $\mathfrak{m}$ and so $(N :_R M) \cap (h(R) - \mathfrak{m}) = \emptyset$. Thus, $N$ is a graded $(h(R) - \mathfrak{m})$-strongly prime submodule of $M$ by Proposition 2.3.

$(ii) \Rightarrow (i)$ Suppose that $N$ is a graded prime submodule of $M$ and $N$ is a graded $(h(R)-\mathfrak{m})$-strongly prime submodule of $M$ for each graded maximal ideal $\mathfrak{m}$. Let $((N + Rx_g) :_R M)y_h \subseteq N$ and $y_h \notin N$ for some $x_g, y_h \in h(M)$. Let $\mathfrak{m}$ be a graded maximal ideal of $R$. Since $N$ is a graded $(h(R) - \mathfrak{m})$-strongly prime submodule of $M$, there exists $s_{\mathfrak{m}} \in h(R) - \mathfrak{m}$ such that $s_{\mathfrak{m}}x_g \in N$ or $s_{\mathfrak{m}}y_g \in N$. If $s_{\mathfrak{m}}y_h \in N$, then since $N$ is a graded prime submodule of $M$ and $y_h \notin N$, $s_{\mathfrak{m}} \in (N :_R M)$ which is a contradiction. Hence, $s_{\mathfrak{m}}x_g \in N$. Consider the set $Q = \{s_{\mathfrak{m}} \mid \exists \mathfrak{m} \in Max^{gr}(R); s_{\mathfrak{m}} \notin \mathfrak{m} \ and \ s_{\mathfrak{m}}x_g \in N\}$. Suppose that $\langle Q\rangle \neq R$. Take any graded maximal ideal $\mathfrak{m}'$ containing $Q$. Then the definition of $Q$ requires that there exists $s_{\mathfrak{m}'} \in Q$ and $s_{\mathfrak{m}'} \notin \mathfrak{m}'$, which is a contradiction. Thus, $\langle Q\rangle = R$ and $1 = r_1s_{\mathfrak{m}_1} + r_2s_{\mathfrak{m}_2} + \cdots + r_ns_{\mathfrak{m}_n}$ for some $r_i \in R$ and $s_{\mathfrak{m}_i} \notin \mathfrak{m}_i$ with $s_{\mathfrak{m}_i}x_g \in N$, where $\mathfrak{m}_i \in Max^{gr}(R)$ for each $i = 1, 2, \ldots, n$. Therefore, $x_g = r_1s_{\mathfrak{m}_1}x_g + r_2s_{\mathfrak{m}_2}x_g + \cdots + r_ns_{\mathfrak{m}_n}x_g \in N$. Hence $N$ is a graded strongly prime submodule of $M$. ■

By the previous theorem we have the following result:

**Corollary 2.19** Let $M$ be a graded module over a graded quasilocal ring $(R, \mathfrak{m})$. Then the following statements are equivalent:

(i) $N$ is a graded strongly prime submodule of $M$.
(ii) $N$ is a graded prime submodule of $M$ and $N$ is a graded $(h(R) - \mathfrak{m})$-strongly prime submodule of $M$.

Now, we characterize graded $S$-strongly prime submodules of a graded multiplication module.

**Theorem 2.20** Let $M$ be a graded multiplication $R$-module and $N$ be a graded submodule of $M$ provided that $(N :_R M) \cap S = \emptyset$. Then the following statements are equivalent:

(i) $N$ is a graded $S$-strongly prime submodule of $M$.
(ii) $(N :_R M)$ is a graded $S$-prime ideal of $R$.
(iii) $N = IM$ for some graded $S$-prime ideal $I$ of $R$ with $ann(M) \subseteq I$.

**Proof.** $(i) \Rightarrow (ii)$ It follows from Theorem 2.14.

$(ii) \Rightarrow (iii)$ Consider $I = (N :_R M)$.

