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Topology and extensions of Mendelian inheritance

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Abstract. In this paper, we seek to provide an appropriate topology for a set of alleles of a gene that have arbitrary interactions between its alleles. For this purpose, we first define a suitable topology on a set of alleles between which there is a similar interaction. Next, with the help of the topology on the set of alleles of a gene, we equip the set of phenotypes of the desired gene with a suitable topology. In addition, the final section will discuss the dependence relations between the alleles of a gene.

Keywords: Discrete topology, indiscrete topology, dependence relation, influence, impact, alleles of a gene, complete dominant, incomplete dominant, codominant.

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1. Introduction

Hyperstructures are algebraic structures equipped with at least one multi-valued operation, called a hyperoperation. The largest classes of the hyperstructures are the ones called H_v-structures [10]. Hypergroups theory was first introduced in 1934 by Marty [17]. Following the introduction of algebraic hypergroups, as a generalization of classical algebraic groups, many studies have been done by mathematical scientists. Two of the most important books in this field have been written by Corsini and Leoreanu [7], and Davvaz and Leoreanu-Fotea [13] (also, see [4, 6, 8, 11, 15, 18–23]).

One of the topics studied in the field of hyperstructures is the topological properties of these structures. In [3], Ameri presented the concept of topological (transposition) hypergroups. He introduced the concept of a (pseudo, strong pseudo) topological hypergroup and gave some related basic results. In various branches of mathematics we encounter important examples of topologico-algebraical structures like topological groupoids, groups,

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rings, fields etc. See [3] to study the generalization of the concept of topological groupoid to topological hypergroupoid. In this paper, we try to provide biological examples of topological hyperstructures by equipping the biological hyperstructures introduced in [1, 2, 12, 14] with a suitable topology.

In many real-life situations, there are contexts with numerous variables, which somehow depend on one another. Consider, for instance, a group of people sharing several secrets, which are not known to everybody because some subgroups, which overlap arbitrarily, share only some secrets; or we have several measurement stations, such as those used in weather forecasting, which measure some data used for predictions for some other places in some future time. In order to study this issue, from the algebraic point of view, the concepts related to dependence relations were introduced [5].

The history of genetics dates from the classical era with contributions by Pythagoras, Hippocrates, Aristotle, Epicurus and others. Modern genetics began with the Mendel's work. His work on pea plants, published in 1866, established the theory of Mendelian inheritance. The year 1900 marked the "rediscovery of Mendel" by Hugo de Vries, Carl Correns and Erich von Tschermak, and by 1915 the basic principles of Mendelian genetics had been studied in a wide variety of organisms most notably the fruit fly Drosophila melanogaster. Led by Thomas Hunt Morgan and his fellow "drosophilists", geneticists developed the Mendelian model, which was widely accepted by 1925. Alongside experimental work, mathematicians developed the statistical framework of population genetics, bringing genetic explanations into the study of evolution [9]. Various applications of algebraic hyperstructures have been studied in other disciplines. From a biological point of view, for some of the studies conducted in this field, we refer the reader to [1, 2, 12, 14].

2. Basic Definitions

2.1 *Concepts of hyperstructures*

In this subsection, we present some of the definitions and concepts related to hyperstructures that are needed throughout this paper.

Definition 2.1 [13] Considering a non-empty set H and denoting by $P^*(H)$ the set of all non-empty subsets of H, a hyperoperation on H is a mapping $\circ : H \times H \longrightarrow P^*(H)$, and the couple $(H, \circ)$ is called a hypergroupoid. If we replace set $P^*(H)$ with set $P(H)$, meaning that the image of some pairs of elements in H could be the empty set, then $(H, \circ)$ is a partial hypergroupoid.

We now consider the following two axioms:

1. $(a \circ b) \circ c = a \circ (b \circ c)$ for all a, b and c of H (associativity),
2. $a \circ H = H \circ a = H$ for all a of H (reproductivity),

if the hyperoperation $\circ$ satisfies Axiom 2, then the hypergroupoid $(H, \circ)$ is a quasi-hypergroup, and if the hyperoperation $\circ$ satisfies Axiom 1, then the hypergroupoid $(H, \circ)$ is a semihypergroup. Also, if the hyperoperation $\circ$ holds in both Axioms 1 and 2, then the hypergroupoid $(H, \circ)$ is called hypergroup. Note that, for any non-empty subsets A and B of H, there is $A \circ B = \cup\{a \circ b; a \in A, b \in B\}$ and $A \circ a = A \circ \{a\}$ (and similarly, $a \circ A = \{a\} \circ A$) for any $a \in H$. If the weak-associativity holds, meaning that $a \circ (b \circ c) \cap (a \circ b) \circ c \neq \emptyset$ for all $a, b, c \in H$, then $(H, \circ)$ is called an H_v-group. Besides, the commutativity means that $a \circ b = b \circ a$ for any $a, b \in H$.

