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Maximum nullity, zero forcing number and propagation time of ℓ-path graphs

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Abstract. Let G be a graph with each vertex is colored either white or black. A white vertex is changed to a black vertex when it is the only white neighbor of a black vertex (color-change rule). A zero forcing set S of a graph G is a subset of vertices G with black vertices, all other vertices G are white, such that after finitely many applications of the color-change rule all of vertices G becomes black. The zero forcing number of G is the minimum cardinality of a zero forcing set in G, denoted by $Z(G)$. In this paper, we define $\ell-$Path graphs. We give some $\ell-$Path and $\ell-$Ciclo graphs such that their maximum nullity are equal to their zero forcing number. Also, we obtain minimum propagation time and maximum propagation time for them.

Keywords: Propagation time, zero forcing number, maximum nullity, minimum rank.

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1. Introduction

In this paper, all graphs are assumed to be finite, simple and undirected. We will often use the notation $G = (V, E)$ to denote the graph with non-empty vertex set $V = V(G)$ and edge set $E = E(G)$. Order of a graph is the number of vertices in the graph and size of a graph is the number of edges in the graph. An edge of G with endpoints u and v is denoted by uv. For every vertex $x \in V(G)$, the open neighborhood of vertex x is denoted by $N_G(x)$ and defined as $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$. Also, the

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closed neighborhood of vertex $x \in V(G)$, $N_G[x]$, is $N_G[x] = N_G(x) \cup \{x\}$. The degree of a vertex $x \in V(G)$ is $\deg_G(x) = \left|N_G(x)\right|$. The minimum degree and maximum degree of a graph G denoted by $\delta(G)$ and $\Delta(G)$, respectively. We denote the complete graph on n vertices by K_n, the cycle on n vertices by C_n and the path on n vertices byP_n. The union of $G_i = (Vi, Ei)$, for $i = 1, \ldots, h$, is $\bigcup_{i=1}^{h} G_i = (\bigcup_{i=1}^{h} V_i, \bigcup_{i=1}^{h} E_i)$. The set of $n \times n$ real symmetric matrices will be denoted by $S_n(\mathbb{R})$. For $A \in S_n(\mathbb{R})$, the graph of $A = (a_{ij})$, denoted by $\mathcal{G}(A)$, is a graph with vertices $\{1, \ldots, n\}$ and edges $\{ij| \ a_{ij} \neq 0, 1 \leqslant i, j \leqslant n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$. The set of symmetric matrices of graph G is defined by $S(G) = \{A \in S_n(\mathbb{R}) \mid \mathcal{G}(A) = G\}$. The maximum nullity of G is $M(G) = \max\{null(A) \mid A \in S(G)\}$ and the minimum rank of G is $mr(G) = \min\{rank(A) \mid A \in S(G)\}$. It is well-known that if G is a graph of order n, then $mr(G) + M(G) = n$.

Let each vertex of a graph G be given one of two colors "black" and "white". If a white vertex b is the only white neighbor of a black vertex a, then a changes the color of b to black, called color-change rule. Furthermore, we say a forces b or b is forced by a. Let B be the initial black vertices. Then B is said a zero forcing set of G if all of the vertices of G will be turned black after finitely many applications of the color-change rule. The zero forcing number of G, called $Z(G)$, is the minimum cardinality among all zero forcing sets. The notation of a zero forcing set, as well as the associated zero forcing number, of a simple graph was introduced by the "AIM Minimum Rank-Special Graphs Work Group" in 2008 [1]. They used the technique of zero forcing parameter of graph G and found an upper bound for the maximum nullity of G related to zero forcing sets. It is shown that for any graph G, $M(G) \leqslant Z(G)$. Also, the following question has been raised in [1].
What is the class of graphs G for which $M(G) = Z(G)$? As a simple example, the complete graph K_n on n vertices has $Z(K_n) = M(K_n) = n - 1$.

In this paper, we give some families graphs which their maximum nullity and zero forcing number are equal. For more results, see [2, 5–7, 10–14, 16, 19].

Let $G = (V, E)$ be a graph and B a zero forcing set of G. Define $B^{(0)} = B$ and for $\ell \geqslant 0$, $B^{(\ell+1)}$ is the set of vertices w for which there exists a vertex $b \in \bigcup_{i=0}^{\ell} B^{(i)}$ such that w is the only neighbor of b not in $\bigcup_{i=0}^{\ell} B^{(i)}$. The propagation time of B in G, denoted by $Pt(G, B)$, is the smallest integer ℓ_0 such that $V = \bigcup_{i=0}^{\ell_0} B^{(i)}$. The minimum propagation time of G is $Pt(G) = \min\{Pt(G, B) \mid B \ is \ a \ minimum \ zero \ forcing \ set \ of \ G\}$. In other word, the propagation time is the number of steps it take for an initial zero forcing set to force all vertices of a graph to black. The maximum propagation time of G is $PT(G) = \max\{Pt(G, B) \mid B \ is \ a \ minimum \ zero \ forcing \ set \ of \ G\}$. Also, the propagation time discrepancy of G is defined as $pd(G) = PT(G) - Pt(G)$. For more results, see [8, 9, 15, 17, 18].