$(iii) \Rightarrow (i)$ By [15, Proposition 2.8], $N$ is a graded $S$-prime submodule of $M$. Thus, there exists $s \in S$ such that whenever $r_gm_h \in N$, then $sm_h \in N$ or $sr_g \in (N :_R M)$ for all $m_h \in h(M)$ and $r_g \in h(R)$. Now, we show that $N$ is a graded $S$-strongly prime submodule of $M$. Let $((N + Rx_g) :_R M)y_h \subseteq N$ and $sy_h \notin N$ for some $x_g, y_h \in h(M)$. Since $N$ is a graded $S$-prime submodule of $M$, $s((N + Rx_g) :_R M) \subseteq (N :_R M)$. As $M$ is a graded multiplication $R$-module, we have

$$s(N + Rx_g) = s((N + Rx_g) :_R M)M \subseteq (N :_R M)M = N.$$

Therefore, $sx_g \in N$ and $N$ is a graded $S$-strongly prime submodule of $M$. ■

**Lemma 2.21** Let $Q$ be a graded $S$-primary ideal of $R$. Then $Grad(Q)$ is a graded $S$-prime ideal of $R$.

**Proof.** First note that $Grad(Q) \cap S = \emptyset$, because if $s \in Grad(Q) \cap S$, then $s^n \in Q \cap S$ for some $n \in \mathbb{N}$, a contradiction. Let $a_g b_h \in Grad(Q)$ where $a_g, b_h \in h(R)$. Thus, $(a_g b_h)^k \in Q$ for some $k \in \mathbb{N}$. Since $Q$ is a graded $S$-primary ideal of $R$, there exists $s \in S$ such that $sa_g^k \in Q$ or $sb_h^k \in Grad(Q)$. We conclude $sa_g \in Grad(Q)$ or $sb_h \in Grad(Q)$. Hence, $Grad(Q)$ is a graded $S$-prime ideal of $R$. ■

**Lemma 2.22** Let $M$ be a finitely generated multiplication $R$-module and $N$ be a submodule of $M$. Then $(Grad(N) :_R M) = Grad((N :_R M))$.

**Proof.** The proof is similar to Theorem 4 of [11]. ■

**Theorem 2.23** Let $M$ be a graded finitely generated multiplication $R$-module. If $N$ is a graded $S$-strongly prime submodule of $M$, then $Grad(N)$ is a graded $S$-strongly prime submodule of $M$.

**Proof.** Since $N$ is a graded $S$-strongly prime submodule of $M$, $(N :_R M)$ is a graded $S$-prime ideal of $R$ by Theorem 2.14. Thus, by Lemma 2.21, $Grad((N :_R M))$ is a graded $S$-prime ideal of $R$. By Lemma 2.22, we have $(Grad(N) :_R M) = Grad((N :_R M))$. Thus, $(Grad(N) :_R M)$ is a graded $S$-prime submodule of $M$. Now, the result follows from Theorem 2.20. ■

## 3. Behaviour of graded $S$-strongly prime submodules

In this section, we investigate the behaviour of graded $S$-strongly prime submodules under graded module homomorphisms, localizations, quotient graded modules and Cartesian product.

**Proposition 3.1** Let $f : M \to M'$ be a graded $R$-homomorphism. Then the following statements hold:

(i) If $N'$ is a graded $S$-strongly prime submodule of $M'$ such that $(f^{-1}(N') :_R M) \cap S = \emptyset$, then $f^{-1}(N')$ is a graded $S$-strongly prime submodule of $M$.
(ii) If $f$ is a graded epimorphism and $N$ is a graded $S$-strongly prime submodule of $M$ containing $Ker(f)$, then $f(N)$ is a graded $S$-strongly prime submodule of $M'$.