Definition 2.2 [13] A hypergroup $(H, \circ)$ is called cyclic if, for some $h \in H$, there is

$$H = h^1 \cup h^2 \cup ... \cup h^n \cup \ldots, \tag{1}$$

where $h^1 = \{h\}$ and $h^n = h \circ \cdots \circ h$. If there exists $n \in \mathbb{N}$ such that (1) is finite, we say that H is a cyclic hypergroup with finite period; otherwise, H is a cyclic hypergroup with infinite period. The element $h \in H$ in (1) is called the generator of H, and the smallest power n for which (1) is valid is called period of h. If all generators of H have the same period n, then H is called cyclic with period n. If, for a given generator h, (1) is valid, but no such n exists (i.e., (1) cannot be finite), then H is called cyclic with infinite period. If we can write $H = h^n$ for some $h \in H$, then the hypergroup H is called single-power cyclic with a generator h.

2.2 *Concepts of topological hyperstructures*

Here, we present some definitions related to topological hypergroupoids that are used throughout the paper (see [16]).

Definition 2.3 [16] Let (X, τ) be a topological space. Then

1) (X, τ) is a T_0-space if for all $x \neq y \in X$, there exists $U \in \tau$ such that $x \in U$ and y is not in U or $y \in U$ and x is not in U.
2) (X, τ) is a T_1-space if for all $x \neq y \in X$, there exist $U, V \in \tau$ such that $x \in U$ and y is not in U and $y \in V$ and x is not in V.
3) (X, τ) is a T_2-space if for all $x \neq y \in X$, there exist $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

So, every T_2-topological space is a T_1-topological space and every T_1-topological space is a T_0-topological space.

Definition 2.4 [16] Let $(H, \circ)$ be a hypergroupoid and (H, τ) be a topological space. The hyperoperation $\circ$ is called

1) pseudocontinuous (p-continuous) if for every $O \in \tau$, the set $O_\star = \{(x, y) \in H^2 : x \circ y \subseteq O\}$ is open in $H \times H$.
2) strongly pseudocontinuous (sp-continuous) if for every $O \in \tau$, the set $O^\star = \{(x, y) \in H^2 : x \circ y \cap O \neq \emptyset\}$ is open in $H \times H$.

A simple way to prove that a hyperoperation $\circ$ is p-continuous (sp-continuous) is to take any open set O in τ and $(x, y) \in H^2$ such that $x \circ y \subseteq O$ ($x \circ y \cap O \neq \emptyset$) and prove that there exist $U, V \in \tau$ such that $x \in U$, $y \in V$ and $u \circ v \subseteq O$ ($u \circ v \cap O \neq \emptyset$) for all $(u, v) \in U \times V$.

Definition 2.5 [16] Let $(H, \circ)$ be a hypergroupoid, (H, τ) be a topological space and $\tau_\star$ be a topology on $P^*(H)$. The triple $(H, \circ, \tau)$ is called a pseudotopological or strongly pseudotopological hypergroupoid if the hyperoperation $\circ$ is p-continuous or sp-continuous, respectively. The quadruple $(H, \circ, \tau, \tau_\star)$ is called $\tau_\star$-topological hypergroupoid if the hyperoperation $\circ$ is $\tau_\star$-continuous.

Let (H, τ) be a topological space and $V, U_1, \ldots, U_k \in \tau$. We define S_V and I_V as follows:

(1) $S_V = \{U \in P^*(H) : U \subseteq V\} = P^*(V)$,
(2) $I_V = \{U \in P^*(H) : U \cap V \neq \emptyset\}$.

$S_\emptyset = I_\emptyset = \emptyset$. For all $V \neq \emptyset$, we have $S_V = P^*(V)$ and $I_V \supseteq \{H, P^*(V)\}$.

Lemma 2.6 [16] Let (H, τ) be a topological space. Then $\{S_V\}_{V\in\tau}$ forms a base for a topology τ_U on $P^*(H)$. Moreover, τ_U is called the upper topology. Then $\{I_V\}_{V\in\tau}$ forms a subbase for a topology τ_L on $P^*(H)$. Moreover, τ_L is called the lower topology.

Then the following result was proved by S. Hoskova-Mayerova.

Theorem 2.7 [16] Let $(H, \circ)$ be a hypergroupoid and (H, τ) be a topological space. Then the triple $(H, \circ, \tau)$ is a pseudotopological hypergroupoid if and only if the quadruple $(H, \circ, \tau, \tau_U)$ is a τ_U-topological hypergroupoid. The triple $(H, \circ, \tau)$ is a strongly pseudotopological hypergroupoid if and only if the quadruple $(H, \circ, \tau, \tau_L)$ is a τ_L-topological hypergroupoid.

2.3 *Concepts of dependence relations*

Definition 2.8 [5] By a dependence relation, we mean a formula between several variables with a left-hand and right-hand side, expressing that the value of the variable on the left-hand side depends (in an unspecific way) on the values of the variables on the right-hand side.

