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Projectivity and injectivity of G-Hilbert $\Im$-modules

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Abstract. Let G be a discrete group acting on C^*-algebra $\Im$. In this paper, we investigate projectivity and injectivity of G-Hilbert $\Im$-modules and study the equivalent conditions characterizing G-C^*-subalgebras of the algebra of compact operators on G-Hilbert spaces in terms of general properties of G-Hilbert $\Im$-modules. In particular, we show that G-Hilbert $\Im$-(bi)modules on G-C^*-algebra of compact operators are both projective and injective.

Keywords: G-projective, G-projective cover, extremally G-disconnected, G-C^*-algebra, G-Hilbert $\Im$-module, G-injective Hilbert $\Im$-module, G-projective Hilbert $\Im$-module, G-self dual, G-monotone complete, G-$*$-representation.

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1. Introduction and preliminaries

The aim of this paper is to generalize the main results of [8, 16] for actions of discrete groups on Hilbert C^*-modules. Accordingly, we investigate two specific problems:

(i) characterizations of G-C^*-algebras $\Im$ and ζ for each G-Hilbert $\Im$-ζ-bimodule is projective or injective, for appropriate morphisms and subobjects;
(ii) characterizations of injective or projective G-Hilbert $\Im$-ζ-bimodules, for G-C^*-algebras $\Im$ and ζ and appropriate morphisms.

Most of the existing work on injectivity for C^*-algebras are focused on (contractive) completely positive maps. The work of Hamana [9, 10] on G-injective operator system and

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G-injective envelopes, gives the flavor of G-injectivity for operator system endowed with G-actions. In the category of topological spaces with a G-action, Hadwin and Paulsen [11] gave a characterization of G-projective and G-injective spaces. Most of these investigations are restricted to the case of actions of discrete groups. In [3], it has been shown that Fréchet algebra $\cap_{n\in\mathbb{N}} L^\infty(\mathsf{G}, \omega_n^{-1})$ is projective if and only if G is finite and Fréchet algebra $\cap_{n\in\mathbb{N}} C_0(\mathsf{G}, \omega_n^{-1})$ is projective (injective) if and only if G is compact (finite), where G is a locally compact group. Oikhberg [19] proved that every p-multinormed space embeds into (is a quotient of) an injective (resp. a projective) p-multinormed space. Mahmoodi and Mardanbeigi [17] showed that an injective AF-algebra must be finite dimensional, while the question is open for injective and projective G-AF-algebras.

In sections 2 and 3, we consider categories of G-Hilbert $\Im$-modules $\mathcal{X}_\mathsf{G}$ on some fixed G-C^*-algebra $\Im$ and specify another G-C^*-algebra ζ that using module-specific G-$*$-representations acts on $\mathcal{X}_\mathsf{G}$ as a set of adjointable bounded operators. This gives on such a module $\mathcal{X}_\mathsf{G}$ the structure of a G-Hilbert $\Im$-ζ-bimodule with $\langle e\iota_1\flat, \iota_2\rangle = e\langle\iota_1, \iota_2\rangle\flat^*$ for each $e \in \Im$, $\flat \in \zeta$, and $\iota_1, \iota_2 \in \mathcal{X}_\mathsf{G}$. We call $\mathcal{X}_\mathsf{G}$ a G-Hilbert $\Im$-ζ-bimodule. Note that any G-Hilbert $\Im$-ζ-module is automatically a G-Hilbert $\Im$-$\mathbb{C}$-bimodule. Then projective and injective G-Hilbert $\Im$-ζ-module on a fixed G-C^*-algebra are defined for morphisms being bounded G-equivariant maps.

A method for generalizing Hilbert C^*-modules ($\mathcal{H}C^*$-M) is to consider the category whose objects are G-Hilbert $\Im$-module on a fixed G-C^*-algebra $\Im$, subobjects are G-submodules and morphisms are the bounded $\Im$-module G-equivariant maps. As in the case of Banach spaces, if one considers the morphisms to be contractive module maps, the theory of injective $\mathcal{H}C^*$-M is nearly simpler and is large works out in (e.g. [15, 16, 23]). In [15, 16], sometimes the morphisms are adjointable contractive module mappings, objects are $\mathcal{H}C^*$-M and subobjects are $\mathcal{H}C^*$-submodules. There are similar observations about projectivity, which is a kind of dual theory of injectivity. In addition to specifying morphisms, the coefficients must also be specified for projectivity. In Section 3, we provide detailed definitions of it. The sets of morphisms in our study are either bounded bimodule G-equivariant maps, or bimodule G-equivariant maps. We will show these two categories with $B_\mathsf{G}(\Im, \zeta)$ and $B^*_\mathsf{G}(\Im, \zeta)$ respectively. We show that for each G-C^*-algebra $\Im$, any G-Hilbert $\Im$-ζ-bimodule in the category $B^*_\mathsf{G}(\Im, \zeta)$ is projective. When $\Im$ is a G-C^*-algebra of compact operators, we show that any G-Hilbert $\Im$-ζ-bimodule in $B_\mathsf{G}(\Im, \zeta)$ is projective, but we can not solve the question of whether these are the only G-C^*-algebras with this property. Even the question of whether all G-Hilbert $\Im$-modules are projective remains open in the larger category. However, we show that all G-Hilbert $\Im$-ζ-bimodules on a G-C^*-algebra in these categories are projective iff the kernel of any surjective bounded module map between G-Hilbert $\Im$-module is a topological direct summand of the domain. Moreover, we identify a family of G-projective G-Hilbert $\Im$-module on unital G-C^*-algebras. We show that finitely generated G-Hilbert $\Im$-module on unital G-C^*-algebras of both categories are projective objects. The G-C^*-algebras $\Im$ of the form $\Im = \varpi_0\text{-}\sum_\sigma \mathcal{K}(\mathcal{H}_\sigma)$ are of particular importance, where $\mathcal{K}(\mathcal{H}_\sigma)$ represents the G-C^*-algebra of all compact operators on some G-Hilbert space $\mathcal{H}_\sigma$, and when the group G is said to have the fixed point property, the ϖ_0-sum is either a finite block-diagonal sum or a block-diagonal sum with a ϖ_0-convergence condition.

2. G-injective Hilbert $\Im$-modules

An operator system in the category of unital completely positive ($\mathcal{U.C.P.}$) linear maps and C^*-algebras, is a self-adjoint linear subspace $\mathcal{Q}$ of a unital C^*-algebra A containing the identity $\mathbb{I}$ of A move the references. An order isomorphim of two operator systems $\mathcal{Q}$ and $\mathcal{Q}'$ is a $\mathcal{U.C.P}$. linear isomorphism $\Omega : \mathcal{Q} \longrightarrow \mathcal{Q}'$ so that Ω^{-1} is also completely positive. We say that Ω is an automorphism, if $\mathcal{Q}' = \mathcal{Q}$, also we denote the automorphisms group of $\mathcal{Q}$ by $Aut(\mathcal{Q})$. Any $\Omega \in Aut(\mathcal{Q})$ is automatically completely isometric, therefore, if $\mathcal{Q}$ be a (unital) C^*-algebra, the definition of an automorphism coincides to the usual concept of an automorphism of a C^*-algebra (e.g. [2]). We say that $\mathcal{Q}$ is a G-operator system, if an action of discrete group G on an operator system $\mathcal{Q}$ be always assumed by automorphisms. We say that $\mathcal{Q}$ is a G-C^*-algebra, if $\mathcal{Q}$ is a C^*-algebra. The image of ι under ϱ, for every $\varrho \in \mathsf{G}$ and $\iota \in \mathcal{Q}$, is denoted by $\varrho \cdot \iota$. A G-operator system is a G-equivariant $\mathcal{U.C.P}$. maps and G-projective object in the category of G-operator system. Suppose that $\mathcal{X}_{\mathsf{G}}$ be a G-operator system, if $\mathcal{C}_{\mathsf{G}}$ is a G-operator system, then we say that $(\mathcal{C}_{\mathsf{G}}, \Upsilon)$ is a G-cover of $\mathcal{X}_{\mathsf{G}}$ and $\Upsilon : \mathcal{C}_{\mathsf{G}} \longrightarrow \mathcal{X}_{\mathsf{G}}$ is a G-equivariant $\mathcal{U.C.P}$ linear epimorphism on $\mathcal{X}_{\mathsf{G}}$. If $(\mathcal{C}_{\mathsf{G}}, \Upsilon)$ be a G-cover, then we say that $(\mathcal{C}_{\mathsf{G}}, \Upsilon)$ is G-essential cover of $\mathcal{X}_{\mathsf{G}}$, and whenever $\mathcal{Y}_{\mathsf{G}}$ is a G-operator system; $\Gamma : \mathcal{Y}_{\mathsf{G}} \longrightarrow \mathcal{C}_{\mathsf{G}}$ is G-equivariant $\mathcal{U.C.P}$ map and $\Upsilon(\Gamma(\mathcal{Y}_{\mathsf{G}})) = \mathcal{X}_{\mathsf{G}}$, then $\Gamma(\mathcal{Y}_{\mathsf{G}}) = \mathcal{C}_{\mathsf{G}}$. We say that $(\mathcal{C}_{\mathsf{G}}, \Upsilon)$ is a G-rigid cover of $\mathcal{X}_{\mathsf{G}}$, if there exists a G-cover and G-equivariant $\mathcal{U.C.P}$ map $\Gamma : \mathcal{C}_{\mathsf{G}} \longrightarrow \mathcal{C}_{\mathsf{G}}$ satisfying $\Upsilon(\Gamma(\varpi)) = \Upsilon(\varpi)$ for each $\varpi \in \mathcal{C}_{\mathsf{G}}$ is a complete isometric.

Entirely this section, we assume that G is a discrete group. A G-C^*-algebra, equipped with the action of G by automorphisms is a C^*-algebra. In other words, a G-C^*-algebra $\Im$ is a C^*-algebra and a left G-M. Given G-C^*-algebras $\Im$ and ζ, the $\mathcal{U.C.P}$ linear map $\varphi : \Im \longrightarrow \zeta$ is G- equivariant if $\varphi(\varrho \cdot e) = \varrho \cdot \varphi(e)$, for any $\varrho \in \mathsf{G}$ and $e \in \Im$.

A G-C^*-algebra $\Im$ is said to be G-injective if for each G-C^*-algebras ζ and $\mathcal{C}$, any G-equivariant complete isometry $\pi : \zeta \longrightarrow \mathcal{C}$ and any G-equivariant $\mathcal{U.C.P}$ map $\varphi : \zeta \longrightarrow \Im$, there exists a G-equivariant $\mathcal{U.C.P}$. map $\tilde{\varphi} : \mathcal{C} \longrightarrow \Im$ satisfying $\tilde{\varphi} \circ \pi = \varphi$.

Definition 2.1 A G-pre-Hilbert $\Im$-module on a G-C^*-algebra $\Im$ is an $\Im$-module $\mathcal{X}_{\mathsf{G}}$ equipped with an $\Im$-valued map $\langle \cdot, \cdot \rangle : \mathcal{X}_{\mathsf{G}} \times \mathcal{X}_{\mathsf{G}} \longrightarrow \Im$ that in the first argument is $\Im$-linear and has the following properties:

$$\langle \iota, \varsigma \rangle = \langle \varsigma, \iota \rangle^*, \langle \iota, \iota \rangle \geqslant 0 \text{ with equality if and only if } \iota = 0.$$

Then $\langle \cdot, \cdot \rangle$ is said to be the $\Im$-valued inner product ($\Im$-$\mathcal{V.I.P}$) in $\mathcal{X}_{\mathsf{G}}$.

