



Some results on graded S -strongly prime submodules

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Abstract. Let G be a group with identity e and R be a commutative G -graded ring with nonzero identity, $S \subseteq h(R)$ a multiplicatively closed subset of R and M be a graded R -module. A graded submodule N of M with $(N :_R M) \cap S = \emptyset$ is said to be graded S -strongly prime if there exists $s \in S$ such that whenever $((N + Rx_g) :_R M)y_h \subseteq N$, then $sx_g \in N$ or $sy_h \in N$ for all $x_g, y_h \in h(M)$. The aim of this paper is to introduce and investigate some basic properties of the notion of graded S -strongly prime submodules, especially in graded multiplication modules. Moreover, we investigate the behaviour of this structure under graded module homomorphisms, localizations of graded modules, quotient graded modules, Cartesian product.

Keywords: Graded S -prime submodule, graded S -strongly prime submodule, graded multiplication module.

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1. Introduction

In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied in [1, 2, 4–8, 10, 12–15]. In this paper, first, we introduce and study the notions of graded S -strongly prime submodules and graded S -strongly semiprime submodules of a graded R -module M as a generalization of graded prime submodules and we investigate some properties of such graded submodules. For example, we show that if N is a graded S -strongly prime submodule of M , then N is a graded S -strongly semiprime submodule and $(N :_R M)$ is a graded S -prime ideal of R . Also, we give some characterizations of graded S -strongly prime submodules in graded multiplication modules. Second, we

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investigate the behaviour of this structure under graded module homomorphisms, localizations, quotient graded modules, Cartesian product.

Let G be a group with identity e and R be a ring. Then R is said to be a G -graded if $R = \bigoplus_{g \in G} R_g$ such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$ [13]. The elements of R_g are homogeneous of degree g . An element r of R has a unique decomposition as $r = \sum_{g \in G} r_g$ with $r_g \in R_g$ for all $g \in G$, but the sum being a finite sum, i.e. almost all r_g zero. Let $R = \bigoplus_{g \in G} R_g$ be a graded ring and I be an ideal of a graded ring R . Then I is said to be a graded ideal of R , if $I = \bigoplus_{g \in G} (I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g \in G} x_g$, where $x_g \in I$ for all $g \in G$. Moreover, R/I becomes a G -graded ring with g -component $(R/I)_g = (R_g + I)/I$ for $g \in G$ [13]. A graded ring R is called graded quasilocal ring if it has a unique graded maximal ideal [12]. We call $S \subseteq h(R)$ is a multiplicatively closed subset of R if $0 \notin S$, $1 \in S$ and $s_g s'_g \in S$ for all $s_g, s'_g \in S$ [12]. Let R be a graded ring and M an R -module. We say that M is a graded R -module if there exists a family of subgroups $\{M_g\}_{g \in G}$ of M such that $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. The elements of M_g are called homogeneous of degree g . It is clear that M_g is an R_e -submodule of M for all $g \in G$. Moreover, $h(M) = \bigcup_{g \in G} M_g$ [13]. Let N be an R -submodule of a graded R -module M . Then N is said to be a graded R -submodule if $N = \bigoplus_{g \in G} (N \cap M_g)$, i.e. for $m \in N$, $m = \sum_{g \in G} m_g$, where $m_g \in N$ for all $g \in G$. Moreover, M/N becomes a G -graded module with g -component $(M/N)_g = (M_g + N)/N$ for $g \in G$ [13]. A proper graded submodule N of a graded R -module M is said to be graded prime if $r_g m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then $m_h \in N$ or $r_g \in (N : M)$. A graded R -module M is called graded prime if the zero graded submodule is graded prime in M [2]. A proper graded submodules N of a graded R -module M is call graded semiprime if $r_g^k m_h \in N$ for some $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{N}$, then $r_g m_h \in N$ [9]. Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded submodule of a graded R -module M with $(N :_R M) \cap S = \emptyset$. Then N is said to be a graded S -prime submodule if there exists $s \in S$ such that whenever $r_g m_h \in N$, then $sm_h \in N$ or $sr_g \in (N : M)$ for each $r_g \in h(R)$ and $m_h \in h(M)$ [15]. A graded R -module M is called graded finitely generated if $M = Rm_{g_1} + Rm_{g_2} + \dots + Rm_{g_n}$ for some $m_{g_1}, \dots, m_{g_n} \in h(M)$ [2]. Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and M be a graded R -module. Then $S^{-1}M$ is a graded $S^{-1}R$ -module with

$$(S^{-1}M)_g = \left\{ \frac{m}{s} : (\deg m)(\deg s)^{-1} = g \right\}$$

and $(S^{-1}R)_g = \left\{ \frac{r}{s} : (\deg r)(\deg s)^{-1} = g \right\}$ [13] Let $M = \bigoplus_{g \in G} M_g$ and $M' = \bigoplus_{g \in G} M'_g$ be two graded R -modules. A mapping f from M into M' is said to be a graded homomorphism, if for all $m, n \in M$;

- (1) $f(m + n) = f(m) + f(n)$,
- (2) $f(rm) = rf(m)$, for any $r \in R$ and $m \in M$,
- (3) For any $g \in G$; $f(M_g) \subseteq M'_g$ [12].

Let R_1 and R_2 be G -graded rings. Then $R = R_1 \times R_2$ is a G -graded ring with $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Let M_1 be a G -graded R_1 -module, M_2 be a G -graded R_2 -module and $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a G -graded R -module with $M_g = (M_1)_g \times (M_2)_g$ for all $g \in G$. Also, if $S_1 \subseteq h(R_1)$ is a multiplicatively closed subset of R_1 and $S_2 \subseteq h(R_2)$ is a multiplicatively closed subset of R_2 , then $S = S_1 \times S_2$ is a multiplicatively closed subset of R . Furthermore, each graded submodule of M is of the form $N = N_1 \times N_2$ where N_i is a graded submodule of M_i for $i = 1, 2$ [12]. A graded

R -module M is called graded multiplication module, if every graded submodule N of M , $N = IM$ for some graded ideal I of R [3].

Throughout this work, R is a commutative graded rings with identity and M is a graded R -module. Also, $S \subseteq h(R)$ is a multiplicatively closed subset of R .