**Proof.** $(i)$ Let $((f^{-1}(N') + Rx_g) :_R M)y_h \subseteq f^{-1}(N')$ for some $x_g, y_h \in h(M)$. Thus, $f(((f^{-1}(N') + Rx_g) :_R M)y_h) \subseteq f(f^{-1}(N')) \subseteq N'$. Since $f$ is a graded $R$-homomorphism, $((f^{-1}(N') + Rx_g) :_R M)f(y_h) \subseteq N'$. Now, we show that $((N' + Rf(x_g)) :_R M') \subseteq ((f^{-1}(N') + Rx_g) :_R M)$. Take $r \in ((N' + Rf(x_g)) :_R M')$. Then $rM' \subseteq N' + Rf(x_g)$. Since $f(M) \subseteq M'$, we have $f(rM) = rf(M) \subseteq rM' \subseteq N' + Rf(x_g)$. This implies that $rM \subseteq f^{-1}(N' + f(Rx_g))$. It is clear that $f^{-1}(N' + f(Rx_g)) \subseteq f^{-1}(N') + Rx_g$. Thus, $r \in ((N' + Rx_g) :_R M)$ and so $((N' + Rf(x_g)) :_R M')f(y_h) \subseteq N'$. Since $N'$ is a graded $S$-strongly prime submodule of $M'$, there exists $s \in S$ such that $sf(x_g) \in N'$ or $sf(y_h) \in N'$. Therefore, $sx_g \in f^{-1}(N')$ or $sy_h \in f^{-1}(N')$ as needed.

$(ii)$ First note that $(f(N) :_R M') \cap S = \emptyset$. Otherwise, there exists $s \in (f(N) :_R M') \cap S$. Hence, $sM' \subseteq f(N)$ and then $f(sM) = sf(M) = sM' \subseteq f(N)$ and $sM \subseteq N + Ker(f) = N$. That means $s \in (N :_R M)$, which is a contradiction. Let $((f(N) + Rx'_g) :_R M')y'_h \subseteq f(N)$ where $x'_g, y'_h \in h(M')$. Since $f$ is a graded epimorphism, $f(x_g) = x'_g$ and $f(y_h) = y'_h$ for some $x_g, y_h \in h(M)$. Thus $(f(N + Rx_g) :_R M')f(y_h) \subseteq f(N)$. It is easy to see that $((N + Rx_g) :_R M) \subseteq (f(N + Rx_g) :_R M')$. Hence $f(((N + Rx_g) :_R M)y_h) \subseteq f(N)$ and $((N + Rx_g) :_R M)y_h \subseteq N + Ker(f) \subseteq N$. Thus, there exists $s \in S$ such that $sx_g \in N$ or $sy_h \in N$ since $N$ is a graded $S$-strongly prime submodule of $M$. Therefore,

$sf(x_g) \in f(N)$ or $sf(y_h) \in f(N)$, and so $f(N)$ is a graded $S$-strongly prime submodule of $M'$. ■

**Proposition 3.2** Let $N$ and $K$ be graded submodules of $M$ with $K \subseteq N$. Then the following assertions hold:

(i) If $N'$ is a graded $S$-strongly prime submodule of $M$ with $(N' :_R K) \cap S = \emptyset$, then $K \cap N'$ is a graded $S$-strongly prime submodule of $K$.
(ii) $N$ is a graded $S$-strongly prime submodule of $M$ if and only if $N/K$ is a graded $S$-strongly prime submodule of $M/K$.

**Proof.** $(i)$ Consider the injection $i : K \to M$ defined by $i(x) = x$ for all $x \in K$. Then $i^{-1}(N') = K \cap N'$. By $(N' :_R K) \cap S = \emptyset$, we give $(i^{-1}(N') :_R K) \cap S = \emptyset$. Thus, the rest follows from Proposition 3.1$(i)$.