***Example* 2.2** Suppose that $\Im$ is a G-C^*-algebra and $\mathbb{C}$ is the set of all numbers of complex. Then

(i) Each inner product space on action G is a left G-pre-Hilbert module on $\mathbb{C}$;
(ii) If $\mathfrak{I}$ is a (closed) right G-invariant ideal of $\Im$, then $\mathfrak{I}$ is a G-pre-Hilbert $\Im$-module if $\langle e, \flat \rangle := e\flat^*$; especially $\Im$ is a G-pre-Hilbert $\Im$-module;
(iii) Let $\{\mathfrak{M}_\alpha\}_{1 \leqslant \alpha \leqslant \mathfrak{m}}$ be a finite family of G-pre-Hilbert $\Im$-module. Then the vector space direct sum $\bigoplus_{\alpha=1}^{m} \mathfrak{M}_\alpha$ is a G-pre-Hilbert $\Im$-module if we define

$$(\iota_1, \cdots, \iota_{\mathfrak{m}})e = (\iota_1 e, \cdots, \iota_{\mathfrak{m}} e), \langle (\iota_1, \cdots, \iota_{\mathfrak{m}}), (\varsigma_1, \cdots, \varsigma_{\mathfrak{m}}) \rangle = \sum_{\alpha=1}^{\mathfrak{m}} \langle \iota_\alpha, \varsigma_\alpha \rangle.$$

Definition 2.3 A G-pre-Hilbert $\Im$-module $\{\mathcal{X}_{\mathsf{G}}, \langle \cdot, \cdot \rangle\}$ is G-Hilbert $\Im$-module iff it is complete with respect to the norm $\| \cdot \| = \| \langle \cdot, \cdot \rangle \|_{\Im}^{\frac{1}{2}}$.

***Example* 2.4** Suppose that $\{\mathcal{X}_n\}$ is a sequence of G-Hilbert $\Im$-module. Then

$$\bigoplus_{\alpha=1}^{\infty} \xi_{\mathfrak{n}} = \{\{\iota_{\mathfrak{n}}\} | \iota_{\mathfrak{n}} \in \mathcal{X}_{\mathfrak{n}}, \sum_{\mathfrak{n}=1}^{\infty} \langle \iota_{\mathfrak{n}}, \iota_{\mathfrak{n}} \rangle \text{ converges in } \nu\}$$

with operations

$$\{\iota_{\mathfrak{n}}\} + \nu\{\varsigma_{\mathfrak{n}}\} = \{\iota_{\mathfrak{n}} + \nu\varsigma_{\mathfrak{n}}\}, \{\iota_{\mathfrak{n}}\}e = \{\iota_{\mathfrak{n}}e\}, \langle\{\iota_{\mathfrak{n}}\}, \{\varsigma_{\mathfrak{n}}\}\rangle = \sum_{\alpha=1}^{\infty}\langle\iota_{\mathfrak{n}}, \varsigma_{\mathfrak{n}}\rangle$$

is a G-Hilbert $\Im$-module. Note that in $\bigoplus_{\mathfrak{n}=p}^{q}\mathcal{X}_{\mathfrak{n}}$, we have

$$\sum_{\mathfrak{n}=p}^{q}\|\langle\iota_{\mathfrak{n}}, \varsigma_{\mathfrak{n}}\rangle\| = \|\sum_{\mathfrak{n}=p}^{q}\langle\iota_{\mathfrak{n}}, \iota_{n}\rangle\|\|\sum_{\mathfrak{n}=p}^{q}\langle\varsigma_{\mathfrak{n}}, \varsigma_{\mathfrak{n}}\rangle\|.$$

Hence, by the Cauchy criterion, $\sum_{\mathfrak{n}=1}^{\infty}\langle\iota_{\mathfrak{n}}, \varsigma_{\mathfrak{n}}\rangle$ converges. For completeness, let $\{\mathfrak{v}_{\pi}\}_{\pi}$ be a Cauchy sequence in $\bigoplus_{n=1}^{\infty}\mathcal{X}_{n}$ and for all $\pi, \mathfrak{v}_{\pi} = \{\iota_{\mathfrak{n},\pi}\}_{\mathfrak{n}}$. Applying

$$\|\iota_{\mathfrak{n},\pi} - \iota_{\mathfrak{n},l}\| = \|\langle\iota_{\mathfrak{n},\pi} - \iota_{\mathfrak{n},l}, \iota_{\mathfrak{n},\pi} - \iota_{\mathfrak{n},l}\rangle\| = \|\sum_{\mathfrak{n}=1}^{\infty}\langle\iota_{\mathfrak{n},\pi} - \iota_{\mathfrak{n},l}, \iota_{\mathfrak{n},\pi} - \iota_{\mathfrak{n},l}\rangle\| = \|\mathfrak{v}_{\pi} - \mathfrak{v}_{l}\|^{2},$$

we deduce that $\{\iota_{\mathfrak{n},\pi}\}_{\pi}$ is Cauchy, for any $\mathfrak{n} \in \mathbb{N}$. So for any $\mathfrak{n}$, there is $\mathfrak{u}_{n}$ so that $\lim_{\pi}\iota_{\mathfrak{n},\pi} = \mathfrak{u}_{\mathfrak{n}}$. Now, we put $\mathfrak{u} = \{\mathfrak{u}_{\mathfrak{n}}\}$. Then $\lim_{\pi}\mathfrak{v}_{\pi} = \mathfrak{u}$.
G-Hilbert $\Im$-module behaves like Hilbert spaces in some way, for example,

$$\|\iota\| = \sup\{\|\langle\iota, \varsigma\rangle\|, \|\varsigma\| = 1, \varsigma \in \mathcal{X}_{\mathsf{G}}\}.$$

But there exists one fundamental method in which G-Hilbert $\Im$-modules differs from Hilbert spaces. Given a closed submodule $\mathcal{Y}_{\mathsf{G}}$ of a G-Hilbert $\Im$-module $\mathcal{X}_{\mathsf{G}}$, we define

$$\mathcal{Y}_{\mathsf{G}}^{\perp} = \{\varsigma \in \mathcal{X}_{\mathsf{G}} \mid \langle\iota, \varsigma\rangle = 0, \forall\iota \in \mathcal{Y}_{\mathsf{G}}\}.$$

Then $\mathcal{Y}_{\mathsf{G}}^{\perp}$ is a closed submodule, but $\mathcal{X}_{\mathsf{G}} \neq \mathcal{Y}_{\mathsf{G}} + \mathcal{Y}_{\mathsf{G}}$ and $\mathcal{Y}_{\mathsf{G}}^{\perp\perp} \neq \mathcal{Y}_{\mathsf{G}}$.

***Example* 2.5** Let $\Im = C([0,1])$, $\mathcal{X}_{\mathsf{G}} = \Im$ and $\mathcal{Y}_{\mathsf{G}} = \{\Upsilon \in \Im | \Upsilon(\frac{1}{2}) = 0\}$. Then $\mathcal{Y}_{\mathsf{G}}^{\perp} = \{0\}$. Also, for all $\varrho \in \mathcal{Y}_{\mathsf{G}}^{\perp}$, $\varrho(\tau)|\tau - \frac{1}{2}| = 0$. Hence, by continuity, $\varrho \equiv 0$. The equality of Pythagoras stating $\mathcal{E}, \gamma \in \mathcal{H}$ and $\mathcal{E} \perp \gamma$ imply $\|\mathcal{E} + \gamma\|^{2} = \|\mathcal{E}\|^{2} + \|\gamma\|^{2}$ does not hold, in general, for G-Hilbert $\Im$-modules. For example, consider $\Im = C([0,1] \cup [2,3])$ as a G-Hilbert $\Im$-module.

$$\Upsilon(\iota) = \begin{cases} 1 & \iota \in [0,1] \\ 0 & \iota \in [2,3] \end{cases} \quad \text{and} \quad \varrho(\iota) = \begin{cases} 0 & \iota \in [0,1] \\ 1 & \iota \in [2,3] \end{cases}.$$

Then $\langle\Upsilon, \varrho\rangle = \Upsilon\varrho = 0$, $\|\Upsilon + \varrho\| = 1$ and $\|\Upsilon\| = \|\varrho\| = 1$.

Two G-Hilbert $\Im$-modules are G-isomorphic if as Banach $\Im$-modules, they are G-isometrically isomorphic on G.

Definition 2.6 The set of all bounded $\Im$-module G-equivariant $r : \mathcal{X}_{\mathsf{G}} \longrightarrow \Im$ forms a G-Banach $\Im$-module $\mathcal{X}'_{\mathsf{G}}$. The G-Banach $\Im$-module $\mathcal{X}'_{\mathsf{G}}$ is G-*dual* of $\mathcal{X}_{\mathsf{G}}$. The action of module $\Im$ on $\mathcal{X}'_{\mathsf{G}}$ for any $\iota \in \mathcal{X}_{\mathsf{G}}$, $e \in \Im$, $\mathfrak{r}, \mathfrak{s} \in \mathcal{X}'_{\mathsf{G}}$ and $\nu \in \mathbb{C}$ is defined as follows:

(i) $(\mathfrak{r} + \mathfrak{s})(\iota) = \mathfrak{r}(\iota) + \mathfrak{s}(\iota)$;
(ii) $(\nu\mathfrak{r})(\iota) = \nu(\mathfrak{r}(\iota))$;
(iii) $(e \cdot \mathfrak{r})(\iota) = \mathfrak{r}(\iota)e^{*}$.

The map $\wedge : \mathcal{X}_{\mathsf{G}} \longrightarrow \mathcal{X}_{\mathsf{G}}^{'}$, $\iota \longrightarrow \hat{\iota}$, where $\hat{\iota} : \mathcal{X}_{\mathsf{G}} \longrightarrow \Im$, $\hat{\iota}(\varsigma) = \langle \iota, \varsigma \rangle$ is a G-isometric $\Im$-linear map. We may identify $\mathcal{X}_{\mathsf{G}}$ with $\hat{\mathcal{X}}_{\mathsf{G}} = \{\hat{\iota} : \iota \in \mathcal{X}_{\mathsf{G}}\}$ as a submodule of $\mathcal{X}_{\mathsf{G}}$.

Definition 2.7 A G-Hilbert $\Im$-module $\{\mathcal{X}_{\mathsf{G}}, \langle \cdot, \cdot \rangle\}$ on a G-C^*-algebra $\Im$ is called G-*self dual* iff any bounded $\Im$-module G-equivariant $\mathfrak{r} : \mathcal{X}_{\mathsf{G}} \longrightarrow \Im$, for some element $\iota_{\mathfrak{r}} \in \mathcal{X}_{\mathsf{G}}$, is of the form $\langle \cdot, \iota_{\mathfrak{r}} \rangle$.

In fact, $\mathcal{X}_{\mathsf{G}}$ is called G-self dual if $\hat{\mathcal{X}}_{\mathsf{G}} = \mathcal{X}_{\mathsf{G}}^{'}$.