2. Characterizations of graded S -strongly prime submodules

Definition 2.1 (a) A proper graded submodule N of M is said to be a graded strongly prime submodule if $((N + Rx_g) :_R M)y_h \subseteq N$, then $x_g \in N$ or $y_h \in N$ for each $x_g, y_h \in h(M)$.

(b) A graded submodule N of M with $(N :_R M) \cap S = \emptyset$ is said to be graded S -strongly prime if there exists $s \in S$ such that whenever $((N + Rx_g) :_R M)y_h \subseteq N$, then $sx_g \in N$ or $sy_h \in N$ for each $x_g, y_h \in h(M)$.

Note that if we consider R as a graded R -module, then graded S -strongly prime submodules are exactly graded S -prime ideals of R .

The following Lemma is known, but we write it here for the sake of references.

Lemma 2.2 Let M be a graded module over a graded ring R . Then the following hold:

- (i) If I and J are graded ideals of R , then $I + J$ and $I \cap J$ are graded ideals of R .
- (ii) If I is a graded ideal of R , N is a graded submodule of M , $r_g \in h(R)$ and $x_h \in h(M)$, then Rx_h, IN, r_gN and $(0 :_M I)$ are graded submodules of M .
- (iii) If N and K are graded submodules of M , then $N + K$ and $N \cap K$ are also graded submodules of M and $(N :_R M)$ is a graded ideal of R . Also, $Ann_R(M) = (0 :_R M)$ is a graded ideal of R .
- (iv) Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a collection of graded submodules of M . Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodules of M .

Proposition 2.3

- (i) Every graded strongly prime submodule N of M with $(N :_R M) \cap S = \emptyset$ is also a graded S -strongly prime submodule of M .
- (ii) Let $S \subseteq h(R)$ be a multiplicatively closed subset of R consisting of units in R . Then a graded submodule N of M is graded strongly prime if and only if N is graded S -strongly prime.

Proof. The proof is completely straightforward. ■

By setting $S = \{1\}$, we conclude that every graded strongly prime submodule is a graded S -strongly prime submodule by Proposition 2.3. The following example shows that the converse is not true in general.

Example 2.4

- (i) Let us observe $R = \mathbb{Z}$ as a trivially \mathbb{Z}_2 -graded ring and $M = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ be a \mathbb{Z}_2 -graded R -module with $M_0 = \mathbb{Z}/n\mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}/n\mathbb{Z}$ where n is a positive integer with M_0 . Let p be a prime factor of n and $S = \mathbb{Z} - p\mathbb{Z}$. Then the submodule $p\mathbb{Z}/n\mathbb{Z} \times \{0\}$ is a graded S -strongly prime submodule of M .
- (ii) Let $R = \mathbb{Z}[i]$ be \mathbb{Z}_2 -graded R -module with $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$ and $S = \{2^n \mid n \in \mathbb{N} \cup \{0\}\}$. Consider the graded submodule $N = \langle 4i \rangle$ of graded R -module R . Put $s = 4$. It is easy to see that N is a graded S -strongly prime submodule. But N is not a graded strongly prime submodule.

Definition 2.5 (a) Let N be a graded submodule of M such that $(N :_R M) \cap S = \emptyset$. Then N is said to be a graded S -semiprime submodule if there exists $s \in S$ such that whenever $r_g^2 m_h \in N$, then $sr_g m_h \in N$ for all $r_g \in h(R)$ and $m_h \in h(M)$.

(b) Let N be a graded submodule of M such that $(N :_R M) \cap S = \emptyset$. Then N is said to be a graded S -strongly semiprime submodule if there exists $s \in S$ such that whenever $((N + Rx_g) :_R M)x_g \subseteq N$, then $sx_g \in N$ for all $x_g \in h(M)$.

(c) A graded ideal I of R is called graded S -semiprime if $I \cap S = \emptyset$ and there exists $s \in S$ such that whenever $a_g^2 \in I$, then $sa_g \in I$ for all $a_g \in h(R)$.

Lemma 2.6 Every graded S -strongly semiprime submodule is a graded S -semiprime submodule.

Proof. Let N be a graded S -strongly semiprime submodule of M and suppose $r_g^2 m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$. Thus, $((N + R(r_g m_h)) :_R M)(r_g m_h) = r_g((N + R(r_g m_h)) :_R M)m_h \subseteq r_g(N + R(r_g m_h)) \subseteq N$. Since N is a graded S -strongly semiprime submodule, there exists $s \in S$ such that $sr_g m_h \in N$. Therefore, N is a graded S -semiprime submodule. ■

Proposition 2.7 If N is a graded S -strongly semiprime submodule of M , then $(N :_R M)$ is a graded S -semiprime ideal of R .

Proof. Let $a_g^2 \in (N :_R M)$ where $a_g \in h(R)$. Let $m \in M$. Hence $m = \sum_{h \in G} m_h$ where $m_h \in M_h$ for all $h \in G$. Suppose $m_h \in M_h$. Thus $((N + R(a_g m_h)) :_R M)(a_g m_h) = a_g((N + R(a_g m_h)) :_R M)m_h \subseteq a_g(N + R(a_g m_h)) \subseteq N$. Since N is graded S -strongly semiprime, there exists $s \in S$ such that $sa_g m_h \in N$, so $sa_g m \in N$ and $sa_g \in (N :_R M)$. Therefore, $(N :_R M)$ is a graded S -semiprime ideal of R . ■

The following example shows that the converse of Proposition 2.7 is not hold.

Example 2.8 Let $R = \mathbb{Z}$ be a trivially \mathbb{Z}_2 -graded ring and $M = \mathbb{Q} \times \mathbb{Q}$ where \mathbb{Q} is the field of rational numbers be a \mathbb{Z}_2 -graded module with $M_0 = \mathbb{Q} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Q}$. Take the graded submodule $N = \mathbb{Z} \times \{0\}$ and the multiplicatively closed subset $S = \mathbb{Z} - \{0\}$ of \mathbb{Z} . Then the graded ideal $(N :_{\mathbb{Z}} M) = 0$ is a graded S -semiprime, but N is not a graded S -strongly semiprime submodule of M . Let s be an arbitrary element of S . Choose a prime number p with $\gcd(p, s) = 1$. Note that $((N + R(\frac{1}{p}, 0)) :_R M)(\frac{1}{p}, 0) \subseteq N$, but $(\frac{s}{p}, 0) \notin N$.