$(ii)$ Let $N$ be a graded $S$-strongly prime submodule of $M$. Then consider the canonical homomorphism $\pi : M \to M/K$ defined by $\pi(m) = m+K$ for all $m \in M$. Then note that $\pi$ is a graded epimorphism and $Ker(\pi) = K \subseteq N$. Thus by Proposition 3.1$(ii)$, $N/K$ is a graded $S$-strongly prime submodule of $M/K$. Conversely, assume that $N/K$ is a graded $S$-strongly prime submodule of $M/K$. Let $((N+Rx_g) :_R M)y_h \subseteq N$ where $x_g, y_h \in h(M)$. We have $((N + Rx_g)/K :_R M/K) = (R(x_g + K) + N/K :_R M/K) = ((N + Rx_g) :_R M)$. Thus, $((R(x_g + K) + N/K) :_R M/K)(y_h + K) \subseteq N/K$. Since $N/K$ is a graded $S$-strongly prime submodule of $M/K$, there exists $s \in S$ such that $s(x_g + K) \in N/K$ or $s(y_h + K) \in N/K$. Thus, $sx_g \in N$ or $sy_h \in N$, and so $N$ is a graded $S$-strongly prime submodule of $M$. ■

Let $S \subseteq h(R)$ be a multiplicatively closed subset of $R$. The saturation $S^*$ of $S$ is defined as $S^* = \{x \in h(R) \mid \frac{x}{1}$ is a homogeneous unit of $S^{-1}R\}$. Note that $S^* \subseteq h(R)$ is a multiplicatively closed subset of $R$ containing $S$.

**Proposition 3.3**

(i) Let $S_1 \subseteq S_2 \subseteq h(R)$ be multiplicatively closed subsets of $R$. If $N$ is a graded $S_1$-strongly prime submodule and $(N :_R M) \cap S_2 = \emptyset$, then $N$ is a graded $S_2$-strongly prime submodule.
(ii) A graded submodule $N$ of $M$ is a graded $S$-strongly prime submodule if and only if it is a graded $S^*$-strongly prime submodule.
(iii) If $N$ is a graded $S$-strongly prime submodule of $M$, then $S^{-1}N$ is a graded strongly prime submodule of graded $S^{-1}R$-module $S^{-1}M$.

**Proof.** $(i)$ It is clear.

$(ii)$ Let $N$ be a graded $S$-strongly prime submodule. Assume that $(N :_R M) \cap S^* \neq \emptyset$ and $r \in (N :_R M) \cap S^*$. Let $r = \sum_{g\in G} r_g$ where $r_g \in R_g$ for all $g \in G$. Hence, $\frac{r_g}{1}$ is a homogeneous unit of $S^{-1}R$, that is, $\frac{r_g}{1}\frac{a}{s} = \frac{1}{1}$ for some $a \in h(R)$ and $s \in S$. Thus, $us = ur_g a \in S$ for some $u \in S$. Then $us = ur_g a \in (N :_R M) \cap S$, which is a contradiction. Thus, $(N :_R M) \cap S^* = \emptyset$. Since $S \subseteq S^*$, by $(i)$, $N$ is a graded $S^*$-strongly prime submodule of $M$. Conversely, assume that $N$ is a graded $S^*$-strongly prime submodule. Let $((N + Rx_g) :_R M)y_h \subseteq N$ for some $x_g, y_h \in h(M)$. Since $N$ is a graded $S^*$-strongly prime submodule, there exists $s' \in S^*$ so that $s'x_g \in N$ or $s'y_h \in N$. As $\dfrac{s'}{1}$ is a unit of $S^{-1}R$, there exist $u, s \in S$ and $a \in h(R)$ such that $su = us'a$. Put $us' = s'' \in S$. Then note that $s''x_g \in N$ or $s''y_h \in N$. Therefore, $N$ is a graded $S$-strongly prime submodule of $M$.

$(iii)$ Let $N$ be a graded $S$-strongly prime submodule. Thus, we have $(N :_R M) \cap S = \emptyset$

and there exists $s \in S$ such that whenever $((N + Rx_g) : M)y_h \subseteq N$, then $sx_g \in N$ or $sy_h \in N$ for all $x_g, y_h \in h(M)$. Let

$$\Big((S^{-1}N + S^{-1}R(\frac{x_g}{u})) :_{S^{-1}R} S^{-1}M\Big) \frac{y_h}{v} \subseteq S^{-1}N$$