***Example* 2.8** Consider the G-C^*-algebra $\Im = \varpi_0$ of all sequences that converge to zero and put $\mathcal{X}_{\mathsf{G}} = \varpi_0$ with the standard $\Im$-$\mathcal{V}.\mathcal{I}.\mathcal{P}$. Let $\mathcal{X}_{\mathsf{G}}$ both as a G-Hilbert $\Im$-module and a G-Hilbert $\Re_{\mathsf{G}}(\Im)$-module. The G-multiplier G-C^*-algebra of $\Im = \varpi_0$ is $\Re_{\mathsf{G}}(\Im) = \mathfrak{l}^{\infty}$. Then $\mathcal{X}_{\mathsf{G}}^{'}$, as a one-sided $\Im$-module, is independent of choosing a set of coefficients equal to $\mathfrak{l}^{\infty}$.

Theorem 2.9 $\mathcal{X}_{\mathsf{G}}$ is G-self dual, as a G-Hilbert $\Im$-module, if and only if $\mathcal{X}_{\mathsf{G}}$ is unital.

Proof. Let $\mathcal{X}_{\mathsf{G}}$ be unital with unit 1 and $\tau \in \mathcal{X}_{\mathsf{G}}^{'}$. Then

$$\tau(e) = \tau(1.\iota) = \tau(1).\iota = \langle \tau(1)^*, \iota \rangle = (\tau(1)^*)^\wedge(e)$$

for all $\iota \in \mathcal{X}_{\mathsf{G}}$. Hence, $\tau = (\tau(1)^*)^\wedge \in \hat{\mathcal{X}}_{\mathsf{G}} \subseteq \mathcal{X}_{\mathsf{G}}^{'}$. If $\mathcal{X}_{\mathsf{G}} = \mathcal{X}'_{\mathsf{G}}$, then $\alpha : \mathcal{X}_{\mathsf{G}} \longrightarrow \mathcal{X}_{\mathsf{G}}$, $\alpha(\varsigma) = \varsigma$ being bounded G-equivariant $\Im$-linear, for some $\iota \in \mathcal{X}_{\mathsf{G}}$, is of the form $\hat{\iota}$. Hence $\varsigma = \alpha(\varsigma) = \hat{\iota}(\varsigma) = \langle \iota, \varsigma \rangle = \iota^*\varsigma$ for all $\varsigma \in \mathcal{X}_{\mathsf{G}}$. Therefore, ι^* is the unit of $\mathcal{X}_{\mathsf{G}}$. ■

Let $\mathcal{X}_{\mathsf{G}}$ be a G-Hilbert $\Im$-module and $\{\mathfrak{e}_\nu\}$ be an approximate unit for $\Im$. For $\iota \in \xi_{\mathsf{G}}$, we have

$$\langle \iota - \iota\mathfrak{e}_\nu, \iota - \iota\mathfrak{e}_\nu \rangle = \langle \iota, \iota \rangle - \mathfrak{e}_\nu\langle \iota, \iota \rangle - \langle \iota, \iota \rangle\mathfrak{e}_\nu + \mathfrak{e}_\nu\langle \iota, \iota \rangle\mathfrak{e}_\nu \longrightarrow 0.$$

Then $\lim_\nu \iota\mathfrak{e}_\nu = \iota$. As a result, $\mathcal{X}_{\mathsf{G}}\Im$, defined as the linear span of $\{\iota e | \iota \in \mathcal{X}_{\mathsf{G}}, e \in \Im\}$, is dense in $\mathcal{X}_{\mathsf{G}}$ and if $\Im$ is unital, then $\iota \cdot 1 = \iota$. Clearly, $\langle \mathcal{X}_{\mathsf{G}}, \mathcal{X}_{\mathsf{G}} \rangle = span\{\langle \iota, \varsigma \rangle | \iota, \varsigma \in \mathcal{X}_{\mathsf{G}}\}$ is a $*$-G-bi-ideal of $\Im$.

Definition 2.10 If $\langle \mathcal{X}_{\mathsf{G}}, \mathcal{X}_{\mathsf{G}} \rangle$ is dense in $\Im$, then we say that $\mathcal{X}_{\mathsf{G}}$ is G-*full*.

$\Im$ as an $\Im$-module is an example of G-full. Let $\mathcal{X}_{\mathsf{G}}$ and $\mathcal{Y}_{\mathsf{G}}$ be G-Hilbert $\Im$-modules and

$$B(\mathcal{X}_{\mathsf{G}}, \mathcal{Y}_{\mathsf{G}}) = \{\tau : \mathcal{X}_{\mathsf{G}} \longrightarrow \mathcal{Y}_{\mathsf{G}} : \ \exists \tau^* : \mathcal{Y}_{\mathsf{G}} \longrightarrow \mathcal{X}_{\mathsf{G}}, \ \langle \tau\iota, \varsigma \rangle = \langle \iota, \tau^*\varsigma \rangle\},$$

where τ is G-equivariant. Then τ must be $\Im$-linear, since $\langle \tau(\iota e), \varsigma \rangle = \langle \iota e, \tau^*\varsigma \rangle = e^*\langle \iota, \tau^*\varsigma \rangle = e^*\langle \tau\iota, \varsigma \rangle = \langle (\tau\iota)e, \varsigma \rangle$ for all ς. Hence $\langle \tau(\iota e) - (\tau\iota)e, \varsigma \rangle = 0$ and then $\langle \tau(\iota e) - (\tau\iota)e, \tau(\iota e) - (\tau\iota)e \rangle = 0$. It concludes that $\tau(\iota e) - (\tau\iota)e = 0$. Similarly, $\tau(\nu\iota + \varsigma) = \nu\tau\iota + \tau\varsigma$. Also, for every ι in the unit ball of $\mathcal{X}_{\mathsf{G}}$, τ must be bounded, which $\Upsilon_\iota : \mathcal{X}_{\mathsf{G}} \longrightarrow \Im$ is defined by $\Upsilon_\iota(\varsigma) = \langle \tau\iota, \varsigma \rangle = \langle \iota, \tau^*\varsigma \rangle$. Then $\|\Upsilon_\iota(\varsigma)\| = \|\iota\|\|\tau^*\varsigma\| = \|\tau^*\varsigma\|$. Therefore, $\{\|\Upsilon_\iota\| : \iota \in \mathcal{X}_1\}$ is bounded. This and $\|\tau_\iota\| = \sup_{\varsigma \in \mathcal{Y}_1} \|\langle \tau\iota, \varsigma \rangle\| = \sup_{\varsigma \in \mathcal{Y}_1} \|\Upsilon_\iota(\varsigma)\| = \|\Upsilon_\iota\|$ indicate that τ is bounded. Then we say that $B(\mathcal{X}_{\mathsf{G}}, \mathcal{Y}_{\mathsf{G}})$ is the space of adjointable G-maps and we put $B(\mathcal{X}_{\mathsf{G}}) = B(\mathcal{X}_{\mathsf{G}}, \mathcal{X}_{\mathsf{G}})$.

***Example* 2.11** Let $\mathcal{Y}_{\mathsf{G}} = \Lambda = C([0,1])$, $\mathcal{X}_{\mathsf{G}} = \{\Upsilon \longrightarrow \Lambda, \Upsilon(\frac{1}{2}) = 0\}$ and $\alpha : \mathcal{X}_{\mathsf{G}} \longrightarrow \mathcal{Y}_{\mathsf{G}}$, $\Upsilon \longrightarrow \Upsilon$ be the inclusion map. If α is adjointable and 1 represents the identity element of $\Im$, then $\langle \iota, \alpha^*(1) \rangle = \langle \alpha(\iota), 1 \rangle = \langle \iota, 1 \rangle$ for all $\iota \in \mathcal{X}_{\mathsf{G}}$. So $\alpha^*(1) = 1$, but $1 \notin \mathcal{E}_{\mathsf{G}}$ and therefore α cannot be adjointable.

$B(\mathcal{X}_{\mathsf{G}})$ is a G-C^*-algebra. If $\tau \in B(\mathcal{X}_{\mathsf{G}}, \mathcal{Y}_{\mathsf{G}})$, then $\tau^* \in B(\mathcal{Y}_{\mathsf{G}}, \mathcal{X}_{\mathsf{G}})$. If $\mathcal{Z}_{\mathsf{G}}$ is a G-Hilbert $\Im$-module and $s \in B(\mathcal{Y}_{\mathsf{G}}, \mathcal{Z}_{\mathsf{G}})$, then $\mathfrak{s}\tau \in B(\mathcal{X}_{\mathsf{G}}, \mathcal{Z}_{\mathsf{G}})$. Therefore, $B(\mathcal{X}_{\mathsf{G}})$ is a G-$*$-algebra. If $\tau_{\mathfrak{n}} \longrightarrow \tau$, then

$$\begin{aligned}
\|\tau_{\mathfrak{n}}^*\varsigma - \tau_{\mathfrak{m}}^*\varsigma\| &= \sup_{\iota \in \mathcal{X}_{\mathsf{G}}} \|\langle \iota, (\tau_{\mathfrak{n}}^* - \tau_{\mathfrak{m}}^*)\varsigma \rangle\| \\
&= \sup_{\iota \in \mathcal{X}_{\mathsf{G}}} \|\langle (\tau_{\mathfrak{n}} - \tau_{\mathfrak{m}})\iota, \varsigma \rangle\| \\
&= \sup_{\iota \in \mathcal{X}_{\mathsf{G}}} \|(\tau_{\mathfrak{n}} - \tau_{\mathfrak{m}})\iota\| \|\varsigma\| \\
&= \|\tau_{\mathfrak{n}} - \tau_{\mathfrak{m}}\| \|\varsigma\|.
\end{aligned}$$

It concludes that $\{\tau_{\mathfrak{n}}^*\varsigma\}$ converges to $\mathfrak{s}\varsigma$ (say). Hence

$$\langle \tau\iota, \varsigma \rangle = \lim_{\mathfrak{n}} \langle \tau_{\mathfrak{n}}\iota, \varsigma \rangle = \langle \iota, \lim_{\mathfrak{n}} \tau_{\mathfrak{n}}^*\varsigma \rangle = \langle \iota, \mathfrak{s}\varsigma \rangle.$$

Thus, $\tau \in B(\mathcal{X}_{\mathsf{G}})$ and $B(\mathcal{X}_{\mathsf{G}})$ is a closed subset of

$$\{\Delta : \mathcal{X}_{\mathsf{G}} \longrightarrow \mathcal{X}_{\mathsf{G}} : \ \Delta \textit{ is linear and bounded}\}.$$

Hence, $B(\mathcal{X}_{\mathsf{G}})$ is a Banach algebra. Moreover,

$$\|\tau\|^2 = \sup_{\iota \in \mathcal{X}_{\mathsf{G}}} \|\tau\iota\|^2 = \sup_{\iota \in \mathcal{X}_{\mathsf{G}}} \|\langle \tau\iota, \tau\iota \rangle\| = \sup_{\iota \in \mathcal{X}_{\mathsf{G}}} \|\langle \tau^*\tau\iota, \iota \rangle\| = \|\tau^*\tau\|.$$

Hence, $\|\tau\|^2 = \|\tau^*\tau\|$. Thus, $B(\mathcal{X}_{\mathsf{G}})$ is a G-C^*-algebra.