Proposition 2.9

- (i) Every graded S -strongly prime submodule is a graded S -prime submodule.
- (ii) Every graded S -strongly prime submodule is a graded S -strongly semiprime submodule.
- (iii) Every graded maximal submodule N of M with $(N :_R M) \cap S = \emptyset$ is a graded S -strongly prime submodule.

Proof. (i) Let N be a graded S -strongly prime submodule of M . Thus there exists $s \in S$ such that whenever $((N + Rx_g) :_R M)y_h \subseteq N$ for all $x_g, y_h \in h(M)$, implies that $sx_g \in N$ or $sy_h \in N$. Let $r_g m_h \in N$ and $sm_h \notin N$ for some $r_g \in h(R)$ and $m_h \in h(M)$. We show that $sr_g \in (N :_R M)$. Let $x = \sum_{k \in G} x_k \in M$. Thus we have $((N + Rm_h) :_R M)(r_g x_k) = r_g((N + Rm_h) :_R M)x_k \subseteq r_g(N + Rm_h) \subseteq N$ for any $x_k \in M_k$, since $sm_h \notin N$ and N is a graded S -strongly prime submodule of M , we conclude $sr_g x_k \in N$ for any $x_k \in M_k$. Hence $sr_g x \in N$. Therefore $sr_g M \subseteq N$ and so $sr_g \in (N :_R M)$.

(ii) It is clear.

(iii) Let N be a graded maximal submodule of M such that $(N :_R M) \cap S = \emptyset$. Let $x_g, y_h \in h(M)$ and $((N + Rx_g) : M)y_h \subseteq N$. Let $x_g \notin N$. Thus $N + Rx_g = M$, hence $(N + Rx_g :_R M) = R$ and we conclude $y_h \in N$. Therefore N is a graded strongly prime submodule, and since $(N :_R M) \cap S = \emptyset$, then by Proposition 2.3, N is a graded S -strongly prime submodule of M . ■

The following example shows that the concept of graded S -strongly prime submodules is different from the concept of graded S -prime submodules.

Example 2.10 Let R be a G -graded ring, P be a graded prime ideal of R and $S = h(R) - P$. Then $P \times P$ is a graded S -prime submodule of graded R -module $R \times R$, because $P \times P$ is a graded prime submodule of $R \times R$ and $(P \times P :_R R \times R) \cap S = P \cap S = \emptyset$. But it is not a graded S -strongly prime submodule of M . Let s be an arbitrary element of S . Then $((P \times P + R(1, 0)) :_R R \times R)(0, 1) \subseteq P \times P$, but $s(1, 0) \notin P \times P$ and $s(0, 1) \notin P \times P$.

Proposition 2.11 Let M be a graded module over a graded field R and N be a proper graded submodule of M . Then N is a graded maximal submodule of M if and only if N is a graded S -strongly prime submodule of M .

Proof. Let N be a graded maximal submodule of M . We have $(N :_R M) \cap S = \emptyset$, because if $s \in (N :_R M) \cap S$, then $1 = s^{-1}s \in (N :_R M)$, a contradiction. Thus by Proposition 2.9, N is a graded S -strongly prime submodule of M . Conversely, let N be a graded S -strongly prime submodule of M which is not a graded maximal submodule of M . Then there exists $x_g \in h(M) \setminus N$ such that $Rx_g + N \neq M$. Let $y = \sum_{h \in G} y_h \in M$. Hence for any $y_h \in M_h$, we have $((N + Rx_g) :_R M)y_h = \{0\}y_h = \{0\} \subseteq N$. Thus there exists $s \in S$ such that $sx_g \in N$ or $sy_h \in N$ since N is a graded S -strongly prime submodule of M . Since $x_g \notin N$, so $sx_g \notin N$. We conclude that $sy_h \in N$ and $y_h \in N$, so $y \in N$. Thus, $N = M$, which is a contradiction. ■

Corollary 2.12 Let N a graded submodule of M with $(N : M) = P$ and $S = h(R) - P$. If P is a graded maximal ideal of R , then there exists a graded S -strongly prime submodule \mathcal{M} of M with $(\mathcal{M} : M) = P$.

Proof. Note that M/N is a graded module over the graded field R/P , so it has a graded maximal submodule, say \mathcal{M}/N . Then \mathcal{M} is a graded maximal submodule of M containing of N and hence $P = (N : M) \subseteq (\mathcal{M} : M)$, we have $(\mathcal{M} : M) = P$. Since $(\mathcal{M} : M) \cap S = \emptyset$, then by Proposition 2.9, \mathcal{M} is a graded S -strongly prime submodule of M . ■

Definition 2.13 A graded submodule N of a graded R -module M is called graded S - I -maximal if $(N :_R M) = I$ and there exists $s \in S$ such that whenever K is a graded submodule of M containing of N with $(K :_R M) = I$, then $sK \subseteq N$.

Theorem 2.14 Let M be a graded R -module and N be a graded submodule of M .

- (i) N is a graded S -strongly prime submodule of M .
- (ii) N is a graded S -strongly semiprime submodule of M and N is a graded S -prime submodule of M .
- (iii) N is a graded S -strongly semiprime submodule of M and $(N :_R M)$ is a graded S -prime ideal of R .
- (iv) N is a graded S - $(N :_R M)$ -maximal submodule of M .

Then (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).

Proof. (i) \Rightarrow (ii) Apply Proposition 2.9.

(ii) \Rightarrow (iii) Note that for every graded S -prime submodule N of M , the graded ideal $(N :_R M)$ is a graded S -prime ideal of R (see [15, Proposition 2.6]).