In this paper, we give some of $\ell-$Path and $\ell-$Ciclo graphs such that their maximum nullity are equal to their zero forcing number. Also, we obtain minimum and maximum propagation time for them. Specially, we introduce families of graphs with minimum propagation time 1.

2. Preliminary

Ciclo graphs have defined by Almodovar et al. [2]. We define $\ell-$Ciclo graphs and $\ell-$path graphs in the following.

Definition 2.1 [2] Let H be a graph and e be an edge of H. A $\ell-$ciclo of H with e, denoted by $C_\ell(H,e)$, is constructed from a $\ell-$cycle C_ℓ and ℓ copies of H by identifying each edge of C_ℓ with the edge e in one copy of H.

Definition 2.2 Let H be a graph and P_ℓ be a ℓ-path with $V(P_\ell) = \{v_1, v_2, \ldots, v_\ell\}$. Also, let e be an edge of H with end vertices a and b. A ℓ-path graph of H with e, denoted by $P_\ell(H,e)$, is constructed from P_ℓ and ℓ copy of H by merging the two vertices v_i and a also, v_{i+1} and b for every $1 \leq i \leq \ell - 1$ (see Figure 1).

Theorem 2.3 [13] Let G be a connected graph of order $n \geqslant 2$. Then $Z(G) = n - 1$ if and only if G is isomorphic to a complete graph of order n.

Theorem 2.4 [1] Let $G = (V,E)$ be a graph and $Z \subseteq V$ be a zero forcing set for G. Then $M(G) \leqslant Z(G)$.

Theorem 2.5 [1] If $G = \bigcup_{i=1}^{h} G_i$, then $mr(G) \leq \sum_{i=1}^{h} mr(G_i)$.

Theorem 2.6 [15] Let G be a graph of order n. Then $Pt(G) \geq \frac{n-Z(G)}{Z(G)}$.

Theorem 2.7 [15] If G is a graph of order n, then $PT(G) \leqslant n - Z(G)$.

Theorem 2.8 [15] For a graph G of order n, the following are equivalent:

i) $pt(G) = n - 1$.
ii) $pT(G) = n - 1$.
iii) $Z(G) = 1$.
iv) G is a path.

It is well-known that $Pt(K_n) = 1 = PT(K_n)$ and

$$Pt(C_n) = PT(C_n) = \begin{cases} \frac{n-2}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

3. Main results

In this section, we show the maximum nullity of some families graphs are equal to their zero forcing number. Also, we obtain propagation time discrepancy for some of them.

Theorem 3.1 Let G be a graph of order n, $e = ab \in E(G)$ and $M(G) = Z(G)$. Also, let B be a zero forcing set for G with minimum cardinality such that $\{a,b\} \subseteq B$ and every white vertex in $N_G(b)$ can be forced by a vertex except b. Then $M(P_\ell(G,e)) = Z(P_\ell(G,e)) = (\ell-1)(|B|-1)+1$ and $mr(P_\ell(G,e)) = (\ell-1)(n-|B|)$.

Proof. Let $V(P_\ell) = \{v_1, v_2, \ldots, v_\ell\}$, $V(G_i \setminus \{a,b\}) = \{v_{i1}, v_{i2}, \ldots, v_{i(n-2)}\}$, where G_i is the $i-th$ copy of G in $P_\ell(G,e)$ $(1 \leq i \leq \ell-1)$. Since $\{a,b\} \subseteq B$ and a is adjacent to b, so $|N_G(a)| \geqslant 2$ and $|N_G(b)| \geqslant 2$. Let $B_i = B \cap (G_i \setminus \{a,b\})$ and $Z = V(P_\ell) \bigcup_{i=1}^{\ell-1} B_i$ be the initial black vertices. Since every white vertex in $N_G(b)$ can be forced by a black vertex except b, so Z is a zero forcing set for $P_\ell(G,e)$. Hence, $Z(P_\ell(G,e)) \leq (\ell-1)(|B|-1)+1$. It is clear that $P_\ell(G,e) = \bigcup_{i=1}^{\ell-1} G_i$. By Theorem 2.5, $mr(P_\ell(G,e)) \leq (\ell-1)mr(G)$. Since $M(G) = |B|$, so $mr(P_\ell(G,e)) \leq (\ell-1)(n-M(G)) = (\ell-1)(n-|B|)$ and

$$M(P_\ell(G,e)) \geq ((\ell-1)(n-2)+\ell) - (\ell-1)(n-|B|) = (\ell-1)(|B|-1)+1.$$

By Theorem 2.4, $(\ell-1)(|B|-1)+1 \geqslant Z(P_\ell(G,e)) \geq M(P_\ell(G,e)) \geq (\ell-1)(|B|-1)+1$. Therefore, $M(P_\ell(G,e)) = Z(P_\ell(G,e)) = (\ell-1)(|B|-1)+1$ and so $mr(P_\ell(G,e)) = (\ell-1)(n-|B|)$. ■

Corollary 3.2 Let $r \geqslant 4$ and $e = ab \in E(K_r)$. Then $Z(P_\ell(K_r,e)) = M(P_\ell(K_r,e)) = (\ell-1)(r-2)+1$ and $Pt(P_\ell(K_r,e)) = 1$ (see Figure 1).