where $\frac{x_g}{u}, \frac{y_h}{v} \in S^{-1}M$. We show that $((N + Rx_g) :_R M)(sy_h) \subseteq N$. If $r \in ((N + Rx_g) :_R M)$, then we can write $r = \sum_{k\in G} r_k$ where $r_k \in R_k$ for any $k \in G$. Hence, for any $k \in G$, $r_kM \subseteq N + Rx_g$ and $(\frac{r_k}{1})S^{-1}M \subseteq S^{-1}N + S^{-1}R(\frac{x_g}{1}) = S^{-1}N + S^{-1}R(\frac{x_g}{u})$ and so $(\frac{r_k}{1})(\frac{y_h}{v}) \in S^{-1}N$. Hence, there exist $n \in N$ and $t_1, t_2 \in S$ such that $t_2(t_1r_ky_h - vn) = 0$. Thus, $(t_2t_1)r_ky_h \in N$, and since $N$ is a graded $S$-prime submodule, we get $st_1t_2 \in (N :_R M)$ or $sr_ky_h \in N$. As $(N :_R M) \cap S = \emptyset$, we have $sr_ky_h \in N$ for every $k \in G$ and so $sry_h \in N$. Hence, $((N + Rx_g) :_R M)(sy_h) \subseteq N$. It follows that $sx_g \in N$ or $s^2y_h \in N$ since $N$ is a graded $S$-strongly prime submodule. Therefore, $\frac{x_g}{u} = \frac{sx_g}{su} \in S^{-1}N$ or $\frac{y_h}{v} = \frac{s^2y_h}{s^2v} \in S^{-1}N$. Thus, $S^{-1}N$ is a graded strongly prime submodule of $S^{-1}M$. ■

The following example shows that the converse of part $(iii)$ of Proposition 3.3 is not true in general.

***Example* 3.4** Consider $R = \mathbb{Z}$ and $G = \mathbb{Z}_2$. Then $R$ is trivially $G$-graded by $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Consider the $R$-module $U = \mathbb{Q}[i]$. Then $U$ is $G$-graded by $U_0 = Q$ and $U_1 = i\mathbb{Q}$. Thus $M = U \times U$ is a $G$-graded $R$-module with $M_0 = U_0 \times U_0$ and $M_1 = U_1 \times U_1$. Take the graded submodule $N = \mathbb{Z} \times \{0\}$ and the multiplicatively closed subset $S = \mathbb{Z} - \{0\}$ of $\mathbb{Z}$. Then $((N + R(x, y)) :_R M) = 0$ for any $(x, y) \in M$. Let $s$ be an arbitrary element of $S$. Choose prime numbers $p, q$ of $\mathbb{Z}$. Then note that $((N + R(\frac{1}{p}, 0)) :_R M)(0, \frac{1}{q}) \subseteq N$. But $(\frac{s}{p}, 0) \notin N$ and $(0, \frac{s}{q}) \notin N$, it follows that $N$ is not a graded $S$-strongly prime submodule of $M$. Since $S^{-1}\mathbb{Z} = \mathbb{Q}$, $S^{-1}N$ is a graded strongly prime submodule of $S^{-1}M$.

**Proposition 3.5** Let $M$ be a graded finitely generated $R$-module and $N$ be a graded submodule of $M$ satisfying $(N :_R M) \cap S = \emptyset$. Then the following statements are equivalent:

(i) $N$ is a graded $S$-strongly prime submodule of $M$.
(ii) $S^{-1}N$ is a graded strongly prime submodule of $S^{-1}M$ and there is an $s \in S$ satisfying $(N :_M s') \subseteq (N :_M s)$ for all $s' \in S$.