(iii) \Rightarrow (iv) Since N is a graded S -strongly semiprime submodule of M , there exists $s \in S$ such that whenever $((N + Rx_g) :_R M)x_g \subseteq N$, then $sx_g \in N$ for all $x_g \in M$. Let K be a graded submodule of M containing N with $(K :_R M) = (N :_R M)$. We show that $sK \subseteq N$. Let $x = \sum_{g \in G} x_g$ be an arbitrary element of K . Since $N \subseteq N + Rx_g \subseteq K$ for any $g \in G$, then $(N :_R M) \subseteq ((N + Rx_g) :_R M) \subseteq (K :_R M) = (N :_R M)$ and so $((N + Rx_g) :_R M) = (N :_R M)$. Thus, $((N + Rx_g) :_R M)x_g = (N :_R M)x_g \subseteq N$, since N is a graded S -strongly semiprime submodule of M , $sx_g \in N$ and hence $sx \in N$. Therefore, $sK \subseteq N$ as required. ■

Proposition 2.15 Let $\{N_i\}_{i \in I}$ be a family of graded S -strongly prime submodules of M such that $(N_i :_R M) = P$ for all $i \in I$. If $\bigcap_{i \in I} N_i$ is a graded S -strongly prime submodule of M , then there exists $s \in S$ such that $sN_i \subseteq N_j$ for all $i, j \in I$.

Proof. Let $N = \bigcap_{i \in I} N_i$. Thus, $(N :_R M) = \bigcap_{i \in I} (N_i :_R M) = P = (N_j :_R M)$ for each $j \in I$. Since N is graded S -strongly semiprime and by Proposition 2.14((i) \Rightarrow (iv)), N is graded S - $(N :_R M)$ -maximal and $N \subseteq N_j$ with $(N :_R M) = (N_j :_R M)$, so there exists $s \in S$ such that $sN_j \subseteq N$ for all $j \in I$. ■

Lemma 2.16 Let N be a graded S -strongly prime submodule of a graded R -module M . Then the following statements hold for some $s \in S$.

- (i) $(N :_M s') \subseteq (N :_M s)$ for all $s' \in S$.
- (ii) $((N :_R M) :_R s') \subseteq ((N :_R M) :_R s)$ for all $s' \in S$.

Proof. (i) Let $m = \sum_{g \in G} m_g \in (N :_M s')$ where $s' \in S$. Then $s'm_g \in N$ for any $m_g \in h(M)$. Since every graded S -strongly prime is graded S -prime, there exists $s \in S$ such that $sm_g \in N$ or $ss' \in (N :_R M)$. As $(N :_R M) \cap S = \emptyset$, we get $sm_g \in N$ so $sm \in N$, namely $m \in (N :_M s)$.

(ii) It follows from (i). ■

Theorem 2.17 Let N be a graded submodule of a graded R -module M provided $(N :_R M) \cap S = \emptyset$. Then N is a graded S -strongly prime submodule of M if and only if $(N :_M s)$ is a graded strongly prime submodule of M for some $s \in S$.

Proof. Assume that N is a graded S -strongly prime submodule of M . Then there exists $s \in S$ such that whenever $((N + Rx_g) :_R M)y_h \subseteq N$, then $sx_g \in N$ or $sy_h \in N$ for all $x_g, y_h \in h(M)$. We prove that $(N :_M s)$ is a graded strongly prime submodule. Taking $x_g, y_h \in M$ with $((N :_M s) + Rx_g) :_R M)y_h \subseteq (N :_M s)$, we have $((N :_M s) + Rx_g) :_R M)(sy_h) \subseteq s(N :_M s) \subseteq N$. Since $N \subseteq (N :_M s)$, $((N + Rx_g) :_R M)(sy_h) \subseteq N$. Thus, $sx_g \in N$ or $s^2y_h \in N$. If $sx_g \in N$, then $x_g \in (N :_M s)$. If $s^2y_h \in N$, then $y_h \in (N :_M s^2) \subseteq (N :_M s)$ by Lemma 2.16. Hence, $(N :_M s)$ is a graded strongly prime submodule of M . Conversely, assume that $(N :_M s)$ is a graded strongly prime submodule of M . Let $((N + Rx_g) :_R M)y_h \subseteq N$ for some $x_g, y_h \in h(M)$. Since $N \subseteq (N :_M s)$, we have $x_g \in (N :_M s)$ or $y_h \in (N :_M s)$. Thus, $sx_g \in N$ or $sy_h \in N$, and so N is a graded S -strongly prime submodule of M . ■

Theorem 2.18 Let N be a graded submodule of M provided $(N :_R M) \subseteq \text{Jac}^{gr}(R)$, where $\text{Jac}^{gr}(R)$ is the intersection of all graded maximal ideals of R . Then the following statements are equivalent:

- (i) N is a graded strongly prime submodule of M .
- (ii) N is a graded prime submodule of M and N is a graded $(h(R) - \mathfrak{m})$ -strongly prime

submodule of M for each graded maximal ideal \mathfrak{m} .

Proof. (i) \Rightarrow (ii) Let N be a graded strongly prime submodule of M . Then N is a graded prime submodule of M . Since $(N :_R M) \subseteq \text{Jac}^{gr}(R)$, $(N :_R M) \subseteq \mathfrak{m}$ for each graded maximal ideal \mathfrak{m} and so $(N :_R M) \cap (h(R) - \mathfrak{m}) = \emptyset$. Thus, N is a graded $(h(R) - \mathfrak{m})$ -strongly prime submodule of M by Proposition 2.3.

(ii) \Rightarrow (i) Suppose that N is a graded prime submodule of M and N is a graded $(h(R) - \mathfrak{m})$ -strongly prime submodule of M for each graded maximal ideal \mathfrak{m} . Let $((N + Rx_g) :_R M)y_h \subseteq N$ and $y_h \notin N$ for some $x_g, y_h \in h(M)$. Let \mathfrak{m} be a graded maximal ideal of R . Since N is a graded $(h(R) - \mathfrak{m})$ -strongly prime submodule of M , there exists $s_{\mathfrak{m}} \in h(R) - \mathfrak{m}$ such that $s_{\mathfrak{m}}x_g \in N$ or $s_{\mathfrak{m}}y_g \in N$. If $s_{\mathfrak{m}}y_h \in N$, then since N is a graded prime submodule of M and $y_h \notin N$, $s_{\mathfrak{m}} \in (N :_R M)$ which is a contradiction. Hence, $s_{\mathfrak{m}}x_g \in N$. Consider the set $Q = \{s_{\mathfrak{m}} \mid \exists \mathfrak{m} \in \text{Max}^{gr}(R); s_{\mathfrak{m}} \notin \mathfrak{m} \text{ and } s_{\mathfrak{m}}x_g \in N\}$. Suppose that $\langle Q \rangle \neq R$. Take any graded maximal ideal \mathfrak{m}' containing Q . Then the definition of Q requires that there exists $s_{\mathfrak{m}'} \in Q$ and $s_{\mathfrak{m}'} \notin \mathfrak{m}'$, which is a contradiction. Thus, $\langle Q \rangle = R$ and $1 = r_1s_{\mathfrak{m}_1} + r_2s_{\mathfrak{m}_2} + \dots + r_ns_{\mathfrak{m}_n}$ for some $r_i \in R$ and $s_{\mathfrak{m}_i} \notin \mathfrak{m}_i$ with $s_{\mathfrak{m}_i}x_g \in N$, where $\mathfrak{m}_i \in \text{Max}^{gr}(R)$ for each $i = 1, 2, \dots, n$. Therefore, $x_g = r_1s_{\mathfrak{m}_1}x_g + r_2s_{\mathfrak{m}_2}x_g + \dots + r_ns_{\mathfrak{m}_n}x_g \in N$. Hence N is a graded strongly prime submodule of M . ■