Proof. It is well-known that $M(K_r) = Z(K_r) = r-1$. Let Z be a zero forcing set of K_r with $|Z| = r-1$ and $\{a,b\} \subseteq Z$. By Theorem 3.1, $M(P_\ell(K_r,e)) = Z(P_\ell(K_r,e)) = (\ell-1)(r-2)+1$. Let $B = V(P_\ell(K_r,e)) \setminus \{v_{i(r-2)} \mid 1 \leq i \leq \ell-1\}$ be the initial black vertices. Then for every $1 \leq i \leq \ell-1$, $v_{i(r-2)}$ is the only white neighbor of v_{i2}, so $v_{i(r-2)}$ is forced by v_{i2}. Thus, B is a zero forcing set of $P_\ell(K_r,e)$. Furthermore, we have $B^{(0)} = B$, $B^{(1)} = \{v_{i(r-2)} \mid 1 \leq i \leq \ell-1\}$ and $V(P_\ell(K_r,e)) = B^{(0)} \cup B^{(1)}$. Hence, $Pt(P_\ell(K_r,e),B) = 1$. Therefore, $Pt(P_\ell(K_r,e)) = 1$.

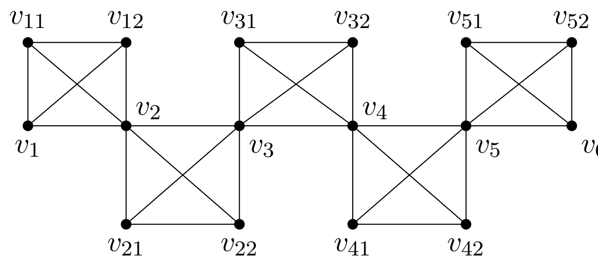

Figure 1. $P_6(K_4,e)$

■

Corollary 3.3 Let $r \geqslant 4$ and $e = ab \in E(K_r)$. Then $Pd(P_\ell(K_r,e)) = \ell-2$.

Proof. By Theorems 2.7 and 3.1, $PT(P_\ell(K_r,e)) \leqslant \ell-1$. Let $V(C_\ell) = \{v_1,\ldots,v_\ell\}$ and $B = V(P_\ell(K_r,e)) \setminus \{v_i \mid 2 \leq i \leq \ell\}$ be the initial black vertices. Then $B^{(1)} = \{v_2\}$, $B^{(2)} = \{v_3\}$, …, $B^{(\ell-1)} = \{v_\ell\}$. So $V(P_\ell(K_r,e)) = B^{(0)} \cup B^{(1)} \cup \ldots \cup B^{(\ell-1)}$. Hence, $Pt(P_\ell(K_r,e),B) = \ell-1$ and so $PT(P_\ell(K_r,e)) \geqslant \ell-1$. Therefore, $PT(P_\ell(K_r,e)) = \ell-1$. By Corollary 3.2, $Pd(P_\ell(K_r,e)) = \ell-2$. ■

Corollary 3.4 Let $r \geqslant 4$, $e = ab \in E(K_r)$ and $G_1 = P_\ell(K_r,e)$. Then $Pt(P_\ell(G_1,e_1)) = 1$, where $e_1 = v_1v_{11}$.

Proof. By Theorem 3.1, the proof is straightforward. ■

Theorem 3.5 Let ℓ and r be even and greater than 4 such that $r \leq \ell+2$. Then $Z(P_\ell(C_r,e)) = M(P_\ell(C_r,e)) = \ell$ and $Pt(P_\ell(C_r,e)) = r-2$, where $e = ab \in E(C_r)$ (see Figure 2).

Proof. It is clear that $\{a,b\}$ is a zero forcing set of C_r. By Theorem 3.1, $M(P_\ell(C_r,e)) = Z(P_\ell(C_r,e)) = \ell$. Let $V(P_\ell) = \{v_1,v_2,\ldots,v_\ell\}$, $V(H_i) \setminus \{a,b\} = \{v_{i1},v_{i2},\ldots,v_{i(r-2)}\}$, where H_i is the $i-th$ copy of C_r and $1 \leq i \leq \ell-1$. Also, let $j = \frac{r}{2}$ and $B = \{v_{(2k)(j-1)}, v_{(2k)j} \mid 1 \leq k \leq \frac{\ell-2}{2}\} \cup \{v_1, v_\ell\}$ be initial black vertices. Then $B^{(1)} = \{v_{(2k)(j-2)}, v_{(2k)(j+1)} \mid 1 \leq k \leq \frac{\ell-2}{2}\}$, …, $B^{(j-1)} = \{v_{2k}, v_{2k+1} \mid 1 \leq k \leq \frac{\ell-2}{2}\}$, $B^{(j)} = \{v_{(2k+1)1}, v_{(2k+1)(r-2)} \mid 0 \leq k \leq \frac{\ell-2}{2}\}$, …, $B^{(2j-2)} = \{v_{(2k+1)(j-1)}, v_{(2k+1)j} \mid 0 \leq k \leq \frac{\ell-2}{2}\}$. Since $2j-2 = r-2$, so $V(P_\ell(C_r,e)) = \bigcup_{i=0}^{r-2} B^{(i)}$. Thus, B is a zero forcing set of $P_\ell(C_r,e)$ with $|B| = \ell$ and $Pt(P_\ell(C_r,e),B) = r-2$. Hence, $Pt(P_\ell(C_r,e)) \leqslant r-2$.