**Proof.** $(i) \Rightarrow (ii)$ It follows from Proposition 3.3 and Lemma 2.16.

$(ii) \Rightarrow (i)$ Let $((N + Rx_g) :_R M)y_h \subseteq N$ for some $x_g, y_h \in h(M)$. We have $((S^{-1}N + S^{-1}R(\frac{x_g}{1})) :_{S^{-1}R} S^{-1}M)\frac{y_h}{1} \subseteq S^{-1}N$. Then $\frac{x_g}{1} \in S^{-1}N$ or $\frac{y_h}{1} \in S^{-1}N$ since $S^{-1}N$ is a graded strongly prime submodule of $S^{-1}M$. Thus, $ux_g \in N$ or $ty_h \in N$ for some $u, t \in S$. By assumption, there exists $s \in S$ so that $(N :_M u) \subseteq (N :_M s)$ and $(N :_M t) \subseteq (N :_M s)$. Thus, $sx_g \in N$ or $sy_h \in N$ and so $N$ is a graded $S$-strongly prime submodule of $M$. ■

**Lemma 3.6** Let $R = R_1 \times R_2$ and $S = (S_1 \times S_2) \cap h(R)$ where $S_i \subseteq h(R_i)$ is a multiplicatively closed subset of $R_i$ for each $i = 1, 2$. Suppose that $P = P_1 \times P_2$ is a graded ideal of $R$. If $P$ is a graded $S$-prime ideal of $R$, then $P_1$ is a graded $S_1$-prime ideal of $R_1$ and $P_2 \cap S_2 \neq \emptyset$ or $P_2$ is a graded $S_2$-prime ideal of $R_2$ and $P_1 \cap S_1 \neq \emptyset$.

**Proof.** Suppose $P$ is a graded $S$-prime ideal of $R$. Since $(1, 0)(0, 1) = (0, 0) \in P$, there exists $s = (s_1, s_2) \in S$ so that $s(1, 0) = (s_1, 0) \in P$ or $s(0, 1) = (0, s_2) \in P$ and thus, $P_1 \cap S_1 \neq \emptyset$ or $P_2 \cap S_2 \neq \emptyset$. We may assume that $P_1 \cap S_1 \neq \emptyset$. As $P \cap S = \emptyset$, we have $P_2 \cap S_2 = \emptyset$. Let $x_gy_h \in P_2$ for some $x_g, y_h \in R_2$. Since $(0, x_g)(0, y_h) \in P$ and $P$ is a graded $S$-prime ideal, we get either $s(0, x_g) = (0, s_2x_g) \in P$ or $s(0, y) = (0, s_2y) \in P$ and this yields $s_2x_g \in P_2$ or $s_2y_h \in P_2$. Therefore, $P_2$ is a graded $S_2$-prime ideal of $R_2$. In the other case, one can easily show that $P_1$ is a graded $S_1$-prime ideal of $R_1$. ■

**Theorem 3.7** Let $M = M_1 \times M_2$ be a graded $R = R_1 \times R_2$-module and $S = (S_1 \times S_2) \cap h(R)$ be a multiplicatively closed subset of $R$ where $M_i$ is a graded $R_i$-module and $S_i \subseteq h(R_i)$ is a multiplicatively closed subset of $R_i$ for each $i = 1, 2$. Suppose that $N_1$ is a graded submodule of $M_1$ and $N_2$ is a graded submodule of $M_2$ and $N = N_1 \times N_2$. If $N$ is a graded $S$-strongly prime submodule of $M$, then $N_1$ is a graded $S_1$-strongly prime submodule of $M_1$ and $(N_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$ or $N_2$ is a graded $S_2$-strongly prime submodule of $M_2$ and $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$.