By the previous theorem we have the following result:

Corollary 2.19 Let M be a graded module over a graded quasilocal ring (R, \mathfrak{m}) . Then the following statements are equivalent:

- (i) N is a graded strongly prime submodule of M .
- (ii) N is a graded prime submodule of M and N is a graded $(h(R) - \mathfrak{m})$ -strongly prime submodule of M .

Now, we characterize graded S -strongly prime submodules of a graded multiplication module.

Theorem 2.20 Let M be a graded multiplication R -module and N be a graded submodule of M provided that $(N :_R M) \cap S = \emptyset$. Then the following statements are equivalent:

- (i) N is a graded S -strongly prime submodule of M .
- (ii) $(N :_R M)$ is a graded S -prime ideal of R .
- (iii) $N = IM$ for some graded S -prime ideal I of R with $\text{ann}(M) \subseteq I$.

Proof. (i) \Rightarrow (ii) It follows from Theorem 2.14.

(ii) \Rightarrow (iii) Consider $I = (N :_R M)$.

(iii) \Rightarrow (i) By [15, Proposition 2.8], N is a graded S -prime submodule of M . Thus, there exists $s \in S$ such that whenever $r_gm_h \in N$, then $sm_h \in N$ or $sr_g \in (N :_R M)$ for all $m_h \in h(M)$ and $r_g \in h(R)$. Now, we show that N is a graded S -strongly prime submodule of M . Let $((N + Rx_g) :_R M)y_h \subseteq N$ and $sy_h \notin N$ for some $x_g, y_h \in h(M)$. Since N is a graded S -prime submodule of M , $s((N + Rx_g) :_R M) \subseteq (N :_R M)$. As M is a graded multiplication R -module, we have

$$s(N + Rx_g) = s((N + Rx_g) :_R M)M \subseteq (N :_R M)M = N.$$

Therefore, $sx_g \in N$ and N is a graded S -strongly prime submodule of M . ■

Lemma 2.21 Let Q be a graded S -primary ideal of R . Then $\text{Grad}(Q)$ is a graded S -prime ideal of R .

Proof. First note that $\text{Grad}(Q) \cap S = \emptyset$, because if $s \in \text{Grad}(Q) \cap S$, then $s^n \in Q \cap S$ for some $n \in \mathbb{N}$, a contradiction. Let $a_g b_h \in \text{Grad}(Q)$ where $a_g, b_h \in h(R)$. Thus, $(a_g b_h)^k \in Q$ for some $k \in \mathbb{N}$. Since Q is a graded S -primary ideal of R , there exists $s \in S$ such that $sa_g^k \in Q$ or $sb_h^k \in \text{Grad}(Q)$. We conclude $sa_g \in \text{Grad}(Q)$ or $sb_h \in \text{Grad}(Q)$. Hence, $\text{Grad}(Q)$ is a graded S -prime ideal of R . ■

Lemma 2.22 Let M be a finitely generated multiplication R -module and N be a submodule of M . Then $(\text{Grad}(N) :_R M) = \text{Grad}((N :_R M))$.

Proof. The proof is similar to Theorem 4 of [11]. ■

Theorem 2.23 Let M be a graded finitely generated multiplication R -module. If N is a graded S -strongly prime submodule of M , then $\text{Grad}(N)$ is a graded S -strongly prime submodule of M .

Proof. Since N is a graded S -strongly prime submodule of M , $(N :_R M)$ is a graded S -prime ideal of R by Theorem 2.14. Thus, by Lemma 2.21, $\text{Grad}((N :_R M))$ is a graded S -prime ideal of R . By Lemma 2.22, we have $(\text{Grad}(N) :_R M) = \text{Grad}((N :_R M))$. Thus, $(\text{Grad}(N) :_R M)$ is a graded S -prime submodule of M . Now, the result follows from Theorem 2.20. ■

3. Behaviour of graded S -strongly prime submodules

In this section, we investigate the behaviour of graded S -strongly prime submodules under graded module homomorphisms, localizations, quotient graded modules and Cartesian product.

Proposition 3.1 Let $f : M \rightarrow M'$ be a graded R -homomorphism. Then the following statements hold:

- (i) If N' is a graded S -strongly prime submodule of M' such that $(f^{-1}(N') :_R M) \cap S = \emptyset$, then $f^{-1}(N')$ is a graded S -strongly prime submodule of M .
- (ii) If f is a graded epimorphism and N is a graded S -strongly prime submodule of M containing $\text{Ker}(f)$, then $f(N)$ is a graded S -strongly prime submodule of M' .