By Theorem 2.6, $Pt(P_\ell(C_r,e)) \geq \frac{(r-2)(\ell-1)}{\ell}$. Since propagation time of a graph is integer and $r \leq \ell+2$, $Pt(P_\ell(C_r,e)) \geq r-2$, $Pt(P_\ell(C_r,e)) = r-2$.

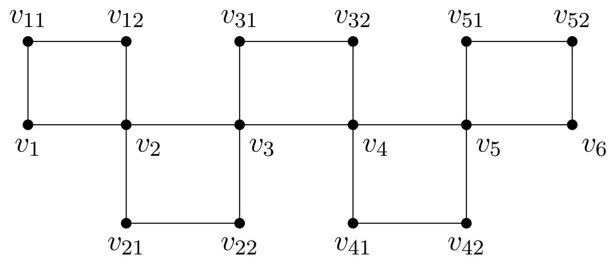

Figure 2. $P_6(C_4,e)$

■

Following Ashrafi et al. [3], a link of graphs G_1 and G_2 by vertices $a \in V(G_1)$ and $b \in V(G_2)$ is defined as the graph $(G_1 \sim G_2)(a,b)$ obtained by joining a and b by an edge in the union of these graphs. For example, see Figure 3, where $V(K_1) = \{a\}$ and $b \in V(C_4)$.

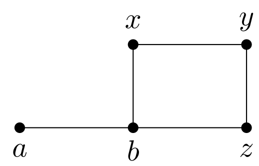

Figure 3. $(K_1 \sim C_4)(a,b)$

Theorem 3.6 Let $V(K_1) = \{a\}$, $b \in V(C_r)$, $H = (K_1 \sim C_r)(a,b)$ and $e = ab$ (see Figure 4). Then $Z(P_\ell(H,e)) = M(P_\ell(H,e)) = \ell$. Also,

$$Pd(P_\ell(H,e)) = \begin{cases} \frac{r-2}{2} & \text{if } r \text{ is even} \\ \frac{r-3}{2} & \text{if } r \text{ is odd.} \end{cases}$$

Proof. Let $V(P_\ell) = \{v_1, v_2, \ldots, v_\ell\}$, $V(H_i) \setminus \{a,b\} = \{v_{i1}, v_{i2}, \ldots, v_{i(r-1)}\}$, where H_i is the $i-th$ copy of H and $1 \leq i \leq \ell-1$. Let $Z = \{v_1\} \cup \{v_{i1} \mid 1 \leq i \leq \ell-1\}$ be the initial black vertices. Then v_1 forces v_2, v_{12} is forced by v_{11}. So v_{12} forces v_{13}. Similarly, all vertices of the H_1 are forced. Now, v_2 forces v_3. With the above method, all vertices H_i, $(1 \leq i \leq \ell-1)$ are forced. So, Z is a zero forcing set for $P_\ell(H,e)$. Hence, $Z(P_\ell(H,e)) \leq \ell$. It is easy to see that $P_\ell(H,e) = \bigcup_{i=1}^{\ell-1} C_r \cup P_\ell$. By Theorem 2.5, $mr(P_\ell(H,e)) \leq (\ell-1)(r-2) + (\ell-1)$. Thus, $M(P_\ell(H,e)) \geq (r(\ell-1)+1)) - ((\ell-1)(r-2)+(\ell-1)) = \ell$. By Theorem 2.4, $\ell \geqslant Z(P_\ell(H,e)) \geq M(P_\ell(H,e)) \geq \ell$. Therefore, $Z(P_\ell(H,e)) = M(P_\ell(H,e)) = \ell$. Now, let B be a zero forcing set for $P_\ell(H,e)$ with $|B| = \ell$. Then $|B \cap (V(H_i) \setminus \{a,b\})| \geq 1$ for every $1 \leq i \leq \ell-1$. Since $|B| = \ell$,

$$B = \{v_{i1} \mid 1 \leq i \leq \ell-1\} \cup \{v_1\} = B_1 \text{ or } B = \{v_{i1} \mid 1 \leq i \leq \ell-1\} \cup \{v_\ell\} = B_2.$$

For zero forcing set B_1, since $Pt(P_\ell) = \ell-1$, with $\ell-1$ steps the forcing process, all of the vertices of P_ℓ are forced. For every $1 \leqslant i \leqslant \ell-2$, with $r-2$ steps the forcing process, all of the vertices of H_i are forced. Finally, if r is even, then with $\frac{r-2}{2}$ steps the forcing process, all of the vertices of $H_{\ell-1}$ are forced. Therefore, if r is even, then $Pt(P_\ell(H,e),B_1) = (\ell-2)(r-1)+\frac{r}{2}$. Similarly, if r is odd, then $Pt(P_\ell(H,e),B_1) = (\ell-2)(r-1)+\frac{r+1}{2}$. Also, for zero forcing set B_2, we have $B_2^{(1)} = \{v_{(\ell-1)2}\}$, $B_2^{(2)} = \{v_{(\ell-1)3}\}, \dots, B_2^{(r-2)} = \{v_{(\ell-1)(r-1)}\}$ and $B_2^{(r-1)} = \{v_{\ell-1}\}$. In the other word, with $\ell-1$ steps the forcing process, all of the vertices of P_ℓ are forced. For every $1 \leqslant i \leqslant \ell-1$, with $r-2$ steps the forcing process, all of the vertices of H_i are forced. Thus, $Pt(P_\ell(H,e),B_2) = (\ell-1)(r-1)$. It is clear that $Pt(P_\ell(H,e),B_1) \leqslant Pt(P_\ell(H,e),B_2)$. Therefore, $PT(P_\ell(H,e)) = (\ell-1)(r-1)$ and

$$Pt(P_\ell(H,e))) = \begin{cases} (\ell-2)(r-1)+\frac{r}{2} & \text{if } r \text{ is even} \\ (\ell-2)(r-1)+\frac{r+1}{2} & \text{if } r \text{ is odd.} \end{cases}$$