For example $x \sim D(y, z, t)$, which can be written as $(x, y, z, t) \in D$ (however, with x always in the first position), suggests that we work with a relation of arity four. we always have $x \sim D(x)$, i.e., $(x, x) \in D$. If, for a certain element, there exists only the dependence $x \sim D(x)$, we call x an isolated element. If we have the dependence $a_1 \sim D(a_2, a_3, a_4, a_5)$, we read it as a_1 depends on a_2, a_3, a_4 and a_5 without any order preferences. Such a statement is obviously equivalent to saying $\{a_2, a_3, a_4, a_5\}$ is the set of components of the dependence that has a certain influence on the element a_1. Besides, we always consider that an element has a certain influence on itself, even though this is not clearly evident in the written form of the dependence, so we will avoid writing $a_1 \sim D(a_1, a_2, a_3, a_4, a_5)$. Thus, in this respect, we will speak of influential elements of a_1 and write $Infl(a_1) = \{a_1, a_2, a_3, a_4, a_5\}$. Notice that the influential elements appear on the right-hand side of the relation. Elements that do not appear on the right-hand side of any of the dependencies, but appear on the left-hand side of at least one dependence, will be called non-influential elements. Now, consider two dependencies: $a_1 \sim D(a_2, a_3, a_4, a_5)$ and $a_3 \sim D(a_2, a_6)$. We can see that a_2 is an influential element of both a_1 and a_3 or, in other words, that a_2 has some impact on a_1 and a_3. We can better describe this property by introducing a new set, $Imp(a_2) = \{a_1, a_3, a_2\}$ as the set of all elements on which a_2 has an impact. Thus, such elements will be called elements influenced by a_2. Notice that in the relations, such elements appear on the left-hand side. Obviously, both of these sets are non-empty because there is always $x \sim D(x)$, i.e., $x \in Infl(x) \cap Imp(x)$. With respect to the above-discussed isolated elements, it is obvious that x is isolated if and only if $|Infl(x)| = 1 = |Imp(x)|$, while there is $|Imp(y)| = 1$ and $|Infl(y)| > 1$ for non-influential elements.

3. Topology and Biology

All biological concepts in this section is taken from the reference [9].

3.1 *Topology and alleles*

Here we try to define a topology using the relationships between the alleles of a gene and their effect on each other. Alleles can interact with each other in complex ways like

(1) Complete dominance
(2) Codominance
(3) Incomplete dominance
(4) Multiple alleles.

(1) occurs when the heterozygote phenotype is indistinguishable from that of the homozygous parent. In fact, dominance is the phenomenon of one variant (allele) of a gene on a chromosome masking or overriding the effect of a different variant of the same gene on the other copy of the chromosome.

Suppose $a_1, a_2, \ldots, a_n$ are alleles of a gene such that for $i = 1, 2, \ldots, n-1$, the allele a_{i+1} masks the effect of alleles $a_1, a_2, \ldots, a_i$. Consider the set of alleles $A = \{a_1, a_2, \ldots, a_n\}$ and define the following mapping $d : A \times A \longrightarrow \mathbb{R}$ as following.

$$d(a_i, a_j) = \begin{cases} 1, & \text{if there is a dominant/recessiverelationship between } a_i \text{ and } a_j, \\ 0, & \text{otherwise.} \end{cases}$$

It can be clearly seen that d defines the discrete metric on set A. In this way we can consider the discrete topology induced by the discrete metric on this set of alleles.

An equivalent argument can be made to justify that discrete topology is a suitable topology for this set of alleles. Let us define the hyperoperation $\circ : A \times A \longrightarrow P^*(A)$ as:

$$a_i \circ a_j = \text{the allele is dominant}, \quad i, j \in \{1, 2, \ldots, n\}.$$

Here, the hyperoperation $\circ$ is actually a function of the interaction between alleles in a gene. In fact, both homozygous parents will have heterozygous offspring in the first generation, in which only the dominant allele effect will be expressed. Obviously, the hyperoperation image contains all the singletone sets in the power set A. Since we intend to define a topology based on the interaction between alleles, only (and of course the smallest) suitable topology, which includes all single-member sets, is the discrete topology.

***Example* 3.1** A classic example of dominance is the inheritance of seed shape in peas. Peas may be round, associated with allele R, or wrinkled, associated with allele r. So, for the set $\{R, r\}$, we have $\tau = \Big\{\emptyset, \{R\}, \{r\}, \{R, r\}\Big\}$. The set $\{R, r\}$ with topology τ is a T_2-space.

(2) is a relationship between two versions of a gene. Individuals receive one version of a gene, called an allele, from each parent. In codominance, however, neither allele is recessive and the phenotypes of both alleles are expressed. Thus codominance means that neither allele can mask the expression of the other allele.

We now intend to provide a suitable topology for codominant alleles of a gene. For this purpose, in the same way as we did for complete dominance, we provide a definition of topology based on how the alleles interact.