Proof. (i) Let $((f^{-1}(N') + Rx_g) :_R M)y_h \subseteq f^{-1}(N')$ for some $x_g, y_h \in h(M)$. Thus, $f(((f^{-1}(N') + Rx_g) :_R M)y_h) \subseteq f(f^{-1}(N')) \subseteq N'$. Since f is a graded R -homomorphism, $((f^{-1}(N') + Rx_g) :_R M)f(y_h) \subseteq N'$. Now, we show that $((N' + Rf(x_g)) :_R M') \subseteq ((f^{-1}(N') + Rx_g) :_R M)$. Take $r \in ((N' + Rf(x_g)) :_R M')$. Then $rM' \subseteq N' + Rf(x_g)$. Since $f(M) \subseteq M'$, we have $f(rM) = rf(M) \subseteq rM' \subseteq N' + Rf(x_g)$. This implies that $rM \subseteq f^{-1}(N' + f(Rx_g))$. It is clear that $f^{-1}(N' + f(Rx_g)) \subseteq f^{-1}(N') + Rx_g$. Thus, $r \in ((N' + Rx_g) :_R M)$ and so $((N' + Rf(x_g)) :_R M')f(y_h) \subseteq N'$. Since N' is a graded S -strongly prime submodule of M' , there exists $s \in S$ such that $sf(x_g) \in N'$ or $sf(y_h) \in N'$. Therefore, $sx_g \in f^{-1}(N')$ or $sy_h \in f^{-1}(N')$ as needed.

(ii) First note that $(f(N) :_R M') \cap S = \emptyset$. Otherwise, there exists $s \in (f(N) :_R M') \cap S$. Hence, $sM' \subseteq f(N)$ and then $f(sM) = sf(M) = sM' \subseteq f(N)$ and $sM \subseteq N + \text{Ker}(f) = N$. That means $s \in (N :_R M)$, which is a contradiction. Let $((f(N) + Rx'_g) :_R M')y'_h \subseteq f(N)$ where $x'_g, y'_h \in h(M')$. Since f is a graded epimorphism, $f(x_g) = x'_g$ and $f(y_h) = y'_h$ for some $x_g, y_h \in h(M)$. Thus $(f(N + Rx_g) :_R M')f(y_h) \subseteq f(N)$. It is easy to see that $((N + Rx_g) :_R M) \subseteq (f(N + Rx_g) :_R M')$. Hence $f(((N + Rx_g) :_R M)y_h) \subseteq f(N)$ and $((N + Rx_g) :_R M)y_h \subseteq N + \text{Ker}(f) \subseteq N$. Thus, there exists $s \in S$ such that $sx_g \in N$ or $sy_h \in N$ since N is a graded S -strongly prime submodule of M . Therefore,

$sf(x_g) \in f(N)$ or $sf(y_h) \in f(N)$, and so $f(N)$ is a graded S -strongly prime submodule of M' . ■

Proposition 3.2 Let N and K be graded submodules of M with $K \subseteq N$. Then the following assertions hold:

- (i) If N' is a graded S -strongly prime submodule of M with $(N' :_R K) \cap S = \emptyset$, then $K \cap N'$ is a graded S -strongly prime submodule of K .
- (ii) N is a graded S -strongly prime submodule of M if and only if N/K is a graded S -strongly prime submodule of M/K .

Proof. (i) Consider the injection $i : K \rightarrow M$ defined by $i(x) = x$ for all $x \in K$. Then $i^{-1}(N') = K \cap N'$. By $(N' :_R K) \cap S = \emptyset$, we give $(i^{-1}(N') :_R K) \cap S = \emptyset$. Thus, the rest follows from Proposition 3.1(i).

(ii) Let N be a graded S -strongly prime submodule of M . Then consider the canonical homomorphism $\pi : M \rightarrow M/K$ defined by $\pi(m) = m + K$ for all $m \in M$. Then note that π is a graded epimorphism and $\text{Ker}(\pi) = K \subseteq N$. Thus by Proposition 3.1(ii), N/K is a graded S -strongly prime submodule of M/K . Conversely, assume that N/K is a graded S -strongly prime submodule of M/K . Let $((N + Rx_g) :_R M)y_h \subseteq N$ where $x_g, y_h \in h(M)$. We have $((N + Rx_g)/K :_R M/K) = (R(x_g + K) + N/K :_R M/K) = ((N + Rx_g) :_R M)$. Thus, $((R(x_g + K) + N/K) :_R M/K)(y_h + K) \subseteq N/K$. Since N/K is a graded S -strongly prime submodule of M/K , there exists $s \in S$ such that $s(x_g + K) \in N/K$ or $s(y_h + K) \in N/K$. Thus, $sx_g \in N$ or $sy_h \in N$, and so N is a graded S -strongly prime submodule of M . ■

Let $S \subseteq h(R)$ be a multiplicatively closed subset of R . The saturation S^* of S is defined as $S^* = \{x \in h(R) \mid \frac{x}{1}$ is a homogeneous unit of $S^{-1}R\}$. Note that $S^* \subseteq h(R)$ is a multiplicatively closed subset of R containing S .

Proposition 3.3

- (i) Let $S_1 \subseteq S_2 \subseteq h(R)$ be multiplicatively closed subsets of R . If N is a graded S_1 -strongly prime submodule and $(N :_R M) \cap S_2 = \emptyset$, then N is a graded S_2 -strongly prime submodule.
- (ii) A graded submodule N of M is a graded S -strongly prime submodule if and only if it is a graded S^* -strongly prime submodule.
- (iii) If N is a graded S -strongly prime submodule of M , then $S^{-1}N$ is a graded strongly prime submodule of graded $S^{-1}R$ -module $S^{-1}M$.

Proof. (i) It is clear.

(ii) Let N be a graded S -strongly prime submodule. Assume that $(N :_R M) \cap S^* \neq \emptyset$ and $r \in (N :_R M) \cap S^*$. Let $r = \sum_{g \in G} r_g$ where $r_g \in R_g$ for all $g \in G$. Hence, $\frac{r_g}{1}$ is a homogeneous unit of $S^{-1}R$, that is, $\frac{r_g}{1} \frac{a}{s} = \frac{1}{1}$ for some $a \in h(R)$ and $s \in S$. Thus, $us = ur_g a \in S$ for some $u \in S$. Then $us = ur_g a \in (N :_R M) \cap S$, which is a contradiction. Thus, $(N :_R M) \cap S^* = \emptyset$. Since $S \subseteq S^*$, by (i), N is a graded S^* -strongly prime submodule of M . Conversely, assume that N is a graded S^* -strongly prime submodule. Let $((N + Rx_g) :_R M)y_h \subseteq N$ for some $x_g, y_h \in h(M)$. Since N is a graded S^* -strongly prime submodule, there exists $s' \in S^*$ so that $s'x_g \in N$ or $s'y_h \in N$. As $\frac{s'}{1}$ is a unit of $S^{-1}R$, there exist $u, s \in S$ and $a \in h(R)$ such that $su = us'a$. Put $us' = s'' \in S$. Then note that $s''x_g \in N$ or $s''y_h \in N$. Therefore, N is a graded S -strongly prime submodule of M .