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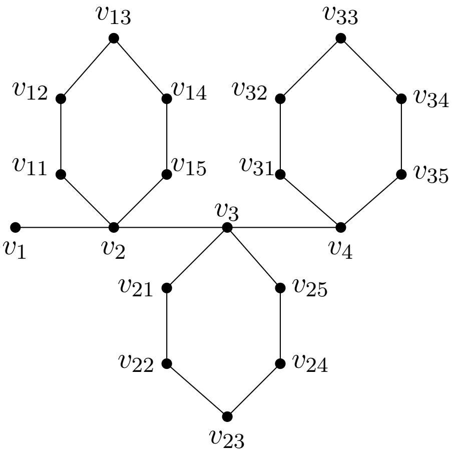

Figure 4. $P_4((K_1 \sim C_6)(a,b),e)$

Theorem 3.7 Let ℓ be even, $b \in V(K_r)$, $V(K_1) = \{a\}$, $H = (K_1 \sim K_r)(a,b)$ and $e = ab$. Then $Z(C_\ell(H,e)) = M(C_\ell(H,e)) = \ell(r-2)+2$. Also, $Pd(C_\ell(H,e)) = 0$ (see Figure 5).

Proof. Let $V(C_\ell) = \{v_1, v_2, \dots, v_\ell\}$ and $V(H_i)\setminus\{a,b\} = \{v_{i1}, v_{i2}, \dots, v_{i(r-1)}\}$, where H_i is the $i-th$ copy of H and $1 \leq i \leq \ell$. Let $Z = \{v_1, v_2\} \cup \{v_{jk} \mid 1 \leq j \leq \ell,\ 2 \leq k \leq r-1\}$ be the initial black vertices. Since v_{x1} is the only white neighbor of v_{x2}, for $x \in \{1,\ell\}$, so v_{x2} forces v_{x1}. Since v_3 is the only white neighbor of v_2, so v_3 is forced by v_2. Similarly, v_1 forces v_ℓ. We see that v_{21} and $v_{(\ell-1)1}$ are forced by v_{22} and $v_{(\ell-1)2}$, respectively. By similar argument, Z is a zero forcing set of $C_\ell(H,e)$. Thus, $Z(C_\ell(H,e)) \leq |Z| = \ell(r-2)+2$. Since $C_\ell(H,e) = \bigcup_{i=1}^{\ell} K_r \cup C_\ell$, by Theorem 2.5, $mr(C_\ell(H,e)) \leq \ell mr(K_r) + mr(C_\ell) = \ell + \ell - 2 = 2\ell - 2$. Hence, $M(C_\ell(H,e)) \geq (\ell(r-1)+\ell) - (2\ell-2) = \ell(r-2)+2$. By Theorem 2.4, $\ell(r-2)+2 \leq M(C_\ell(H,e)) \leq Z(C_\ell(H,e)) \leq \ell(r-2)+2$. Therefore, $Z(C_\ell(H,e)) = \ell(r-2)+2 = M(C_\ell(H,e))$ and $mr(C_\ell(H,e)) = 2\ell-2$. Now, let B be a zero forcing set of $C_\ell(H,e)$ with $|B| = \ell(r-2)+2$. Then, for every $1 \leq i \leq \ell$, $|B \cap \{v_{ij} \mid 1 \leq j \leq r-1\}| \geq r-2$ and so we have three following cases:
Case 1: $B \cap \{v_1, v_2, \dots, v_\ell\} = \emptyset$. In this case, without loss of generality, we assume that $B = \{v_{1j}, v_{\ell j} \mid 1 \leq j \leq r-1\} \bigcup_{i=2}^{\ell-1} \{v_{ij} \mid 2 \leq j \leq r-1\} = B_1$.Then $B_1^{(1)} = \{v_1, v_2\}$, $B_1^{(2)} = \{v_\ell, v_3\}$, $B_1^{(3)} = \{v_{(\ell-1)1}, v_{21}\}$, $B_1^{(4)} = \{v_{\ell-1}, v_4\}$, $B_1^{(5)} = \{v_{(\ell-2)1}, v_{31}\}$,

$\ldots$, $B_1^{(\ell-2)} = \{v_{\frac{\ell+2}{2}}, v_{\frac{\ell+4}{2}}\}$, $B_1^{(\ell-1)} = \{v_{(\frac{\ell+2}{2})1}, v_{(\frac{\ell+4}{2})1}\}$. Thus, $V(C_\ell(H,e)) = \bigcup_{i=0}^{\ell-1} B_1^{(i)}$ and $Pt(C_\ell(H,e), B_1) = \ell - 1$.