Let $\Im$ and ζ be G-C^*-algebras. Consider two categories. In both categories, the objects will be G-Hilbert $\Im$-ζ-bimodules. We study the sets of morphisms that include of either all bounded bimodule G-equivariant between the objects, or all bounded bimodule G-equivariant, adjointable between them. In both states, norm closed subspaces are invariant under the both module actions, that is, the subobjects will be the set of all G-Hilbert $\Im$-ζ-subbimodules. We will represent these two categories, along with the specified sets of subobjects, with $B_{\mathsf{G}}(\Im, \zeta)$ and $B^*_{\mathsf{G}}(\Im, \zeta)$, respectively. Note that any left G-Hilbert $\Im$-module is always equipped with a (right) action by $\mathbb{C}$, and any G-C^*-algebra is left G-module. Therefore, $B_{\mathsf{G}}(\Im, \mathbb{C})$ (resp. $B^*_{\mathsf{G}}(\Im, \mathbb{C})$) is just the category of bounded (resp. adjointable, bounded) G-equivariant maps and left G-Hilbert $\Im$-module.

Definition 2.12 A ξ_{G} is G-injective G-Hilbert $\Im$-module in $B_{\mathsf{G}}(\Im, \zeta)$, (resp. $B^*_{\mathsf{G}}(\Im, \zeta)$) if and only if for every G-Hilbert $\Im$-ζ-bimodule, $\mathcal{X}_{\mathsf{G}}$ of $\mathcal{Y}_{\mathsf{G}}$, and any bounded, (resp. bounded, adjointable) bimodule G-equivariant $\Omega : \mathcal{X}_{\mathsf{G}} \longrightarrow \xi_{\mathsf{G}}$, there exists a bounded (resp. bounded, adjointable), bimodule G-equivariant $\mho : \mathcal{Y}_{\mathsf{G}} \longrightarrow \xi_{\mathsf{G}}$ that extends Ω. In other words, a G-Hilbert $\Im$-ζ-bimodule ξ_{G} is G-injective Hilbert $\Im$-module iff the diagram

$$\begin{array}{ccc}
\mathcal{Y}_{\mathsf{G}} & & \\
\Delta \uparrow & & \\
\mathcal{X}_{\mathsf{G}} & \xrightarrow{\ \Omega\ } & \xi_{\mathsf{G}}
\end{array}$$

(2.1)

can be completed to a commutative diagram by an $\Im$-ζ-bimodule G-equivariant $\mho$: $\mathcal{Y}_{\mathsf{G}} \longrightarrow \xi_{\mathsf{G}}$ of the selected category.

$End^*_{\Im}(\mathcal{X}_{\mathsf{G}})$ represents G-C^*-algebra of all bounded $\Im$-linear adjointable operators on G-Hilbert $\Im$-module $\mathcal{X}_{\mathsf{G}}$ and $End_{\Im}(\mathcal{X}_{\mathsf{G}})$ represents Banach algebra of all bounded $\Im$-linear operators on $\mathcal{X}_{\mathsf{G}}$. In [16] was shown to be a G-Hilbert $\Im$-module is injective iff it as a G-Hilbert $\Im$-module, be orthogonally comparable. Can be shown that expanding the morphisms to $\Im$-ζ-bimodule maps, rather significantly change the picture, but necessitate the inclusion maps to be morphisms.

Theorem 2.13 Let $\Im$ be an arbitrary G-C^*-algebra, $\{\xi_{\mathsf{G}}, \langle \cdot, \cdot \rangle\}$ be a G-Hilbert $\Im$-module and ζ be a G-C^*-algebra that admits a G-$*$-representation in $End^*_{\Im}(\xi_{\mathsf{G}})$. Then ξ_{G} is a G-injective object in the category whose objects are the G-Hilbert $\Im$-ζ-bimodules, morphisms are either the (adjointable) bounded bimodoul G-equivariants or (adjointable) contractive, subobjects are the $\Im$-ζ-subbimodules and inclusion G-maps are adjointable. As a result, any element of those categories is G-injective Hilbert $\Im$-module.

Proof. By hypothesis, the inclusion $\Delta : \mathcal{X}_{\mathsf{G}} \hookrightarrow \mathcal{Y}_{\mathsf{G}}$ is an $\mathcal{A}.\mathcal{B}.\Im$-$\zeta$-bimodule G-equivariant, the G-equivariant Δ^* is a surjective bounded $\Im$-ζ-bimodule G-equivariant and by Theorem 15.3.8 from [22], the image set $\Delta(\mathcal{X}_{\mathsf{G}}) \subseteq \mathcal{Y}_{\mathsf{G}}$ is a subset orthogonal summand of $\mathcal{Y}_{\mathsf{G}}$. Furthermore, the G-equivariant $\Delta^{-1} : \Delta(\mathcal{X}_{\mathsf{G}}) \longrightarrow \mathcal{X}_{\mathsf{G}}$ defined by $\Delta^{-1}(\Delta(\iota)) = \iota$ for $\iota \in \mathcal{X}_{\mathsf{G}}$ is everywhere defined on $\Delta(\mathcal{X}_{\mathsf{G}}) \subseteq \mathcal{Y}_{\mathsf{G}}$ and bijective. Thus, by definition, it is bounded G-equivariant and $\Im$-ζ-bilinear. It can be developed to a map defined on $\mathcal{Y}_{\mathsf{G}}$ simply placed on the zero maps in the orthogonal complement of $\Delta(\mathcal{X}_{\mathsf{G}})$ in $\mathcal{Y}_{\mathsf{G}}$. By preserving the concept Δ^{-1} for this development, we put $\mho = \Omega \circ \Delta^{-1}$ that implies the desired development of Ω to $\mathcal{Y}_{\mathsf{G}}$. As a result, the G-Hilbert $\Im$-ζ-bimodule ξ_{G}, in the category under investigation, is automatically G-injective G-Hilbert $\Im$–module. ■

Now, for further progress in identifying the G-injective G-Hilbert $\Im$–module objects of category $B_{\mathsf{G}}(\Im, \zeta)$, we consider the results of the definition of G-injective G-Hilbert $\Im$–module.

The G-C^*-module of all bounded $\Im$-module G-maps from $\mathcal{X}'_{\mathsf{G}}$ into $\Im$, shown by $\mathcal{X}''_{\mathsf{G}}$. Let Ω_{G} be the G-map G-Hilbert $\Im$-module $\Omega_{\mathsf{G}} : \xi_{\mathsf{G}} \longrightarrow \mathcal{X}''_{\mathsf{G}}$, $\Omega_{\mathsf{G}}(\iota)(\tau) = \tau(\iota)^*$, $\iota \in \mathcal{X}_{\mathsf{G}}$, $\tau \in \mathcal{X}'_{\mathsf{G}}$.

Definition 2.14 A G-Hilbert $\Im$-module $\mathcal{X}_{\mathsf{G}}$ is called G-$\Im$-*reflexive* if Ω_{G} is a G-isomorphism of $\Im$-modules.

Let $\mathcal{X}_{\mathsf{G}}$ and $\mathcal{Y}_{\mathsf{G}}$ be a G-Hilbert $\Im$-module. For $\iota \in \mathcal{X}_{\mathsf{G}}$ and $\varsigma \in \mathcal{Y}_{\mathsf{G}}$, define $\Theta_{\iota,\varsigma} : \mathcal{Y}_{\mathsf{G}} \longrightarrow \mathcal{X}_{\mathsf{G}}$ by $\Theta_{\iota,\varsigma}(\vartheta) = \iota\langle \varsigma, \vartheta \rangle$ for $\vartheta \in \mathcal{Y}_{\mathsf{G}}$. Then

$$\Theta^*_{\iota,\varsigma} = \Theta_{\varsigma,\iota}\Theta_{\iota,\varsigma}\Theta_{\mathfrak{v},z} = \Theta_{\iota\langle \varsigma,\mathfrak{v}\rangle,z}, \quad \tau\Theta_{\iota,\varsigma} = \Theta_{\tau\iota,\varsigma} \quad (\mathfrak{v} \in \mathcal{X}_{\mathsf{G}}, \quad z \in \mathcal{Y}_{\mathsf{G}}).$$

Suppose that the set of "compact" G-operators $\mathcal{K}(\mathcal{Y}_{\mathsf{G}}, \mathcal{X}_{\mathsf{G}})$ is the closed linear span of $\{\Theta_{\iota,\varsigma} | \iota \in \mathcal{X}_{\mathsf{G}}, \varsigma \in \mathcal{Y}_{\mathsf{G}}\}$. Let $\mathcal{X}_{\mathsf{G}}$ be a G-Hilbert $\Im$-module and $\iota \in \mathcal{X}_{\mathsf{G}}$.

$$\Theta_{e,\vartheta}(\varsigma) = e\langle \vartheta, \varsigma \rangle = \langle \vartheta e^*, \varsigma \rangle = (\vartheta e^*)^\wedge(\varsigma) \qquad (e \in \Im) \quad (*)$$

and so $\hat{\iota} \in \mathcal{K}(\mathcal{X}_{\mathsf{G}},)$, where ι is of the form ϑe^*. Since $\mathfrak{M}_{\mathsf{G}}\Im$ is dense in $\mathfrak{M}_{\mathsf{G}}$ for every $\iota \in \mathfrak{M}_{\mathsf{G}}$ there exists a sequence $\{\iota_n\}$ in $\mathfrak{M}_{\mathsf{G}}\Im$ so that $\lim_n \iota_n = \iota$. But $\mathfrak{M}_{\mathsf{G}} \longrightarrow L(\xi_{\mathsf{G}}, \Im)$, $\iota \longrightarrow \hat{\iota}$ is continuous (isometry). Therefore, since $\mathcal{K}(\mathfrak{M}_{\mathsf{G}}, \Im)$ is closed in $L(\mathfrak{M}_{\mathsf{G}}, \Im)$, then $\hat{\iota} = \lim_n \hat{\iota}_n \in \mathcal{K}(\mathfrak{M}_{\mathsf{G}}, \Im)$.

Condition $(*)$ shows that each element of $\mathcal{K}(\mathfrak{M}_{\mathsf{G}}, \Im)$ is of the from $\hat{\iota} = \langle \iota, \cdot \rangle$ a Riesz theorem for G-Hilbert $\Im$-modules, for some $\iota \in \mathcal{X}_{\mathsf{G}}$.

Lemma 2.15 Let $\Im$ and ζ be G-C^*-algebras and $\{\xi_\mathsf{G}, \langle\cdot,\cdot\rangle\}$ be a G-injective Hilbert $\Im$-ζ-bimodule in one of the two categories under investigation. If $\xi_\mathsf{G} \subseteq \mathcal{Y}_\mathsf{G}$ is an $\Im$-ζ-subbimodule, then the Hilbert $\Im$-ζ-bimodule ξ_G is a topological summand of the Hilbert $\Im$-ζ-bimodule $\mathcal{Y}_\mathsf{G}$. Furthermore, ξ_G as a G-Hilbert $\Im$-module is G-$\Im$-reflexive and when ξ_G is a G-$\mathcal{H}\Im$-submodule of another G-Hilbert $\Im$-module $\mathcal{X}_\mathsf{G}$ with $\xi_\mathsf{G}^\perp = \{0\}$, then $\xi_\mathsf{G} = \xi_\mathsf{G}^{\perp\perp}$ in $\mathcal{X}_\mathsf{G}$.