If we want to define the hyperoperation resulting from the interaction between distinct codominant alleles, then: if we are dealing with a diploid organism and a bi-allelic gene, then the offspring of homozygous parents in the first generation all show the effects of both alleles in their phenotype, even if we have an n-allelic gene with codominant alleles

$\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, by defining the n-ary hyperoperation $\circ : H^n \longrightarrow P^*(A)$, so that the image of each member of H^n is equal to the set of all alleles that occur in the phenotype, we can see that whenever we have n codominant alleles that for any $j \neq k$, $A_{i_j} \neq A_{i_k}$, its image will be equal to set $\mathcal{A}$. In fact, all alleles will show their effect in a n-ploidy organism. Therefore, since for any $n \in \mathbb{N}$, the image of each member of H^n, with distinct two-by-two components in the hyperoperation $\circ$, is equal to the set of all codominant alleles, we define the indiscrete topology on this set of alleles.

***Example* 3.2** Coat colour in short-horned cattle is an example of codominance. If a cattle with black coat, associated with allele B, is crossed with a cattle with white coat, associated with allele W, the F_1 hybrids possess neither black nor white coat colour, but have roan coat colour, where black and white patches appear separately. Therefore, the appropriate topology for this example would be the indiscrete topology $\tau = \Big\{\emptyset, \{B, W\}\Big\}$. The set $\{B, W\}$ with topology τ is not a T_0-space, and therefore not a T_1-space and T_2-space.

(3) happens when a dominant allele, or form of a gene, does not completely mask the effects of a recessive allele, and the organism's resulting physical appearance shows a blending of both alleles. It is also called semi-dominance or partial dominance.

Using the same argument we used for codominant alleles, we propose the indiscrete topology for alleles with incomplete dominance. In fact, as a result of the interaction of such alleles, the effects of none of these alleles are observed in the offspring, and in F_1 we are faced with an empty subset of the set of alleles of homozygous parents.

***Example* 3.3** A child born to a parent with straight hair, associated with allele S, and a parent with curly hair, associated with allele C, will usually have wavy hair, or hair that is a little curled, due to the expression of both curly and straight alleles. The topology intended here will be the indiscrete topology $\tau = \Big\{\emptyset, \{S, C\}\Big\}$. The set $\{S, C\}$ with topology τ is not a T_0-space, and therefore not a T_1-space and T_2-space.

Now suppose we are dealing with a set of (4) between which there is a combination of the interactions of complete dominance, codominance and incomplete dominance. For this purpose, we draw a diagram in which the dominant-recessive alleles are located along each other and other interactions across. Cross-linked alleles will have a indiscrete topology. Therefore, we consider them as an allele and define the discrete topology on the alleles that are along the diagram. In this way, we obtain a topology on the set of existing alleles.

Remark 1 *Note that in this model, the set of alleles, that are incomplete in a dominant relationship to each other or codominant, should all be under the complete dominance of the same set of alleles.*

To illustrate the model we have proposed, we provide a hypothetical example.

***Example* 3.4** Suppose $A = \{a, b, c, d, e, f\}$ is a set of alleles for a gene in which there is an incomplete dominant relationship between the two members e and f. Also, allele a is complete dominant over all alleles. On the other hand, there is a codominant relationship between the alleles b, c and d, and also the allele c is complete dominant over the alleles e and f. In this case, we will have Figure 1. In this diagram, three alleles b, c and d are in a row and two alleles e and f are in a row. Therefore, we accept each of the sets $\{b, c, d\}$ and $\{e, f\}$ as a member (or a phenotype in a diploid and triploid organism, respectively) and consider a discrete topology for the three members a, $\{b, c, d\}$, and $\{e, f\}$. In this

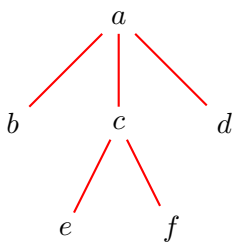

Figure 1. Multiple alleles

case, with the proposed model, we will have the following topology on this allele set:

$$\tau = \Big\{\emptyset, \{a\}, \{b, c, d\}, \{e, f\}, \{a, b, c, d\}, \{a, e, f\}, \{b, c, d, e, f\}, \{a, b, c, d, e, f\}\Big\}.$$

The set A with topology τ is not a T_0-space, and therefore not a T_1-space and T_2-space. Indeed, for two elements b and c, there is no open set that contains only one of these two elements.

Remark 2 *In fact, in the case of multiple alleles, we are looking for the smallest topology that contains the basic topologies defined for dominant alleles and incomplete dominant-codominant alleles.*

***Example* 3.5** In humans, blood type is determined by 3 alleles A, B, and O. But each human can only inherit 2 alleles.

Codominant: A and B,
Recessive: O.

If we take the discrete topology on $\{A, B\}$, we have $\tau = \Big\{\emptyset, \{A, B\}\Big\}$. Now, if we use the introduced pattern on the allele set $\{A, B, O\}$, we get $\tau' = \Big\{\{A, B, O\}, \emptyset, \{A, B\}, \{O\}\Big\}$ and this is clearly a topology on the set $\{A, B, O\}$. The set $\{A, B, O\}$ with topology τ' is not a T_0-space, and therefore not a T_1-space and T_2-space. Indeed, for two elements A and B, there is no open set that contains only one of these two elements.