Case 2: $|B \cap \{v_1, v_2, \ldots, v_\ell\}| = 1$. In this case, without loss of generality, we may assume that $|B \cap \{v_1, v_2, \ldots, v_\ell\}| = \{v_1\}$ and $B = \{v_1\} \cup V(H_1) \setminus \{a,b\} \bigcup_{i=2}^{\ell} \{v_{ij} \mid 2 \leq j \leq r-1\} = B_2$. Then $B_2^{(1)} = \{v_2, v_{\ell 1}\}$, $B_2^{(2)} = \{v_3, v_\ell\}$, $B_2^{(3)} = \{v_{21}, v_{(\ell-1)1}\}$, $\ldots$, $B_2^{(\ell-2)} = \{v_{\frac{\ell+2}{2}}, v_{\frac{\ell+4}{2}}\}$, $B_2^{(\ell-1)} = \{v_{(\frac{\ell+2}{2})1}, v_{(\frac{\ell+4}{2})1}\}$. Hence, $V(C_\ell(H,e)) = \bigcup_{i=0}^{\ell-1} B_2^{(i)}$ and so $Pt(C_\ell(H,e), B_2) = \ell - 1$.

Case 3: $|B \cap \{v_1, v_2, \ldots, v_\ell\}| = 2$ and $B \cap \{v_1, v_2, \ldots, v_\ell\} = \{v_i, v_j\}$. Since B is a zero forcing set of $P_t(C_\ell, e)$), so $|i-j| = 1$. Without loss of generality, we may assume that $i = 1$, $j = 2$. Then $B = \{v_1, v_2\} \bigcup_{i=1}^{\ell} \{v_{ij} \mid 2 \leq j \leq r-1\} = B_3$. It is easy to see that $B_3^{(1)} = \{v_{11}, v_{\ell 1}\}$, $B_3^{(2)} = \{v_3, v_\ell\}$, $B_3^{(3)} = \{v_{21}, v_{(\ell-1)1}\}$, $\ldots$, $B_3^{(\ell-2)} = \{v_{\frac{\ell+2}{2}}, v_{\frac{\ell+4}{2}}\}$, $B_3^{(\ell-1)} = \{v_{(\frac{\ell+2}{2})1}, v_{(\frac{\ell+4}{2})1}\}$. Hence, $V(C_\ell(H,e)) = \bigcup_{i=0}^{\ell-1} B_3^{(i)}$ and so $Pt(C_\ell(H,e), B_3) = \ell - 1$. Therefore, $Pt(C_\ell(H,e)) = \ell - 1 = PT(C_\ell(H,e))$ and so $Pd(C_\ell(H,e)) = 0$.

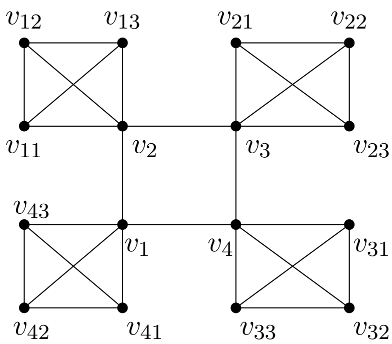

Figure 5. $C_4((K_1 \sim K_4)(a,b), e)$

■

Theorem 3.8 Let $V(K_1) = \{a\}$, $b \in V(K_r)$, $H = (K_1 \sim K_r)(a,b)$ and $e = ab$. Then $Z(P_\ell(H,e)) = M(P_\ell(H,e)) = (\ell-1)(r-2)+1$, $mr(P_\ell(H,e)) = 2\ell - 2$ and $Pd(P_\ell(H,e)) = 0$ (see Figure 6).

Proof. Let $V(P_\ell) = \{v_1, v_2, \ldots, v_\ell\}$, $V(H_i) \setminus \{a,b\} = \{v_{i1}, v_{i2}, \ldots, v_{i(r-1)}\}$, where H_i is the $i-th$ copy of H and $1 \leq i \leq \ell - 1$. Also, let $Z = \{v_1\} \cup \{v_{jk} \mid 1 \leq j \leq \ell-1,\ 2 \leq k \leq r-1\}$, be initial black vertices.Then v_1 forces v_2. The vertex v_{11} is forced by v_{12}. Now v_2 forces v_3. It is easy to see that Z is a zero forcing set of $P_\ell(H,e)$. Hence, $Z(P_\ell(H,e)) \leq |Z| = (\ell-1)(r-2)+1$. Since $P_\ell(H,e) = \bigcup_{i=1}^{\ell-1} K_r \cup P_\ell$, by Theorem 2.5, $mr(P_\ell(H,e)) \leq (\ell-1)mr(K_r) + mr(P_\ell) = 2\ell - 2$. So, $M(P_\ell(H,e)) \geq r(\ell-1)+1-2\ell+2 = (r-2)(\ell-1)+1$.