Proof. According to the definition of G-injectivity, let $\mathcal{X}_\mathsf{G} = \xi_\mathsf{G}$, $\Delta : \xi_\mathsf{G} \hookrightarrow \mathcal{Y}_\mathsf{G}$ indicates inclusion and $\Omega = id_{\xi_\mathsf{G}}$. By assumption there is an $\Im$-ζ-bimodule G-equivariant $\mho : \mathcal{Y}_\mathsf{G} \longrightarrow \xi_\mathsf{G}$ so that $\mho \circ \Delta = id_{\xi_\mathsf{G}}$. By Lemma 3.1.8(2) from [12], we have the set identities $\mathcal{Y}_\mathsf{G} = \mho^{-1}(\xi_\mathsf{G}) = Im(\Delta) + Ker(\mho)$ and $\{0\} = \Delta(Ker(id_{\xi_\mathsf{G}})) = Im(\Delta) \cap Ker(\mho)$. Thus, $\mathcal{Y}_\mathsf{G} = \Delta(\xi_\mathsf{G}) + Ker(\mho)$, that is, ξ_G with topological complement $Ker(\mho)$ there, must be a topological summand.

To extract the G-$\Im$-reflexivity of G-injective G-Hilbert $\Im$-module, consider the definition of G-injectivity with $\mathcal{X}_\mathsf{G} = \xi_\mathsf{G}$, $\mathcal{Y}_\mathsf{G} = \xi_\mathsf{G}''$ and $\Omega = id_{\xi_\mathsf{G}}$. By Proposition 2.1 from [6], the $\Im$-$\mathcal{V.I.P}$ on ξ_G expands to an $\Im$-$\mathcal{V.I.P}$ on its $\Im$-bidual Banach $\Im$-module ξ_G''. Furthermore, since any bounded G-module operator on ξ_G expands to a bounded G-module operator on ξ_G'' in a unique method [21], the G-$*$-representation of ζ on ξ_G turns into a G-$*$-representation of ζ on ξ_G'' via the canonical isometric embedding $\xi_\mathsf{G} \subseteq \xi_\mathsf{G}''$. However, the embedded copy of ξ_G is a topological summand of ξ_G'' iff both they coincide. Actually, since by [20, 21], we have $\xi_\mathsf{G} \subseteq \xi_\mathsf{G}'' \subseteq \xi_\mathsf{G}'$ as the chain of isometric embedding, the supposition of ξ_G being a non-trivial topological summand of ξ_G'' leads to the non-uniqueness of the representation of the zero maps on ξ_G in ξ_G', which contradict the definition of this set. The statement above is a result of the G-$\Im$-reflexivity, G-injectivity of ξ_G and Lemma 3.1 from [4]. ■

By Lemma 2.15, if an object is complemented in any object that it is a subobject, then it is injective. According to [8], this holds for G-injective.

Inspired by [8], we have the following proposition for G-Hilbert $\Im$-module with the fact that unital G-C^*-algebras $\Im$, as G-Hilbert $\Im$-modules, are ever orthogonally comparable. The same is true for some non-unital G-C^*-algebras $\Im$ with the condition $\Re_G(\Im) = L\Re_G(\Im)$.

Proposition 2.16 Let $\Im$ be a G-C^*-algebra and $\Im^{\mathfrak{N}}$ be the G-Hilbert $\Im$-module of all $\mathfrak{N}$-tuples of elements of $\Im$ for $\mathfrak{N} \in \mathbb{N}$. The following conditions are equivalent:

(i) $\Im^{\mathfrak{N}}$ is G-injective in $B_\mathsf{G}(\Im, \mathbb{C})$ for $\mathfrak{N} \in \mathbb{N}$;
(ii) $\Im^{\mathfrak{N}}$ is G-injective in $B_\mathsf{G}(\Im, \mathbb{C})$ for any $\mathfrak{N} \in \mathbb{N}$;
(iii) $\Im$ is G-injective in $B_\mathsf{G}(\Im, \mathbb{C})$;
(iv) $\Re_G(\Im)$ is a G-monotone complete (G-m.c.) G-C^*-algebra.

Proof. Let $\mathcal{X}_\mathsf{G} \subseteq \mathcal{Y}_\mathsf{G}$ be a subobject and $\Omega : \mathcal{X}_\mathsf{G} \longrightarrow \Im^{\mathfrak{N}}$ be a bounded $\Im$-module map. We have $\Omega = (\Omega_1, ..., \Omega_{\mathfrak{N}})$, where $\Omega_\alpha : \mathcal{X}_\mathsf{G} \longrightarrow \Im$ are bounded $\Im$-module maps. $\mho : \mathcal{Y}_\mathsf{G} \longrightarrow \Im^{\mathfrak{N}}$ that extends Ω exists iff there are bounded $\Im$-module maps $\mho_\alpha : \mathcal{Y}_\mathsf{G} \longrightarrow \Im_{(\alpha)}$ coinciding with Ω_α on $\mathcal{X}_\mathsf{G}$, the index (α) denotes α-th coordinate of $\Im^{\mathfrak{N}}$. In this case, $(1), (2)$ and (3) are equivalence. We also see that such an expansion exists iff a generalized Hahn-Banach theorem is credible for arbitrary pairs of G-Hilbert $\Im$-modules $\mathcal{X}_\mathsf{G} \subseteq \mathcal{Y}_\mathsf{G}$ and arbitrary bounded $\Im$-linear functionals $r : \mathcal{X}_\mathsf{G} \longrightarrow \Im$. By, Theorem 2 from [4], this happens iff $\Re_G(\Im)$ is G-m.c. ■

Definition 2.17 A G-Hilbert $\Im$-module $\mathcal{X}_\mathsf{G}$ is G-*m.c.* if underlying Hilbert $\Im$-module $\Im$ is a m.c.

Definition 2.18 A G-Hilbert $\Im$-module $\mathcal{X}_{\mathsf{G}}$ is G-*simple* when it has no non-trivial 2-sided G-invariant ideals.

Proposition 2.19 Let $\Im$ be a unital G-C^*-algebra. If there is every G-full $\Im$-module that in $B_{\mathsf{G}}(\Im,\mathbb{C})$ is G-injective, then $\Im$ is G-m.c. Therefore, if $\Im$ is G-simple, not G-m.c and unital, then there exists no non-zero injective G-Hilbert $\Im$-module in $B_{\mathsf{G}}(\Im,\mathbb{C})$.

Proof. Let ξ_{G} be an injective G-full $\Im$-module. By Lemma 2.4.3 from [18], there is a finite positive integer $\mathfrak{n}$ and a subset of elements $\{\mathfrak{e}_1,...,\mathfrak{e}_{\mathfrak{n}}\}$ of E so that $\sum_{\alpha=1}^{\mathfrak{n}}\langle\mathfrak{e}_\alpha,\mathfrak{e}_\alpha\rangle = 1_{\Im}$ since the G-Hilbert $\Im$-module ξ_{G} is full. Note that $\xi_{\mathsf{G}}^{\mathfrak{n}}$ is G-injective Hilbert $\Im$-module whenever ξ_{G} is G-injective G-Hilbert $\Im$-module and $\mathfrak{n}$ is a finite. Hence, one has an isometric left $\Im$-module G-equivariant, $\Omega : \Im \longrightarrow \xi_{\mathsf{G}}^{\mathfrak{n}}$ defined by $\Omega(e) = \sum_{\alpha}^{\mathfrak{n}} e\mathfrak{e}_{\alpha=1}$. Because, $\Im$ is orthogonally comparable by, Proposition 6.2 and Theorem 6.3 from [6], there is a bounded $\Im$-module G-equivariant, $\mho : \xi_{\mathsf{G}}^{\mathfrak{n}} \longrightarrow \Omega(\Im)$. So, we simply conclude that $\Omega(\Im)$ in $B_{\mathsf{G}}(\Im,\mathbb{C})$ is G-injective G-Hilbert $\Im$-module. Therefore, by Proposition 3.3 from [8], $\Re_G(\Im) = \Im$ is G-m.c. For the final claim, since $\Im$ is G-simple and unital, each non-zero G-Hilbert $\Im$-module is G-full, since the range of its $\Im$-$\mathcal{V.I.P}$. is norm-closed 2-sided ideal in $\Im$. ■

In the following example, we show that when $\Im$ is unital, not G-simple and G-m.c., it may be G-injective in $B_{\mathsf{G}}(\Im,\mathbb{C})$. However, we show that when $\Im$ is unital but not G-m.c., in this case, there exist not enough G-injectives so that in a G-injection, any G-Hilbert $\Im$-module can be embedded.

***Example* 2.20** Let $\Im = \mathbb{C} \oplus \zeta$, where ζ be a unital G-C^*-algebra which is not G-m.c. Therefore, $\Im$ is unital and not G-m.c. Note that any G-Hilbert space $\mathcal{K}_{\mathsf{G}}$ is a (non-full) G-Hilbert $\Im$-module with $(0 \oplus \zeta)\mathcal{K} = 0$. We assert $\mathcal{K}_{\mathsf{G}}$ is an injective G-Hilbert $\Im$-module in $B_{\mathsf{G}}(\Im,\mathbb{C})$. Indeed, ξ_{G} is a G-Hilbert $\Im$-module and $\mathcal{H}_{\mathsf{G}} = (\mathbb{C}\oplus 0)\xi_{\mathsf{G}}$ and $\mathcal{Z}_{\mathsf{G}} = (0\oplus\zeta)\xi_{\mathsf{G}}$ are its submodules, then $\xi_{\mathsf{G}} = \mathcal{H}_{\mathsf{G}} \oplus \mathcal{Z}_{\mathsf{G}}$ is an orthogonal direct sum decomposition. In addition, each $\Im$-module G-equivariant from ξ_{G} into $\mathcal{K}_{\mathsf{G}}$ is zero on $\mathcal{Z}_{\mathsf{G}}$, and it is a linear map on $\mathcal{H}_{\mathsf{G}}$. Given the fact that $\mathcal{K}_{\mathsf{G}}$ in $B_{\mathsf{G}}(\Im,\mathbb{C})$ is G-injective, it is easy to conclude that it is G-injective in the category of G-Hilbert space and bounded linear G-equivariant maps.

The following theorem shows that G-C^*-algebras for which any G-Hilbert $\Im$-module is G-injective in $B_{\mathsf{G}}(\Im,\mathbb{C})$.

Theorem 2.21 Let $\Im$ be a compact operators G-C^*-algebra on some G-Hilbert space. Let $\{\xi_{\mathsf{G}},\langle\cdot,\cdot\rangle\}$ be a G-Hilbert $\Im$-module and ζ be another G-C^*-algebra admitting a G-$*$-representation on ξ_{G}. Then ξ_{G} is a G-injective G-Hilbert $\Im$-module object in $B_{\mathsf{G}}(\Im,\zeta)$. Contrariwise, suppose that $\Im$ is a G-C^*-algebra. If any G-Hilbert $\Im$-module is G-injective in $B_{\mathsf{G}}(\Im,\mathbb{C})$, then $\Im$ is $*$-isomorphic to a compact operators G-C^*-algebra on some G-Hilbert space.

Proof. By Theorem 2.1 and Proposition 2.2 from [8], we observe any bounded $\Im$-linear map between G-Hilbert $\Im$-modules on a G-C^*-algebra $\Im$ of type $\varpi_0 - \sum_\alpha \bigoplus \mathcal{K}(\mathcal{H}_\alpha)$ has an adjoint. Therefore, each inclusion map is adjointable and according to Theorem 2.13, the first claim hold.