(iii) Let N be a graded S -strongly prime submodule. Thus, we have $(N :_R M) \cap S = \emptyset$

and there exists $s \in S$ such that whenever $((N + Rx_g) : M)y_h \subseteq N$, then $sx_g \in N$ or $sy_h \in N$ for all $x_g, y_h \in h(M)$. Let

$$\left((S^{-1}N + S^{-1}R\left(\frac{x_g}{u}\right)) :_{S^{-1}R} S^{-1}M \right) \frac{y_h}{v} \subseteq S^{-1}N$$

where $\frac{x_g}{u}, \frac{y_h}{v} \in S^{-1}M$. We show that $((N + Rx_g) :_R M)(sy_h) \subseteq N$. If $r \in ((N + Rx_g) :_R M)$, then we can write $r = \sum_{k \in G} r_k$ where $r_k \in R_k$ for any $k \in G$. Hence, for any $k \in G$, $r_k M \subseteq N + Rx_g$ and $(\frac{r_k}{1})S^{-1}M \subseteq S^{-1}N + S^{-1}R\left(\frac{x_g}{1}\right) = S^{-1}N + S^{-1}R\left(\frac{x_g}{u}\right)$ and so $(\frac{r_k}{1})(\frac{y_h}{v}) \in S^{-1}N$. Hence, there exist $n \in N$ and $t_1, t_2 \in S$ such that $t_2(t_1 r_k y_h - vn) = 0$. Thus, $(t_2 t_1) r_k y_h \in N$, and since N is a graded S -prime submodule, we get $st_1 t_2 \in (N :_R M)$ or $sr_k y_h \in N$. As $(N :_R M) \cap S = \emptyset$, we have $sr_k y_h \in N$ for every $k \in G$ and so $sry_h \in N$. Hence, $((N + Rx_g) :_R M)(sy_h) \subseteq N$. It follows that $sx_g \in N$ or $s^2 y_h \in N$ since N is a graded S -strongly prime submodule. Therefore, $\frac{x_g}{u} = \frac{sx_g}{su} \in S^{-1}N$ or $\frac{y_h}{v} = \frac{s^2 y_h}{s^2 v} \in S^{-1}N$. Thus, $S^{-1}N$ is a graded strongly prime submodule of $S^{-1}M$. ■

The following example shows that the converse of part (iii) of Proposition 3.3 is not true in general.

Example 3.4 Consider $R = \mathbb{Z}$ and $G = \mathbb{Z}_2$. Then R is trivially G -graded by $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Consider the R -module $U = \mathbb{Q}[i]$. Then U is G -graded by $U_0 = \mathbb{Q}$ and $U_1 = i\mathbb{Q}$. Thus $M = U \times U$ is a G -graded R -module with $M_0 = U_0 \times U_0$ and $M_1 = U_1 \times U_1$. Take the graded submodule $N = \mathbb{Z} \times \{0\}$ and the multiplicatively closed subset $S = \mathbb{Z} - \{0\}$ of \mathbb{Z} . Then $((N + R(x, y)) :_R M) = 0$ for any $(x, y) \in M$. Let s be an arbitrary element of S . Choose prime numbers p, q of \mathbb{Z} . Then note that $((N + R(\frac{1}{p}, 0)) :_R M)(0, \frac{1}{q}) \subseteq N$. But $(\frac{s}{p}, 0) \notin N$ and $(0, \frac{s}{q}) \notin N$, it follows that N is not a graded S -strongly prime submodule of M . Since $S^{-1}\mathbb{Z} = \mathbb{Q}$, $S^{-1}N$ is a graded strongly prime submodule of $S^{-1}M$.

Proposition 3.5 Let M be a graded finitely generated R -module and N be a graded submodule of M satisfying $(N :_R M) \cap S = \emptyset$. Then the following statements are equivalent:

- (i) N is a graded S -strongly prime submodule of M .
- (ii) $S^{-1}N$ is a graded strongly prime submodule of $S^{-1}M$ and there is an $s \in S$ satisfying $(N :_M s') \subseteq (N :_M s)$ for all $s' \in S$.

Proof. (i) \Rightarrow (ii) It follows from Proposition 3.3 and Lemma 2.16.

(ii) \Rightarrow (i) Let $((N + Rx_g) :_R M)y_h \subseteq N$ for some $x_g, y_h \in h(M)$. We have $((S^{-1}N + S^{-1}R\left(\frac{x_g}{1}\right)) :_{S^{-1}R} S^{-1}M) \frac{y_h}{1} \subseteq S^{-1}N$. Then $\frac{x_g}{1} \in S^{-1}N$ or $\frac{y_h}{1} \in S^{-1}N$ since $S^{-1}N$ is a graded strongly prime submodule of $S^{-1}M$. Thus, $ux_g \in N$ or $ty_h \in N$ for some $u, t \in S$. By assumption, there exists $s \in S$ so that $(N :_M u) \subseteq (N :_M s)$ and $(N :_M t) \subseteq (N :_M s)$. Thus, $sx_g \in N$ or $sy_h \in N$ and so N is a graded S -strongly prime submodule of M . ■

Lemma 3.6 Let $R = R_1 \times R_2$ and $S = (S_1 \times S_2) \cap h(R)$ where $S_i \subseteq h(R_i)$ is a multiplicatively closed subset of R_i for each $i = 1, 2$. Suppose that $P = P_1 \times P_2$ is a graded ideal of R . If P is a graded S -prime ideal of R , then P_1 is a graded S_1 -prime ideal of R_1 and $P_2 \cap S_2 \neq \emptyset$ or P_2 is a graded S_2 -prime ideal of R_2 and $P_1 \cap S_1 \neq \emptyset$.