**Proof.** Assume that $N$ is a graded $S$-strongly prime submodule of $M$. First, note that $(N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ is a graded $S$-prime ideal of $R$ by Theorem 2.14. Hence, by Lemma 3.6, $(N_1 :_R M_1) \cap S_1 \neq \emptyset$ or $(N_2 :_R M_2) \cap S_2 \neq \emptyset$. We may assume that $(N_1 :_R M_1) \cap S_1 \neq \emptyset$. We will show that $N_2$ is a graded $S$-strongly prime submodule of $M_2$. Let $((N_2 + R_2(x_2)_g) :_{R_2} M_2)(y_2)_h \subseteq N_2$ for some $(x_2)_g, (y_2)_h \in h(M_2)$. We have $((N_1 \times N_2 + R(0, (x_2)_g)) :_R M_1 \times M_2)(0, (y_2)_h) \subseteq N_1 \times N_2$ because if $(r_1, r_2) \in ((N_1 \times N_2 + R(0, (x_2)_g)) :_R M_1 \times M_2)$, then $(r_1, r_2)(M_1 \times M_2) \subseteq N_1 \times N_2 + R(0, (x_2)_g)$. We get $r_2M_2 \subseteq N_2 + R(x_2)_g$. Thus, $(r_1, r_2)(0, (y_2)_h) = (0, r_2(y_2)_h) \in N_1 \times N_2$ and so $((N_1 \times N_2 + R(0, (x_2)_g)) :_R M_1 \times M_2)(0, (y_2)_h) \subseteq N_1 \times N_2$. Then there exists $s = (s_1, s_2) \in S$ such that $(s_1, s_2)(0, (x_2)_g) \in N_1 \times N_2$ or $(s_1, s_2)(0, (y_2)_h) \in N_1 \times N_2$ since $N_1 \times N_2$ is a graded $S$-strongly prime submodule of $M$, hence $s_2(x_2)_g \in N_2$ or $s_2(y_2)_h \in N_2$. Therefore, $N_2$ is a graded $S$-strongly prime submodule of $M_2$. In the other case, it can be similarly shown that $N_1$ is a graded $S_1$-strongly prime submodule of $M_1$. ■

**Corollary 3.8** Let $M = M_1 \times M_2 \times \cdots \times M_n$ be a graded $R = R_1 \times R_2 \times \cdots \times R_n$-module and $S = S_1 \times S_2 \times \cdots \times S_n \cap h(R)$ be a multiplicatively closed subset of $R$ where $M_i$ is a graded $R_i$-module and $S_i \subseteq h(R_i)$ is a multiplicatively closed subset of $R_i$ for each $i = 1, 2, \ldots, n$. Suppose that $N = N_1 \times N_2 \times \cdots \times N_n$ is a graded submodule of $M$. If $N$ is a graded $S$-strongly prime submodule of $M$, then $N_i$ is a graded $S_i$-strongly prime submodule of $M_i$ for some $i \in \{1, 2, \ldots, n\}$ and $(N_j :_{R_j} M_j) \cap S_j \neq \emptyset$ for all $j \in \{1, 2, \ldots, n\} - \{i\}$.

**Proof.** We apply induction on $n$. For $n = 1$, the result is true. If $n = 2$, then it follows from Theorem 3.7. Let it hold when $k < n$. Now, we will prove if $k = n$. Let $N = N_1 \times N_2 \times \cdots \times N_n$. Put $N' = N_1 \times N_2 \times \cdots \times N_{n-1}$ and $S' = S_1 \times S_2 \times \cdots \times S_{n-1} \cap h(R_1 \times R_2 \times \cdots \times R_{n-1})$. Then, by Theorem 3.7, for $N = N' \times N_n$ is a graded $S$-strongly prime submodule of $M$ that $N'$ is a graded $S'$-strongly prime submodule of $M'$ and $(N_n :_{R_n} M_n) \cap S_n \neq \emptyset$ or $N_n$ is a graded $S_n$-strongly prime submodule of $M_n$ and $(N' :_{R'} M') \cap S' \neq \emptyset$ where $M' = M_1 \times M_2 \times \cdots \times M_{n-1}$ and $R' = R_1 \times R_2 \times \cdots \times R_{n-1}$. The rest follows from the induction hypothesis. ■

## 4. Conclusions

In this article, we introduced the concept of graded $S$-strongly prime submodules of a graded module over a commutative graded ring. In fact, the concept of graded $S$-strongly prime submodules is different from the concept of graded strongly prime submodules and many results for graded strongly prime submodules do not apply to graded $S$-strongly prime submodules. Several properties, examples and characterizations of graded $S$-strongly prime submodules, especially in graded multiplication modules, have been investigated. Moreover, we explored the behaviour of graded $S$-strongly prime submodules under graded module homomorphisms, localizations, quotient graded modules, Cartesian product.

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