By Theorem 2.4, $(\ell-1)(r-2)+1 \geqslant Z(P_\ell(H,e)) \geq M(P_\ell(H,e)) \geq (r-2)(\ell-1)+1$. Therefore, $Z(P_\ell(H,e)) = M(P_\ell(H,e)) = (r-2)(\ell-1)+1$ and so $mr(P_\ell(H,e)) = 2\ell-2$. Let B be a zero forcing set of $P_\ell(H,e)$ with $|B| = (r-2)(\ell-1)+1$. Then

$$B = \{v_1\} \cup \{v_{jk} \mid 1 \leq j \leq \ell-1,\ 2 \leq k \leq r-1\} = B_1$$

or

$$B = \{v_\ell\} \cup \{v_{jk} \mid 1 \leq j \leq \ell-1,\ 2 \leq k \leq r-1\} = B_2.$$

It is easy to see that $B_1^{(1)} = \{v_2\}$, $B_1^{(2)} = \{v_{11}\}$, $B_1^{(3)} = \{v_3\}$, $B_1^{(4)} = \{v_{21}\}$, $\ldots$,

$B_1^{(2\ell-3)} = \{v_\ell\}$ and $B_1^{(2\ell-2)} = \{v_{(\ell-1)1}\}$. Hence, $V(P_\ell(H,e)) = \bigcup_{i=0}^{2\ell-2} B_1^{(i)}$. Thus, $Pt(P_\ell(H,e),B_1) = 2(\ell-1)$. Also, we have $B_2^{(1)} = \{v_{(\ell-1)1}\}$, $B_2^{(2)} = \{v_{\ell-1}\}$, $B_2^{(3)} = \{v_{(\ell-2)1}\}$, $B_2^{(4)} = \{v_{\ell-2}\}, \dots, B_2^{(2\ell-4)} = \{v_2\}$, $B_2^{(2\ell-3)} = \{v_{11}\}$ and $B_2^{(2\ell-2)} = \{v_1\}$. Hence, $V(P_\ell(H,e)) = \bigcup_{i=0}^{2\ell-2} B_2^{(i)}$ and $Pt(P_\ell(H,e),B_2) = 2(\ell-1)$. Therefore, $Pt(P_\ell(H,e)) = 2(\ell-1) = PT(P_\ell(H,e))$ and $Pd(P_\ell(H,e)) = 0$.

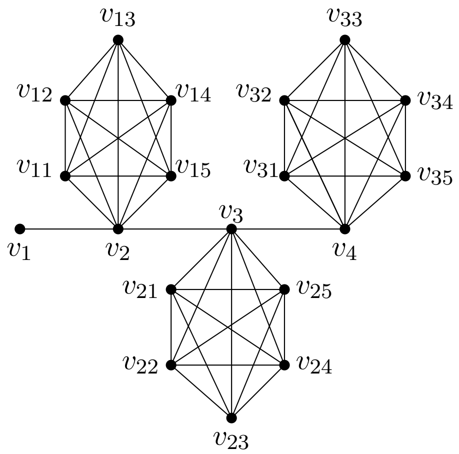

Figure 6. $P_4((K_1 \sim K_6)(a,b),e)$

■

Theorem 3.9 Let ℓ and r be even, $b \in V(C_r)$, $V(K_1) = \{a\}$, $H = (K_1 \sim C_r)(a,b)$ and $e = ab$. Then $Z(C_\ell(H,e)) = M(C_\ell(H,e)) = \ell + 2$ and $Pt(C_\ell(H,e)) = \frac{(r-1)(\ell-2)}{2}$ (see Figure 7).

Proof. Let $V(C_\ell) = \{v_1, v_2, \dots, v_\ell\}$ and $V(H_i \setminus \{a,b\}) = \{v_{i1}, v_{i2}, \dots, v_{i(r-1)}\}$, where H_i is the $i-th$ copy of H and $1 \le i \le \ell$. Let $Z = \{v_1, v_2\} \cup \{v_{i1} \mid 1 \le i \le \ell\}$ be the initial black vertices. Since $v_{\ell 2}$ and v_{12} are the only white neighbor of $v_{\ell 1}$ and v_{11}, respectively, so $v_{\ell 2}$ is forced by $v_{\ell 1}$ and v_{12} is forced by v_{11}. It is easy to see that $v_{\ell j}$ and v_{1j} are forced by $v_{\ell(j-1)}$ and $v_{1(j-1)}$, respectively for $(3 \le j \le r-1)$. Also, v_3 is forced by v_2 and v_ℓ is forced by v_1. Since v_{22} and $v_{(\ell-1)2}$ are the only white neighbor of v_{21} and $v_{(\ell-1)1}$, so v_{21} forces v_{22} and $v_{(\ell-1)1}$ forces $v_{(\ell-1)2}$. Also, v_{2j} and $v_{(\ell-1)j}$ are forced by $v_{2(j-1)}$ and $v_{(\ell-1)(j-1)}$ for $(3 \le j \le r-1)$. By similar argument, we see that Z is a zero forcing set for $C_\ell(H,e)$. Hence, $Z(C_\ell(H,e)) \le |Z| = \ell + 2$. Since $C_\ell(H,e) = (\bigcup_{i=1}^{\ell} C_r) \cup C_\ell$, by Theorem 2.5, $mr(C_\ell(H,e)) \le \ell(mr(C_r)) + mr(C_\ell) = \ell(r-2) + \ell - 2$. So $M(C_\ell(H,e)) \ge \ell r - (\ell(r-2)+\ell-2) = \ell+2$. By Theorem 2.4, $\ell+2 \geqslant Z(C_\ell(H,e)) \ge M(C_\ell(H,e)) \ge \ell+2$. Therefore, $Z(C_\ell(H,e)) = M(C_\ell(H,e)) = \ell + 2$. Now, let B be a zero forcing set for $C_\ell(H,e)$ with $|B| = \ell + 2$. Then $|B \cap (V(H_i) \setminus \{a,b\})| \ge 1$ for every $1 \le i \le \ell$. Since $|B| = \ell + 2$, we have three following cases:

Case 1: Let $B \cap \{v_1, v_2, \dots, v_\ell\} = \emptyset$. Then there exist $(1 \le j \le \ell)$ such that $|B \cap V(H_j) \setminus \{a,b\}| = |B \cap V(H_{j+1}) \setminus \{a,b\}| = 2$. Without loss of generality we may assume that $j = 1$. Let $B = \{v_{1(\frac{r-2}{2})}, v_{1(\frac{r}{2})}, v_{2(\frac{r-2}{2})}, v_{2(\frac{r}{2})}\} \cup \{v_{i1} \mid 3 \le i \le \ell\} = B_1$. If $k \ne \frac{r-2}{2}$ and $B = \{v_{1k}, v_{1(k+1)}, v_{2k}, v_{2(k+1)}\} \cup \{v_{i1} \mid 3 \le i \le \ell\} = B_2$, then since $Pt(C_r) = \frac{r-2}{2}$,

$$Pt(C_\ell(H,e),B_1) \le Pt(C_\ell(H,e),B_2).$$

For zero forcing set $B_1 = \{v_{1(\frac{r-2}{2})}, v_{1(\frac{r}{2})}, v_{2(\frac{r-2}{2})}, v_{2(\frac{r}{2})}\} \cup \{v_{i1} \mid 3 \le i \le \ell\}$, with $\frac{r-2}{2}$ steps the forcing process, all of the vertices of $H_1 \cup H_2 \cup \{v_2, v_3\}$ are forced. Since $Pt(C_\ell) = \frac{\ell-2}{2}$, with $\frac{\ell-2}{2}$ steps the forcing process, all of the vertices of C_ℓ are forced. For every $3 \leqslant i \leqslant \frac{\ell}{2}$, $i \neq \frac{\ell+2}{2}$ and $i \neq \frac{\ell+4}{2}$, with $r-2$ steps the forcing process, all of the vertices of $H_i \cup H_{\ell-i+3}$ are forced. Finally, with $\frac{r-2}{2}$ steps the forcing process, all of the vertices of $H_{\frac{\ell+2}{2}} \cup H_{\frac{\ell+4}{2}}$ are forced. Thus, $Pt(C_\ell(H,e), B_1) = \frac{(r-1)(\ell-2)}{2}$.

Case 2: Let $|B \cap \{v_1, v_2, \ldots, v_\ell\}| = 2$. Without loss of generality, we assume that $B = \{v_2, v_3\} \cup \{v_{i1} \mid 1 \leq i \leq \ell\} = B_3$. Then with $r-2$ steps the forcing process, all of the vertices of $H_1 \cup H_2$ are forced. Since $Pt(C_\ell) = \frac{\ell-2}{2}$, with $\frac{\ell-2}{2}$ steps the forcing process, all of the vertices of C_ℓ are forced. Also, for every $3 \leqslant i \leqslant \frac{\ell}{2}$, with $r-2$ steps the forcing process, all of the vertices of $H_i \cup H_{\ell-i+3}$ are forced. with $\frac{r-2}{2}$ steps the forcing process, all of the vertices of $H_{\frac{\ell+2}{2}} \cup H_{\frac{\ell+4}{2}}$ are forced. Thus, $Pt(C_\ell(H,e), B_3) = \frac{(r-1)(\ell-1)-1}{2}$.

Case 3: Let $|B \cap \{v_1, v_2, \ldots, v_\ell\}| = 1$. Without loss of generality, we assume that $|B \cap \{v_1, v_2, \ldots, v_\ell\}| = \{v_1\}$. Then $B \cap V(H_1) = \{v_{1\frac{r}{2}}, v_{1\frac{r+2}{2}}\}$ and $B = \{v_1\} \cup \{v_{1\frac{r-2}{2}}, v_{1\frac{r}{2}}\} \cup \{v_{i1} \mid 2 \leq i \leq \ell\} = B_4$. It is easy to see that $Pt(C_\ell(H,e), B_1) \leqslant Pt(C_\ell(H,e), B_3)$ and $Pt(C_\ell(H,e), B_1) \leqslant Pt(C_\ell(H,e), B_4)$. Therefore, $Pt(C_\ell(H,e)) = \frac{(r-1)(\ell-2)}{2}$.

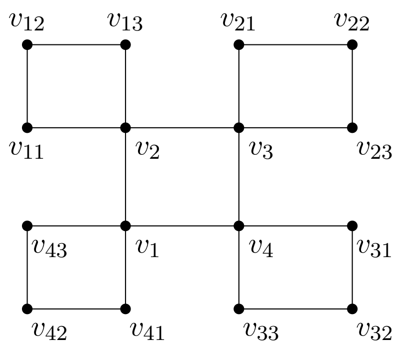

Figure 7. $C_4((K_1 \sim C_4)(a,b), e)$

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Acknowledgments

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