To illustrate the reverse, consider a maximal G-invariant left-sided $\mathfrak{I}_{\mathsf{G}}$ of the G-C^*-algebra $\Im$. Put $\xi_{\mathsf{G}} = \mathcal{X}_{\mathsf{G}} = \mathfrak{I}_{\mathsf{G}}$, $\mathcal{Y}_{\mathsf{G}} = \Im$, $\Omega = id_{\mathfrak{I}_{\mathsf{G}}}$ and in the definition of G-injectivity, take $\Im$-linear embedding of $\mathfrak{I}_{\mathsf{G}}$ into $\Im$. Then the existence of an $\Im$-module G-equivaraint map $\mho : \Im \longrightarrow \mathfrak{I}_{\mathsf{G}}$ expanding Ω is G-equivariant to the existence of $\mathcal{P}_{\mathfrak{I}_{\mathsf{G}}} \in \Re_{\mathsf{G}}(\Im)$ as an orthogonal projection so that $\mathfrak{I}_{\mathsf{G}} = \Im_{\mathcal{P}\mathcal{I}_{\mathsf{G}}}$. Therefore, by Proposition 2.2 of [6], the

G-C^*-algebra $\Im$ has the form ϖ_0-$\sum_\alpha \bigoplus \mathcal{K}(\mathcal{H}_\alpha)$. ■

Theorem 2.22 Let $\Im$ be G-m.c. G-C^*-algebra and $\{\mathcal{X}_\mathsf{G}, \langle\cdot,\cdot\rangle\}$ be a G-Hilbert $\Im$-module. Let ζ be a G-C^*-algebra admitting a G-$*$-representation in $End^*_\Im(\mathcal{X}_\mathsf{G})$. Then $\mathcal{X}_\mathsf{G}$ is G-injective in $B_\mathsf{G}(\Im,\zeta)$ iff $\mathcal{X}_\mathsf{G}$, as G-Hilbert $\Im$-module, is G-self-dual.

Proof. Assume that $\mathcal{X}_\mathsf{G}$ is G-injective in $B_\mathsf{G}(\Im,\zeta)$, and let the canonical isometric embedding of $\mathcal{X}_\mathsf{G}$ into its $\Im$-dual G-Banach $\Im$-module $\mathcal{X}'_\mathsf{G}$. By Theorem 4.7 of [7], the $\Im$-$\mathcal{V}.\mathcal{I}.\mathcal{P}.$ on $\mathcal{X}_\mathsf{G}$ can be extended to an $\Im$-$\mathcal{V}.\mathcal{I}.\mathcal{P}.$ on $\mathcal{X}'_\mathsf{G}$ in a manner compatible with $\mathcal{X}_\mathsf{G} \hookrightarrow \mathcal{X}'_\mathsf{G}$, as the canonical embedding. The G-$*$-representation of ζ on $\mathcal{X}_\mathsf{G}$ on the right with the canonical embedding induces a G-$*$-representation of ζ on $\mathcal{X}'_\mathsf{G}$, because any bounded module operator on $\mathcal{X}_\mathsf{G}$ expands to a unique bounded module operator on $\mathcal{X}'_\mathsf{G}$ ([20]). Eventually, the copy of $\mathcal{X}_\mathsf{G}$ in $\mathcal{X}'_\mathsf{G}$ is a topological summand of them iff both the sets coincide, because otherwise, the zero functional on $\mathcal{X}_\mathsf{G}$ accepts multiple G-$*$-representations in $\mathcal{X}'_\mathsf{G}$. Thus, $\mathcal{X}_\mathsf{G}$ should be G-self-dual. To create the inverse notion, consider the following diagram with a bounded $\Im$-ζ-bilinear map Ω and an isometric $\Im$-ζ-bilinear embedding Δ:

$$\begin{array}{ccc} \mathcal{L}_\mathsf{G} & & \\ \Delta \uparrow & & \\ \mathcal{K}_\mathsf{G} & \xrightarrow{\ \Omega\ } & \mathcal{X}_\mathsf{G} \end{array}$$

(2.2)

In this diagram, Ω can be replaced with $\frac{\Omega}{||\Omega||}$, a contractive map. Then, by Theorem 2.2 of [16], there is a bounded $\Im$-linear G-equivariant $\mho : \mathcal{L}_\mathsf{G} \longrightarrow \mathcal{X}_\mathsf{G}$ so that $(\frac{\Omega}{||\Omega||}) = \mho \circ \Delta$. The G-equivariant $\mho$ is also ζ-linear, because Ω and Δ are ζ-linear. By multiplying the constant $||\Omega||$ on both sides, we get the map $||\Omega||\mho$, which completes the above diagram to a commutative one. Therefore, in the selected category, $\mathcal{X}_\mathsf{G}$ is G-injective G-Hilbert $\Im$-module. ■

When the G-C^*-algebra of coefficients of a G-Hilbert $\Im$-module ξ_G is not a unital G-C^*-algebra and the G-Hilbert $\Im$-module ξ_G is full, that is, its G-C^*-algebra of coefficients $\Im$ is the minimal admissible one, so we can consider G-Hilbert $\Im$-module ξ_G on larger G-C^*-algebras, logically, as an ideal, on G-C^*-algebras containing the G-C^*-algebra of coefficients $\Im$ and belonging to the multiplier algebra $\Re_G(\Im)$ of $\Im$. The construction introduced in [1] gives us the opportunity to institute the necessary terms on those G-Hilbert $\Im$-modules to be G-injective in the G-Hilbert $\Re_G(\Im)$-module category.

Suppose that $\Im$ equipped with an $\Im$-$\mathcal{V}.\mathcal{I}.\mathcal{P}.$ $\langle\cdot,\cdot\rangle$ is a (non-unital) G-C^*-algebra and $\mathcal{X}_\mathsf{G}$ is a full G-Hilbert $\Im$-module. If $\Im$ is equipped with the standard $\Im$-$\mathcal{V}.\mathcal{I}.\mathcal{P}.$ defined by the rule $\langle e, \flat\rangle_\Im = e\flat^*$, then $\mathcal{X}_{\mathsf{G}_d}$ represents the G-Hilbert $\Re_G(\Im)$-module $End^*_\Im(\Im, \mathcal{X}_\mathsf{G})$ of all $\Im$-linear maps from $\Im$ to $\mathcal{X}_\mathsf{G}$. The $\Re_G(\Im)$-$\mathcal{V}.\mathcal{I}.\mathcal{P}.$ on $\mathcal{X}_{\mathsf{G}_d}$ is defined by $\langle \mathfrak{r}, \mathfrak{s}\rangle = \mathfrak{s}^* \circ \mathfrak{r}$ for every $\mathfrak{r}, \mathfrak{s} \in \mathcal{X}_{\mathsf{G}_d}$. One of the significant features of this structure is the existence of an isometric embedding Γ of $\mathcal{X}_\mathsf{G}$ into $\mathcal{X}_{\mathsf{G}_d}$. It is defined by $\Gamma(\iota)(e) = e\iota$ for every $e \in \Im$, and $\iota \in \mathcal{X}_\mathsf{G}$. The image $\Gamma(\mathcal{X}_\mathsf{G}) \subseteq \mathcal{X}_{\mathsf{G}_d}$ coincides with the subset $\Im \cdot \mathcal{X}_{\mathsf{G}_d}$. Note that the structure depends on the unitary equivalence classes of both the $\Im$-$\mathcal{V}.\mathcal{I}.\mathcal{P}.$ on $\Im$ and $\mathcal{X}_\mathsf{G}$. In addition, $\mathcal{X}_{\mathsf{G}_d}$ can be specified topologically as the linear hull of the completion of the unit ball of $\mathcal{X}_\mathsf{G}$ with respect to the strict topology, which is induced by the set of semi-norms $\{\|\langle\cdot,\iota\rangle\|_\Im : \iota \in \mathcal{X}_\mathsf{G}\} \bigcup \{\|\varsigma \cdot\|_{\mathcal{X}_\mathsf{G}} : \varsigma \in \Im\}$. Thus, $\mathcal{X}_{\mathsf{G}_d} \equiv \mathcal{X}_{\mathsf{G}_{dd}}$, that is, the described extension is a closure operation for every G-Hilbert $\Im$-module $\mathcal{X}_\mathsf{G}$. Eventually, $(\mathcal{X}_\mathsf{G} \bigoplus \mathcal{Y}_\mathsf{G})_d = \mathcal{X}_{\mathsf{G}_d} \bigoplus \mathcal{Y}_{\mathsf{G}_d}$, that is, the closure operation obeys orthogonal decompositions, and the sets of all adjointable bounded module maps over $\mathcal{X}_{\mathsf{G}}$ and $\mathcal{X}_{\mathsf{G}_d}$ are always G-$*$-isomorphic, by limiting operators over $\mathcal{X}_{\mathsf{G}_d}$ in the $\Re_G(\Im)$-invariant subset $\Gamma(\mathcal{X}_{\mathsf{G}}) \subseteq \mathcal{X}_{\mathsf{G}_d}$ which is isometrically isomorphic in $\mathcal{X}_{\mathsf{G}}$. See [1] for more information.

Proposition 2.23 Suppose that $\Im$ is a non-unital G-C^*-algebra and ξ_{G} a full G-Hilbert $\Im$-module. Let ζ be another G-C^*-algebra that admits a G-$*$-representation on ξ_{G}. If ξ_{G} is G-injective in $B_{\mathsf{G}}(\Re_G(\Im), \zeta)$, then $\xi_{\mathsf{G}} \equiv \xi_d$.

Proof. The isomorphism of the sets of all adjointable bounded module maps on both the G-Hilbert $\Im$-module and its strict closure turns the strict closure into a $\Re_G(\Im)$-ζ-bimodule, too. Thus, ξ_d is included in the same category under consideration. By definition of G-injectivity, set $\mathcal{X}_{\mathsf{G}} = \xi_{\mathsf{G}}$, $\mathcal{Y}_{\mathsf{G}} = \xi_d$, we specify ξ_{G} with its image, $\Gamma(\xi_{\mathsf{G}}) \subseteq \xi_d$ and $\Omega = id_{\xi_{\mathsf{G}}}$. Because $\Gamma(E)$ is G-injective, there exists a bounded $\Re_G(\Im)$-ζ-bimodule map, $\mho : \xi_d \longrightarrow \Gamma(\xi_{\mathsf{G}})$ expanding the identity map. In addition, by [5, Theorem 6.4], we have the canonical isometric inclusions $\xi_{\mathsf{G}} \hookrightarrow \xi_d \hookrightarrow \xi_{\mathsf{G}}^{'}$, and the $\Re_G(\Im)$-linear bounded identity operator over ξ_{G} has a unique extension to the identity operator over ξ' preserving the norm. Also, the identity operator over ξ_{G} expands uniquely to the identity operator over ξ_d. Hence, $\xi_{\mathsf{G}} \equiv \xi_d$. ■

One has $\xi_{\mathsf{G}} \equiv \xi_d$ for a G-Hilbert $\Im$-module ξ_{G} provided that either the G-C^*-algebra $\Im$ of coefficients or the G-C^*-algebra $\mathcal{K}_{\Im}(\xi_{\mathsf{G}})$ is unital.

Corollary 2.24 Let $\Im$ be a G-C^*-algebra. If $\Im$ is G-injective in the category $B_{\mathsf{G}}(\Re_G(\Im), \mathbb{C})$, then $\Im$ should be unital (*that is,* $\Im = \Re_G(\Im)$) and G-m.c. In addition, if $\Im^{\mathfrak{N}}$ is G-injective for some $\mathfrak{N} \in \mathbb{N}$, then $\Im^{\mathfrak{N}}$ is G-injective for every $\mathfrak{N} \in \mathbb{N}$, in particular, for $\mathfrak{N} = 1$.