***Example* 3.6** Let us consider a gene that specifies coat color in rabbits, called the C gene. The C gene comes in four common alleles: C, c^{ch}, c^h, and c:

A CC rabbit has black or brown fur
A $c^{ch}c^{ch}$ rabbit has chinchilla coloration (grayish fur).
A c^hc^h rabbit has Himalayan (color-point) patterning, with a white body and dark ears, face, feet, and tail
A cc rabbit is albino, with a pure white coat.

Multiple alleles makes for many possible dominance relationships. In this case, the black C allele is completely dominant to all the others; the chinchilla c^{ch} allele is completely dominant to the Himalayan c^h and albino c alleles; and the Himalayan c^h allele is completely dominant to the albino c allele. So we have Dominance $= C > c^{ch} > c^h > c$.

As a result, we consider the following discrete topology for this biological example:

$$\tau = \Big\{\emptyset, \{C\}, \{c^{ch}\}, \{c^h\}, \{c\}, \{c^{ch}, c\}, \{c^{ch}, c^h\}, \{c^h, c\}, \{C, c^{ch}\}, \{C, c^h\}, \{C, c\},$$
$$\{C, c^{ch}, c^h\}, \{C, c^{ch}, c\}, \{c^h, c^{ch}, c\}, \{C, c^h, c\}, \{C, c^{ch}, c^h, c\}\Big\}.$$

The set $\{C, c^{ch}, c^h, c\}$ with topology τ is a T_2-space.

3.2 *Topology and phenotypes*

In this section, we try to get a suitable idea for presenting topology on phenotypes with the help of topologies provided for alleles.

Complete Dominance

We first consider a gene with two alleles a_1 and a_2 in which allele a_1 dominates over allele a_2.

$$\begin{aligned} P &: a_1a_1 \times a_2a_2, \\ F_1 &: a_1a_2, \\ F_1 \times F_1 &: a_1a_2 \times a_1a_2, \\ F_2 &: \widehat{A} = \{a_1a_1, a_1a_2\}, \widehat{a} = \{a_2a_2\}. \end{aligned}$$

Here the number of phenotypes is the same as the number of alleles, and in fact the interaction between the phenotypes will result in all single-member sets of them. Therefore, the smallest topology produced by these sets will be the discrete topology. In fact, for a n-allelic gene in which for $i = 1, 2, \ldots, n-1$, the allele a_{i+1} masks the effect of alleles a_1, a_2, ... and a_i, the result of interaction between phenotypes is similar to the interaction between alleles. Therefore, for these phenotypes, we will consider the discrete topology.

Incomplete Dominance and Codominance

In the previous section, we saw that when faced with a set of incomplete dominant or codominant alleles, the indiscrete topology will be the smallest topology produced by the results of the interaction between the alleles. Note that in this case the number of phenotypes will be bigger than the number of alleles. Therefore, to determine the topology on the resulting phenotypes, we are looking for the smallest topology that will be produced by adding new resulting phenotypes to the indiscrete topology.

***Example* 3.7** Suppose A_1 and A_2 are two alleles of the same gene. Then we have

$$\begin{aligned} P &: A_1A_1 \times A_2A_2 \\ F_1 &: a_1a_2 \\ F_1 \times F_1 &: A_1A_2 \times A_1A_2 \\ F_2 &: A = \{A_1A_1\}, B = \{A_1A_2\}, C = \{A_2A_2\}. \end{aligned}$$

If we denote the topology for the set $\{A, B, C\}$ by τ, then we have

$$\tau = \Big\langle \emptyset, \{A, C\}, \{B\} \Big\rangle = \Big\{ \emptyset, \{A, C\}, \{B\}, \{A, B, C\} \Big\},$$

where $\Big\langle \emptyset, \{A, C\}, \{B\} \Big\rangle$ is the topology generated by $\emptyset, \{A, C\}, \{B\}$. The set $\{A, B, C\}$ with topology τ is not a T_0-space, and therefore not a T_1-space and T_2-space. Indeed, for two elements A and C, there is no open set that contains only one of these two elements.

Multiple alleles

We now consider the following biological example. In this example, we are dealing with a three-allelic gene that has four phenotypes and try to construct a topology related to its phenotypes.

***Example* 3.8** The ABO blood type is controlled by a single gene (the ABO gene) with three types of alleles inferred from classical genetics: i, I^A, and I^B. The I^A allele gives type A, I^B gives type B, and i gives type O. As both I^A and I^B are dominant over i, only ii people have type O blood. Individuals with I^AI^A or I^Ai have type A blood, and individuals with I^BI^B or I^Bi have type B. I^AI^B people have both phenotypes, because A and B express codominance relationship. In Subsection 3.1 we saw that the topology defined for the set of alleles $\{A, B, O\}$ is equal to $\tau = \Big\{ \emptyset, \{O\}, \{A, B\}, \{A, B, O\} \Big\}$. Now, if we denote the topology of the set of phenotypes $\{A, B, O, AB\}$ with the symbol τ', we have $\tau' = \Big\langle \tau \cup \{AB\} \Big\rangle$. So it can be clearly seen that

$$\tau' = \Big\{ \emptyset, \{O\}, \{A, B\}, \{A, B, O\}, \{AB\}, \{AB, O\}, \{A, B, AB\}, \{A, B, O, AB\} \Big\}.$$

The set $\{A, B, O, AB\}$ with topology τ' is not a T_0-space, and therefore not a T_1-space and T_2-space. Indeed, for two elements A and B, there is no open set that contains only one of these two elements.