Proof. Suppose P is a graded S -prime ideal of R . Since $(1, 0)(0, 1) = (0, 0) \in P$, there exists $s = (s_1, s_2) \in S$ so that $s(1, 0) = (s_1, 0) \in P$ or $s(0, 1) = (0, s_2) \in P$ and thus, $P_1 \cap S_1 \neq \emptyset$ or $P_2 \cap S_2 \neq \emptyset$. We may assume that $P_1 \cap S_1 \neq \emptyset$. As $P \cap S = \emptyset$, we have $P_2 \cap S_2 = \emptyset$. Let $x_g y_h \in P_2$ for some $x_g, y_h \in R_2$. Since $(0, x_g)(0, y_h) \in P$ and P is a graded S -prime ideal, we get either $s(0, x_g) = (0, s_2 x_g) \in P$ or $s(0, y) = (0, s_2 y) \in P$ and

this yields $s_2x_g \in P_2$ or $s_2y_h \in P_2$. Therefore, P_2 is a graded S_2 -prime ideal of R_2 . In the other case, one can easily show that P_1 is a graded S_1 -prime ideal of R_1 . ■

Theorem 3.7 Let $M = M_1 \times M_2$ be a graded $R = R_1 \times R_2$ -module and $S = (S_1 \times S_2) \cap h(R)$ be a multiplicatively closed subset of R where M_i is a graded R_i -module and $S_i \subseteq h(R_i)$ is a multiplicatively closed subset of R_i for each $i = 1, 2$. Suppose that N_1 is a graded submodule of M_1 and N_2 is a graded submodule of M_2 and $N = N_1 \times N_2$. If N is a graded S -strongly prime submodule of M , then N_1 is a graded S_1 -strongly prime submodule of M_1 and $(N_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$ or N_2 is a graded S_2 -strongly prime submodule of M_2 and $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$.

Proof. Assume that N is a graded S -strongly prime submodule of M . First, note that $(N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ is a graded S -prime ideal of R by Theorem 2.14. Hence, by Lemma 3.6, $(N_1 :_R M_1) \cap S_1 \neq \emptyset$ or $(N_2 :_R M_2) \cap S_2 \neq \emptyset$. We may assume that $(N_1 :_R M_1) \cap S_1 \neq \emptyset$. We will show that N_2 is a graded S -strongly prime submodule of M_2 . Let $((N_2 + R_2(x_2)_g) :_{R_2} M_2)(y_2)_h \subseteq N_2$ for some $(x_2)_g, (y_2)_h \in h(M_2)$. We have $((N_1 \times N_2 + R(0, (x_2)_g)) :_R M_1 \times M_2)(0, (y_2)_h) \subseteq N_1 \times N_2$ because if $(r_1, r_2) \in ((N_1 \times N_2 + R(0, (x_2)_g)) :_R M_1 \times M_2)$, then $(r_1, r_2)(M_1 \times M_2) \subseteq N_1 \times N_2 + R(0, (x_2)_g)$. We get $r_2M_2 \subseteq N_2 + R(x_2)_g$. Thus, $(r_1, r_2)(0, (y_2)_h) = (0, r_2(y_2)_h) \in N_1 \times N_2$ and so $((N_1 \times N_2 + R(0, (x_2)_g)) :_R M_1 \times M_2)(0, (y_2)_h) \subseteq N_1 \times N_2$. Then there exists $s = (s_1, s_2) \in S$ such that $(s_1, s_2)(0, (x_2)_g) \in N_1 \times N_2$ or $(s_1, s_2)(0, (y_2)_h) \in N_1 \times N_2$ since $N_1 \times N_2$ is a graded S -strongly prime submodule of M , hence $s_2(x_2)_g \in N_2$ or $s_2(y_2)_h \in N_2$. Therefore, N_2 is a graded S -strongly prime submodule of M_2 . In the other case, it can be similarly shown that N_1 is a graded S_1 -strongly prime submodule of M_1 . ■

Corollary 3.8 Let $M = M_1 \times M_2 \times \dots \times M_n$ be a graded $R = R_1 \times R_2 \times \dots \times R_n$ -module and $S = S_1 \times S_2 \times \dots \times S_n \cap h(R)$ be a multiplicatively closed subset of R where M_i is a graded R_i -module and $S_i \subseteq h(R_i)$ is a multiplicatively closed subset of R_i for each $i = 1, 2, \dots, n$. Suppose that $N = N_1 \times N_2 \times \dots \times N_n$ is a graded submodule of M . If N is a graded S -strongly prime submodule of M , then N_i is a graded S_i -strongly prime submodule of M_i for some $i \in \{1, 2, \dots, n\}$ and $(N_j :_{R_j} M_j) \cap S_j \neq \emptyset$ for all $j \in \{1, 2, \dots, n\} - \{i\}$.

Proof. We apply induction on n . For $n = 1$, the result is true. If $n = 2$, then it follows from Theorem 3.7. Let it hold when $k < n$. Now, we will prove if $k = n$. Let $N = N_1 \times N_2 \times \dots \times N_n$. Put $N' = N_1 \times N_2 \times \dots \times N_{n-1}$ and $S' = S_1 \times S_2 \times \dots \times S_{n-1} \cap h(R_1 \times R_2 \times \dots \times R_{n-1})$. Then, by Theorem 3.7, for $N = N' \times N_n$ is a graded S -strongly prime submodule of M that N' is a graded S' -strongly prime submodule of M' and $(N_n :_{R_n} M_n) \cap S_n \neq \emptyset$ or N_n is a graded S_n -strongly prime submodule of M_n and $(N' :_{R'} M') \cap S' \neq \emptyset$ where $M' = M_1 \times M_2 \times \dots \times M_{n-1}$ and $R' = R_1 \times R_2 \times \dots \times R_{n-1}$. The rest follows from the induction hypothesis. ■

4. Conclusions

In this article, we introduced the concept of graded S -strongly prime submodules of a graded module over a commutative graded ring. In fact, the concept of graded S -strongly prime submodules is different from the concept of graded strongly prime submodules and many results for graded strongly prime submodules do not apply to graded S -strongly prime submodules. Several properties, examples and characterizations of graded S -strongly prime submodules, especially in graded multiplication modules, have been investigated. Moreover, we explored the behaviour of graded S -strongly prime sub-

modules under graded module homomorphisms, localizations, quotient graded modules, Cartesian product.

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