Proof. This is a result of the reality that $\Im_d = \Re_G(\Im)$ and $(\Im^{\mathfrak{N}})_d = \Re_G(\Im)^{\mathfrak{N}}$ for every $\mathfrak{N} \in \mathbb{N}$ according to the structure. Thus, $\Im = \Re_G(\Im)$ with the prior proposition. Moreover, Proposition 3.3 and Theorem 3.9 enforce $\Im$ to be G-m.c. ■

3. G-projective G-Hilbert $\Im$-modules

As in the pervious section, let $\Im$ and ζ be two fixed G-C^*-algebras, and suppose $B_{\mathsf{G}}(\Im, \zeta)$ (resp. $B^*_{\mathsf{G}}(\Im, \zeta)$) is the sets of all bounded bimodule G-equivariant maps between the G-Hilbert $\Im$-ζ-bimodules, (resp. all adjointable, bounded bimodule G-equivariants between them).

Definition 3.1 By definition, a G-Hilbert $\Im$-ζ-bimodule $\mathcal{Z}_{\mathsf{G}}$ is G-*projective* G-*Hilbert* $\Im$-*module* in $B_{\mathsf{G}}(\Im, \zeta)$ (resp. $B^*_{\mathsf{G}}(\Im, \zeta)$) iff the following diagram

$$\begin{array}{ccc} & & \mathcal{Z}_{\mathsf{G}} \\ & & \big\downarrow \Delta \\ \mathcal{Y}_{\mathsf{G}} & \xrightarrow{\ \Omega\ } & \mathcal{X}_{\mathsf{G}} \end{array} \tag{3.1}$$

where Δ is a surjective $\Im$-ζ-bimodule (resp. adjointable) morphism and Ω is a (resp. adjointable) $\Im$-ζ-bimodule morphism among G-Hilbert $\Im$-ζ-bimodules, can be completed to a commutative diagram by an $\Im$-ζ-bimodule (resp. adjointable) morphism $\mho : \mathcal{Z}_{\mathsf{G}} \longrightarrow \mathcal{Y}_{\mathsf{G}}$.

The proof is relatively easy (and we do) that any object is G-projective in $B^*_{\mathsf{G}}(\Im, \zeta)$. We do not know that this holds for $B_{\mathsf{G}}(\Im, \zeta)$, but we will identify a family of G-C^*-algebras that holds in $B_{\mathsf{G}}(\Im, \zeta)$.

Theorem 3.2 Let $\Im$ be an arbitrary G-C^*-algebra, $\{\mathcal{F}_{\mathsf{G}}, \langle\cdot,\cdot\rangle\}$ a G-Hilbert $\Im$-module and ζ another G-C^*-algebra that admits a G-$*$-representation in $End^*_{\Im}(\mathcal{Z}_{\mathsf{G}})$. Then $\mathcal{Z}_{\mathsf{G}}$ is a G-projective object in $B^*_{\mathsf{G}}(\Im, \zeta)$.

Proof. Let $\Delta : \mathcal{Y}_{\mathsf{G}} \longrightarrow \mathcal{X}_{\mathsf{G}}$, where $\mathcal{X}_{\mathsf{G}}$ and $\mathcal{Y}_{\mathsf{G}}$ are G-Hilbert $\Im$-ζ-bimodules, be an adjointable surjective bounded $\Im$-ζ-bimodule map. By definition, since Δ has closed range, the range of $\Delta^* : \mathcal{X}_{\mathsf{G}} \longrightarrow \mathcal{Y}_{\mathsf{G}}$ in $\mathcal{Y}_{\mathsf{G}}$ is closed and an orthogonal summand by Proposition 1.1 of [8]. Since Δ is surjective, Δ^* should be G-injective, and one has the decomposition $\mathcal{Y}_{\mathsf{G}} = \Delta^*(\mathcal{X}_{\mathsf{G}}) \bigoplus Ker(\Delta)$. By construction, both these orthogonal summands are $\Im$-ζ-invariant. Each element $\iota \in \mathcal{X}_{\mathsf{G}}$ has a unique G-pre-image $\Delta^{-1}(\iota) \in \Delta^*(\mathcal{X}_{\mathsf{G}})$. The operator $\Delta^{-1} : \mathcal{X}_{\mathsf{G}} \longrightarrow \Delta^*(\mathcal{X}_{\mathsf{G}}) \subseteq \mathcal{Y}_{\mathsf{G}}$ defined as this is defined everywhere on $\mathcal{X}_{\mathsf{G}}$ and has a closed range, so it is bounded. In addition, it is $\Im$-ζ-linear. If $\mho : \mathcal{Z}_{\mathsf{G}} \longrightarrow \mathcal{Y}_{\mathsf{G}}$ is defined with the rule $\mho(\Upsilon) = \Delta^{-1}(\Omega(z)) \in \Delta^*(\mathcal{X}_{\mathsf{G}}) \subseteq \mathcal{Y}_{\mathsf{G}}$ for $z \in \mathcal{Z}_{\mathsf{G}}$, then we obtain a bounded $\Im$-ζ-bilinear map $\mho$ that completes diagram (3.1) to the commutative diagram. ■

The following theorem shows a way to find non G-projective G-Hilbert $\Im$-module if such G-Hilbert $\Im$-modules are available.

Theorem 3.3 Let $\Im$ and ζ be arbitrary G-C^*-algebras and $\{\mathcal{Z}_{\mathsf{G}}, \langle\cdot,\cdot\rangle\}$ be a G-Hilbert $\Im$-ζ-bimodule. Then the following equivalent conditions hold:

(i) $\mathcal{Z}_{\mathsf{G}}$ is G-projective in $B_{\mathsf{G}}(\Im, \zeta)$;
(ii) each surjective, bounded bimodule G-equivariant, $\Delta : \mathcal{Y}_{\mathsf{G}} \longrightarrow \mathcal{Z}_{\mathsf{G}}$ has a right inverse, $\mathcal{Q} : \mathcal{Z}_{\mathsf{G}} \longrightarrow \mathcal{X}_{\mathsf{G}}$ which is a bounded bimodule G-equivariant;
(iii) if $\Delta : \mathcal{Y}_{\mathsf{G}} \longrightarrow \mathcal{Z}_{\mathsf{G}}$ is a surjective, bounded bimodule G-equivariant, then $Ker(\Delta)$ is a topological bimodule summand.

Proof. (ii) = (iii) is clear. We show that (i) $\Rightarrow$ (iii). Suppose that $\mathcal{Z}_{\mathsf{G}}$ is G-projective G-Hilbert $\Im$-module. By definition, there exists an $\Im$-ζ-bimodule G-equivariant map $\mho : \mathcal{Z}_{\mathsf{G}} \longrightarrow \mathcal{Y}_{\mathsf{G}}$ so that $\Delta \circ \mho = id_{\mathcal{Z}_{\mathsf{G}}}$. Using [12, Lemma 3.1.8(2)], we have the set identities $\mathcal{Y}_{\mathsf{G}} = \Delta^{-1}(\mathcal{Z}_{\mathsf{G}}) = Im(\mho) + Ker(\Delta)$ and $\{0\} = \mho(Ker(id_{\mathcal{Z}_{\mathsf{G}}})) = Im(\mho) \bigcap Ker(\Delta)$. Hence, the G-Hilbert $\Im$-ζ-bimodule $Ker(\Delta) \subseteq \mathcal{Y}_{\mathsf{G}}$ is a topological summand with topological complement $Im(\mho)$ there, that is, $\mathcal{Y}_{\mathsf{G}} = \mho(\mathcal{Z}_{\mathsf{G}}) + Ker(\Delta)$. The invariance of $Ker(\Delta)$ under the action of ζ is due to the $\Im$-ζ-bilinearity of the operator Δ.

To show (iii) $\Rightarrow$ (i), suppose that (ii) holds and according to the diagram (2.1), let $\mathcal{L}_{\mathsf{G}} = \{(z, \varsigma) \in \mathcal{Z}_{\mathsf{G}} \bigoplus \mathcal{Y}_{\mathsf{G}} : \Omega(z) = \Delta(\varsigma)\}$, that is an $\Im$-ζ-submodule of $\mathcal{Z}_{\mathsf{G}} \bigoplus \mathcal{Y}_{\mathsf{G}}$. $\mathfrak{Q} : \mathcal{L}_{\mathsf{G}} \longrightarrow \mathcal{Z}_{\mathsf{G}}$ defined by $\mathfrak{Q}((z, \varsigma)) = z$ is a bounded bimodule surjection G-equivariant and so has a right inverse, $\mathcal{Q} : \mathcal{Z}_{\mathsf{G}} \longrightarrow \mathcal{L}_{\mathsf{G}}$. Assume that $\mathfrak{P} : \mathcal{L}_{\mathsf{G}} \longrightarrow \mathcal{Y}_{\mathsf{G}}$ is defined by $\mathfrak{P}((z, \varsigma)) = \varsigma$, therefore $\mathfrak{P}$ is a bounded bimodule G-equivariant map and $\mho = \mathfrak{P} \circ \mathcal{Q} : \mathcal{Q}_{\mathsf{G}} \longrightarrow \mathcal{Y}_{\mathsf{G}}$ is the desired lifting of Ω. ■

Theorem 3.4 Let $\Im$ be a G-C^*-algebra of type $\varpi_0 - \sum_\alpha \bigoplus \mathcal{K}(\mathcal{H}_\alpha)$, that is, a compact operators G-C^*-algebra on a G-Hilbert space. Let $\{\mathcal{Z}_{\mathsf{G}}, \langle\cdot,\cdot\rangle\}$ be a G-Hilbert $\Im$-module and ζ be another G-C^*-algebra admitting a G-$*$-representation in $End^*_{\Im}(\mathcal{Z}_{\mathsf{G}})$. Then $\mathcal{Z}_{\mathsf{G}}$ is a G-projective G-Hilbert $\Im$-module in $B_{\mathsf{G}}(\Im, \zeta)$.

Proof. By Theorem 2.1 and Proposition 2.2 of [8], we can complete the proof. ■

Corollary 3.5 Let $\Im$ be G-C^*-algebra. Each G-Hilbert $\Im$-module is G-projective G-Hilbert $\Im$-module in the category $B_{\mathsf{G}}(\Im, \mathbb{C})$ iff the kernel of any surjective bounded

$\Im$-linear G-equivariant maps between G-Hilbert $\Im$-modules is a topological summand.

Proof. Apply condition (iii) of 3.3. ■

Now, we investigate the relationship between G-projectivity of G-Hilbert $\Im$-modules in the case of unital G-C^*-algebras and Kasparovs stabilization theorem.

Proposition 3.6 Let $\Im$ be a unital G-C^*-algebra. Then, for any $\mathfrak{N} \in \mathbb{N}$ the G-Hilbert $\Im$-module, $\Im^{\mathfrak{N}}$ is G-projective G-Hilbert $\Im$-module in $B_{\mathsf{G}}(\Im, \mathbb{C})$.