4. Topological hyperstructures

In this section, we study some biological examples in the form of a topological hyperstructures.

Remark 3 *Since we have equipped a set of phenotypes of a n-allelic gene in which for $i = 1, 2, \ldots, n-1$, the allele a_{i+1} masks the effect of alleles a_1, a_2, ... and a_i, with the discrete topology; therefore, by defining any desired hyperoperation on this set of phenotypes, we will have a topological hyperstructure.*

***Example* 4.1** Incomplete dominance is seen when the two alleles mix together to create an entirely different phenotype. The flowers below are an example of incomplete dominance because the red, denoted by R, and white, denoted by W, alleles mix together in certain individuals to create a pink phenotype. The pink phenotype is a mixture of both alleles being expressed at the same time in every cell. Considering the phenotype set $H = \{R, W, P\}$, and defining hyperoperation $\circ$ as the mating of the second generation,

we will have the following table

$\circ$	R	W	P
R	R	P	R, P
W		W	W, P
P			R, P, W

Now, considering the topology $\tau = \Big\{\emptyset, H, \{R, W\}, \{P\}\Big\}$ on H, the triple $(H, \circ, \tau)$ is a strongly pseudotopological H_v-semigroup. For instance, $\Big(R \circ R = R\Big) \cap \{R, W\} \neq \emptyset$, and there exist $\{R, W\}, \{P\} \in \tau$ such that for every $(u, v) \in \{R, W\} \times \{P\}$ we have $u \circ v \cap \{R, W\} \neq \emptyset$. Indeed,

$$R \circ P = \{R, P\} \cap \{R, W\} \neq \emptyset \text{ and } W \circ P = \{W, P\} \cap \{R, W\} \neq \emptyset.$$

Other items can simply be shown similarly. Now, due to Theorem 2.7, the quadruple $(H, \circ, \tau, \tau_L)$ is a τ_L-topological hypergroupoid in which according to Lemma 2.6, the following family forms a subbasis for τ_L on $P^*(H)$:

$$\mathcal{F} = \Big\{I_\emptyset, I_{\{R,W\}}, I_{\{P\}}, I_{\{R,W,P\}}\Big\},$$

where

$$\begin{aligned} I_\emptyset &= \emptyset, \\ I_{\{R,W\}} &= \Big\{\{R\}, \{W\}, \{R, W\}, \{R, P\}, \{W, P\}, \{R, W, P\}\Big\}, \\ I_{\{P\}} &= \Big\{\{P\}, \{R, P\}, \{W, P\}, \{R, W, P\}\Big\}, \\ I_{\{R,W,P\}} &= P^*(H). \end{aligned}$$

If we add the member $\Big\{\{R, P\}, \{W, P\}, \{R, W, P\}\Big\}$ to $\mathcal{F}$, then we get a basis for topology τ_L on $P^*(H)$. Also, $(P^*(H), \tau_L)$ is not a T_0-topological space (hence, is not T_1 and T_2) because for the members $\{R\}$ and $\{W\}$, there are no open sets such as U and V such that $\{R\} \in U$ and $\{W\} \notin U$, and $\{W\} \in V$ and $\{R\} \notin V$. $(P^*(H), \tau_L)$ is not even a T_0-topological space. Indeed for two members $\{R\}$ and $\{W\}$, there is no open set that contains only one of them.

Remark 4 *Not every strongly pseudotopological hypergroupoid is a pseudotopological hypergroupoid. In Example 4.1, $(H, \circ, \tau)$ is not pseudotopological hypergroupoid. Consider the open set $\{P\}$ in τ and the members R and W of H. Then. we have $R \circ W = \{P\} \in \tau$, while there are no two open sets U and V such that $u \circ v \subseteq \{P\}$ for all $(u, v) \in U \times V$.*

***Example* 4.2** The ABO blood type. Considering $H = \{O, A, B, AB\}$ with the following table, the couple $(H, \otimes)$ is an H_v-semigroup.

$\otimes$	O	A	B	AB
O	O	O, A	O, B	A, B
A		O, A	A, B, O, AB	A, B, AB
B			O, B	A, B, AB
AB				A, B, AB

Considering the topology defined on $H = \{O, A, B, AB\}$ in Example 3.8, i.e.

$$\tau' = \Big\{\emptyset, \{O\}, \{A, B\}, \{A, B, O\}, \{AB\}, \{AB, O\}, \{A, B, AB\}, \{A, B, O, AB\}\Big\},$$

the triple $(H, \otimes, \tau')$ is a pseudotopological H_v-semigroup. For instance, $O \otimes AB = \{A, B\}$, and $\{A, B\}$ is a subset of sets $\{A, B\}$, $\{A, B, O\}$, $\{A, B, AB\}$ and H. Now, we have

$$\begin{aligned}
\{O\} \otimes \{AB\} &= \{A, B\} \subseteq \{A, B\},\\
\{O\} \otimes \{AB\} &= \{A, B\} \subseteq \{A, B, O\},\\
\{O\} \otimes \{AB\} &= \{A, B\} \subseteq \{A, B, AB\},\\
\{O\} \otimes \{AB\} &= \{A, B\} \subseteq H.
\end{aligned}$$

As another instance, $B \otimes O = \{B, O\}$ and $\{B, O\} \subseteq \{A, B, O\}, H$. Now, we have

$$\begin{aligned}
\{O\} \otimes \{AB\} &= \{A, B\} \subseteq \{A, B, O\},\\
\{O\} \otimes \{AB\} &= \{A, B\} \subseteq H.
\end{aligned}$$

According to the table and the proof of the above two cases, the proof of the other cases is obvious. In the following, due to Theorem 2.7, the quadruple $(H, \otimes, \tau', \tau'_U)$ is a τ'_U-topological hypergroupoid, in which, according to Lemma 2.6, the following family form a basis for τ'_U on $P^*(H)$.

$$\begin{aligned}
\mathcal{F}' = \Big\{&\emptyset, \{O\}, P^*(\{A, B\}), P^*(\{A, B, O\}), \{AB\},\\
&P^*(\{AB, O\}), P^*(\{A, B, AB\}), P^*(\{A, B, O, AB\})\Big\}.
\end{aligned}$$

$(P^*(H), \tau'_U)$ is not a T_0-topological space. Indeed for two members $\{A\}$ and $\{B\}$, there is no open set that contains only one of them. Therefore, $(P^*(H), \tau'_U)$ is not T_1-topological space and T_2-topological space either.

Remark 5 *In Example 4.2, the triple $(H, \otimes, \tau')$ is a strongly pseudotopological H_v-semigroup. To check this, it is sufficient to consider the definition only for the open sets $\{O\}$, $\{A, B\}$ and $\{AB\}$; as other open sets include members of one of these three sets. Each open set including O can be written $U = V = \{O\}$. Also, each open set including A, B can be written $U = \{O\}$ and $V = \{AB\}$. In addition to, each open set including AB can be written $U = V = \{AB\}$.*

5. Hypergroupoids associated with abstract dependencies

Suppose $a_1, a_2, \ldots, a_n$ are alleles of a gene such that for $i = 1, 2, \ldots, n-1$, the allele a_{i+1} masks the effect of alleles $a_1, a_2, \ldots, a_i$. Consider the set of alleles $A = \{a_1, a_2, \ldots, a_n\}$ with the cardinality n. Then there exist dependences $a_i \sim D(a_{i+1}, \ldots, a_n)$ for $i = 1, 2, \ldots, n$, which means that $a_1 \sim D(a_2 \sim D(\cdots \sim D(a_n)))$. Thus, for $i = 1, 2, \ldots, n$, we have

$$Infl(a_i) = \{a_i, a_{i+1}, \ldots, a_n\} \text{ and } Imp(a_i) = \{a_i, a_{i-1}, \ldots, a_1\}.$$

As a result, for any pair (i, j) where $i \leqslant j$, we have

$$\begin{aligned} a_i \circ_1 a_j &= Imp(a_i) \cap Imp(a_j) = \{a_i, a_{i-1}, \ldots, a_1\}, \\ a_i \circ_2 a_j &= Infl(a_i) \cap Infl(a_j) = \{a_j, a_{j+1}, \ldots, a_n\}, \\ a_i \circ_3 a_j &= Imp(a_i) \cup Imp(a_j) = \{a_j, a_{j-1}, \ldots, a_1\}, \\ a_i \circ_4 a_j &= Infl(a_i) \cup Infl(a_j) = \{a_i, a_{i+1}, \ldots, a_n\}. \end{aligned}$$

Obviously, for $k = 1, 2, 3, 4$, the hyperstructures $(A, \circ_k)$ are all commutative, extensive hypergroups.

Remark 6 *If we regard $(a_n)^n = a_n \circ_1 \cdots \circ_1 a_n$, then we get $(a_n)^n = (a_n)^2 = A$ for $n > 2$. To be more precise, the hypergroup $(A, \circ_1)$ is single-power cyclic with infinite period. Now, if we regard $(a_1)^n = a_1 \circ_2 \cdots \circ_2 a_1$, then we get $(a_1)^n = (a_1)^2 = A$ for $n > 2$. To be more precise, the hypergroup $(A, \circ_2)$ is single-power cyclic with infinite period. Also, we have $(a_n)^n = a_n \circ_3 \cdots \circ_3 a_n = (a_n)^2 = A$, and $(a_1)^n = a_1 \circ_4 \cdots \circ_4 a_1 = (a_1)^2 = A$ for $n > 2$. Therefore, the hypergroups $(A, \circ_3)$ and $(A, \circ_4)$ are single-power cyclic with infinite period, too.*

If we have an n-allelic gene with codominant alleles $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, then all of them show their effects, and none of the alleles affects the effect of the other allele. Therefore, for $i = 1, 2, \ldots, n$, $Imp(A_i) = Infl(A_i) = \{A_i\}$. As a result, each of the alleles is isolated. So, for any pair (i, j), we have

$$\begin{aligned} A_i \circ_3 A_j &= Imp(A_i) \cup Imp(A_j) = \{A_i, A_j\}, \\ A_i \circ_4 A_j &= Infl(A_i) \cup Infl(A_j) = \{A_i, A_j\}. \end{aligned}$$

Then the hyperstructures $(\mathcal{A}, \circ_3)$ and $(\mathcal{A}, \circ_4)$ are commutative, extensive hypergroups. If we have the n-allelic set $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ that interact with each other incompletely dominant, then in fact both alleles are affected by each other, in other words, $B_i \sim D(B_1, B_2, \ldots, B_n)$ for $i = 1, 2, \ldots, n$. Then, for $i = 1, 2, \ldots, n$, we have $Imp(B_i) = Infl(B_i) = \{B_1, B_2, \ldots, B_n\}$, which means that each element of H depends on all elements in the set $\mathcal{B}$, which is the extreme case opposite to the case when all elements are isolated. So, for any pair (i, j), the following relationships are established:

$$\begin{aligned} B_i \circ_1 B_j &= Imp(B_i) \cap Imp(B_j) = \mathcal{B}, \\ B_i \circ_2 B_j &= Infl(B_i) \cap Infl(B_j) = \mathcal{B}, \\ B_i \circ_3 B_j &= Imp(B_i) \cup Imp(B_j) = \mathcal{B}, \\ B_i \circ_4 B_j &= Infl(B_i) \cup Infl(B_j) = \mathcal{B}. \end{aligned}$$

In fact, for $k = 1, 2, 3, 4$, the hypergroups $(\mathcal{B}, \circ_k)$ are total hypergroups. It is also clear that the hypergroups $(\mathcal{B}, \circ_k)$ are single-power cyclic with period two. See a biological example below.

***Example* 5.1 ABO blood group system.** Consider the set of alleles $H = \{A, B, O\}$. Then there exists dependence $O \sim D(A, B)$. Therefore, $Imp(O) = \{O\}$ and $Infl(O) = \{O, A, B\}$. So O is a non-influential element. Also $Imp(A) = \{O, A\}$, $Infl(A) = \{A\}$,

and $Imp(B) = \{O, B\}$, $Infl(B) = \{B\}$. See the tables below.

$\circ_1$	O	A	B
O	O	O	O
A	O	O, A	O
B	O	O	O, B

$\circ_2$	O	A	B
O	O, A, B	A	B
A	A	A	$\{\}$
B	B	$\{\}$	B

and

$\circ_3$	O	A	B
O	O	O, A	O, B
A	O, A	O, A	O, A, B
B	O, B	O, A, B	O, B

$\circ_4$	O	A	B
O	O, A, B	O, A, B	O, A, B
A	O, A, B	A	A, B
B	O, A, B	A, B	B

The couple $(H, \circ_1)$ is a non-cyclic commutative semihypergroup. The couple $(H, \circ_2)$ is a partial hypergroupoid and single-power cyclic with a generator O. The couple $(H, \circ_3)$ is a non-cyclic hypergroup, and the couple $(H, \circ_4)$ is a hypergroup and single-power cyclic with a generator O.

***Example* 5.2** Consider the set of alleles in Example 3.4 and the interaction between them. Then there exist the dependencies $b \sim D(a)$, $d \sim D(a)$, $e \sim D(c \sim D(a))$ and $f \sim D(c \sim D(a))$. Therefore, $Imp(a) = \{a, b, c, d, e, f\} = A$ and $Infl(a) = \{a\}$. So the hypergropoid $(A, \circ_3)$ is single-power cyclic with a generator a.

6. Conclusion

If we are dealing with n genes, and each of these genes has alleles with different interactions, it is enough to use a product topology $\tau = \tau_1 \times \tau_2 \times \cdots \times \tau_n$ in which for each $i = 1, 2, \ldots, n$, the corresponding topology τ_i is defined according to what has been stated so far. In this case, it is obvious that the product topology defined on the product of n hyperstructures $(H, \otimes) = (H_1, \otimes_1) \times (H_2, \otimes_2) \times \cdots \times (H_n, \otimes_n)$ will turn it into a strongly pseudo topological hyperstructure. Also, according to what was discussed in Section 5, we saw that if we are dealing with the set $A = \{a_1, a_2, \ldots, a_n\}$ of n alleles in which for $i = 1, 2, \ldots, n-1$, the allele a_{i+1} masks the effect of alleles a_1, a_2, ... and a_i, then for $k = 1, 2, 3, 4$ the commutative hypergroups $(A, \circ_k)$ are all single-power cyclic with finite period. But if we have an n-allelic gene with codominant alleles $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, then each of the alleles is isolated, and if the alleles of the set $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ interact with each other incompletely dominant, then each element of H depends on all elements in the set $\mathcal{B}$.

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