Proof. Let $\mathcal{Y}_{\mathsf{G}}$ be a G-Hilbert $\Im$-module and $\Delta : \mathcal{Y}_{\mathsf{G}} \longrightarrow \Im^{\mathfrak{N}}$ a bounded surjective G-equivariant, we select elements $\iota_\beta \in \mathcal{Y}_{\mathsf{G}}$ so that $\Delta(\iota_\beta) = \nu_j$, where ν_j represents the element that in the β-th component, is $1_\Im$ and elsewhere 0. The mapping $\mathfrak{Q} : \Im^{\mathfrak{N}} \longrightarrow \mathcal{Y}_{\mathsf{G}}$ defined by $\mathfrak{Q}((e_1, ..., e_{\mathfrak{N}})) = \Sigma_\beta e_\beta \iota_\beta$ is a right inverse for Δ. ■

The respective infinite dimensional version of $\Im^{\mathfrak{N}}$ is

$$\mathfrak{l}^2(\Im) = \{(e_1, e_2, ...) : \Sigma_{\mathfrak{n}=1}^{\infty} e_{\mathfrak{n}} e_{\mathfrak{n}}^* \in \Im\},$$

where the convergence is meaning the norm.

Definition 3.7 A closed submodule $\mathcal{Y}_{\mathsf{G}}$ of a G-Hilbert $\Im$-module $\mathcal{X}_{\mathsf{G}}$ is topologically complementable if there exists a closed submodule $\mathcal{H}_{\mathsf{G}}$ so that $\mathcal{Y}_{\mathsf{G}} + \mathcal{H}_{\mathsf{G}} = \mathcal{X}_{\mathsf{G}}$ and $\mathcal{Y}_{\mathsf{G}} \cap \mathcal{H}_{\mathsf{G}} = \{0\}$. We say that $\mathcal{Y}_{\mathsf{G}}$ is G-$\mathcal{O}.\mathcal{C}.$ (G-orthogonally complemented) if we have the condition $\mathcal{Y}_{\mathsf{G}} \perp \mathcal{H}_{\mathsf{G}}$.

***Example* 3.8** Let $\Im = \mathcal{C}([0, 1])$, $\partial = \{\Upsilon \in \Im | \Upsilon(0) = 0\} \simeq \mathcal{C}_0((0, 1])$ and $\mathcal{X}_{\mathsf{G}} = \Im \oplus \partial$ as a G-Hilbert $\Im$-module. If $\mathcal{Y}_{\mathsf{G}} = \{(\Upsilon, \Upsilon) | \Upsilon \in \partial\}$, then $\mathcal{Y}_{\mathsf{G}}^{\perp} = \{(\varrho, -\varrho) | \varrho \in \partial\}$, $\mathcal{Y}_{\mathsf{G}} + \mathcal{Y}_{\mathsf{G}}^{\perp} = \partial + \partial \neq \mathcal{Y}_{\mathsf{G}}$ and $\mathcal{H}_{\mathsf{G}} = \{(\Upsilon, 0) | \Upsilon \in \Im\}$ is a topological complement for $\mathcal{Y}_{\mathsf{G}}$. Therefore, not each topologically complemented is G-$\mathcal{O}.\mathcal{C}.$

Proposition 3.9 If $\mathfrak{l}^2(\Im)$ is G-projective G-Hilbert $\Im$-module in $B_{\mathsf{G}}(\Im, \mathbb{C})$, then each countably generated G-Hilbert $\Im$-module is G-projective G-Hilbert $\Im$-module in $B_{\mathsf{G}}(\Im, \mathbb{C})$.

Proof. If $\mathcal{X}_{\mathsf{G}}$ is countably generated then $\mathcal{X}_{\mathsf{G}} \bigoplus \mathfrak{l}^2(\Im)$ is $\Im$-module G-isomorphic to $\mathfrak{l}^2(\Im)$ by Kasparov's stabilization theorem [13]. Therefore, $\mathcal{X}_{\mathsf{G}}$ is G-isomorphic to a G-$\mathcal{O}.\mathcal{C}.$ submodule of $\mathfrak{l}^2(\Im)$. Therefore an elementary diagram chase presents that a G-$\mathcal{O}.\mathcal{C}.$ submodule of a G-projective G-Hilbert $\Im$-module is G-projective G-Hilbert $\Im$-module. ■

Given every G-Hilbert $\Im$-module $\mathcal{X}_{\mathsf{G}}$, we can represent it on G-Hilbert spaces as operators. This gives us the idea that we can imagine the norms of matrices on G-Hilbert $\Im$-modules, and that these norms depend only on the internal product, in other words, they are canonical. We denote the set of $\infty \times \infty$ matrices on $\Im$ by $M_\infty(\Im)$, that are bounded, that is, $||(e_{\alpha,\beta})|| \equiv \sup_{\mathfrak{n}} ||(e_{\alpha,\beta})_{\alpha,\beta=1}^{\mathfrak{n}}|| < +\infty$ and $\mathfrak{C}_\infty(\mathcal{X}_{\mathsf{G}}) = \{(\mathfrak{m}_1, \mathfrak{m}_2, ...)^\tau : (\langle \mathfrak{m}_\alpha, \mathfrak{m}_\beta \rangle) \in M_\infty(\Im)\}$.

Proposition 3.10 Let $\Omega : \mathfrak{l}^2(\Im) \longrightarrow \mathcal{X}_{\mathsf{G}}$ be defined by $\Omega((e_1, e_2, ...)) = \sum_{\mathfrak{n}} e_{\mathfrak{n}} \mathfrak{m}_{\mathfrak{n}}$. Then Ω defines a bounded $\Im$-module G-equivariant map iff $||(\langle \mathfrak{m}_\alpha, \mathfrak{m}_\beta \rangle)||$ is finite. Furthermore, $||\Omega|| = ||(\langle \mathfrak{m}_\alpha, \mathfrak{m}_\beta \rangle)||$.

Proof. For each finitely supported tuple, one has

$$||\Omega((e_1, ..., e_n, 0, 0...))|| = ||\sum_{\alpha,\beta=1}^{\mathfrak{n}} e_\alpha \langle \mathfrak{m}_\alpha, \mathfrak{m}_\beta \rangle e_\beta^*||.$$

But for each $(p_{\alpha,\beta}) \in M_{\mathfrak{n}}(\Im)$, one has

$$||(p_{\alpha,\beta})|| = \sup\{||\sum_{\alpha,\beta=1}^{\mathfrak{n}} e_\alpha p_{\alpha,\beta} e_\beta^*|| : \sum_{\beta=1}^{\mathfrak{n}} e_\beta e_\beta^* \leqslant 1_\Im\}.$$

Thus, the result is obtained. ■

Theorem 3.11 Let $\Im$ be a unital G-C^*-algebra. Then $\mathfrak{l}^2(\Im)$ is G-projective G-Hilbert $\Im$-module in $\mathcal{X}_\mathsf{G}(\Im,\mathbb{C})$ iff for each G-Hilbert $\Im$-modules $\mathcal{Y}_\mathsf{G}, \mathcal{X}_\mathsf{G}$ and each surjective, bounded module G-equivariant, $\Delta : \mathcal{Y}_\mathsf{G} \longrightarrow \mathcal{X}_\mathsf{G}$, the induced G-equivariant $\Delta_\infty : \mathfrak{C}_\infty(\mathcal{Y}_\mathsf{G}) \longrightarrow \mathfrak{C}_\infty(\mathcal{X}_\mathsf{G})$, is surjective.

Proof. Suppose we are setting of diagram (2.1). Since the G-equivariant $\Omega : \mathfrak{l}^2(\Im) \longrightarrow \mathcal{X}_\mathsf{G}$ is bounded, one has $(\mathfrak{m}_1, \mathfrak{m}_2, ...)^\tau \in \mathfrak{C}_\infty(\mathcal{X}_\mathsf{G})$ with $\Omega((e_1, ...)) = e_1\mathfrak{m}_1 + ...$, and in order to lift Ω to a G-equivariant $\mho$ we have to find $(\mathfrak{n}_1, ...)^\tau \in \mathfrak{C}_\infty(\mathcal{Y}_\mathsf{G})$, with $\Delta_\mathsf{G}(\mathfrak{n}_\alpha) = \mathfrak{m}_\alpha$ for all α. ■

Note that the G-equivariant map Δ_∞ is not necessarily bounded.

Corollary 3.12 Let $\Im$ be a non-unital G-C^*-algebra. If $\Im$ equipped with the canonical $\Im$-$\mathcal{V.I.P.}$ is a G-projective G-Hilbert $\Im$-module in $B_\mathsf{G}(\Im,\mathbb{C})$, then each $\tau \in \mathcal{L}\Re_G(\Im)$ that induces a surjective G-equivariant $\Delta : \Im \longrightarrow \Im$ by $\Delta(e) = e\tau^*$, admits a right inverse which is an element of $\mathcal{L}\Re_G(\Im)$, and the kernel of Δ is a topological summand of $\Im$. In addition, each surjective bounded module mapping $\Delta : \Im \longrightarrow \Im$ is achieved by multiplying by a left multiplication in the manner shown. If for the G-C^*-algebra under consideration $\Re_G(\Im) = \mathcal{L}\Re_G(\Im)$, then these conditions are automatically fulfilled.

Proof. Set $\mathcal{Y}_\mathsf{G} = \mathcal{X}_\mathsf{G} = \Im$ and $\Omega = id_\Im$ by the diagram (3.1). Since $\Im$ is assumed to be a G-projective G-Hilbert $\Im$-module, there is a G-eqivariant $\mho : \Im \longrightarrow \Im$ that is enforced with rule $\mho(e) = e\mathfrak{s}^*$ for some $\mathfrak{s} \in \mathcal{L}\Re_\mathsf{G}(\Im)$ by the existing canonical identification of $End_\Im(\Im)$ with $\mathcal{L}\Re_\mathsf{G}(\Im)$ [14]. Note that $\Delta \circ \mho = \Omega$ by selecting $\mho$. As a result, $1_\Im = 1_{\mathcal{L}\Re_\mathsf{G}(\Im)} = \mathfrak{s}^*\tau^* = \tau\mathfrak{s}$ since $e = 1_\Im$ for the free variable, is a feasible selection. Thus, $\mathfrak{s}\tau\mathfrak{s}\tau = \mathfrak{s}(\tau\mathfrak{s})\tau = \mathfrak{s}\tau$ and $\mathfrak{p} = \mathfrak{s}\tau$ is an idempotent element of $\mathcal{L}\Re_\mathsf{G}(\Im)$. Thus, $\mathfrak{s} \in \mathcal{L}\Re_\mathsf{G}(\Im)$ is the right inverse of $\tau \in \mathcal{L}\Re_\mathsf{G}(\Im)$. Note that the idempotent $(1_\Im - \mathfrak{p}) \in \mathcal{L}\Re_\mathsf{G}(\Im)$ maps $\Im$ onto the kernel of the G-equivariant Δ that becomes a topological summand of the G-Hilbert $\Im$-module $\Im$. The last sentences are derived from the canonical identification of $End_\Im(\Im)$ with $\mathcal{L}\Re_\mathsf{G}(\Im)$ and from spectral decomposition in $\Re_\mathsf{G}(\Im)$ ([14], Proposition 1.1 of [8]). ■

Finally, we show that finitely generated G-Hilbert $\Im$-modules in all the categories of G-Hilbert $\Im$-ζ bimodules are G-projective.

Theorem 3.13 Let $\Im$ be a unital G-C^*-algebra. Then $\mathcal{Z}_\mathsf{G}$ finitely generated G-Hilbert $\Im$-module is a G-projective object in the category that includes all G-C^*-modules on a fixed G-C^*-algebra $\Im$ with morphisms being $\Im$-linear G-equivariant maps. Also, $\mathcal{Z}_\mathsf{G}$ is an orthogonal summand of some G-Hilbert $\Im$-module $\Im^n$, where $n < \infty$, and in particular, $\mathcal{Z}_\mathsf{G}$ is G-projective in $B_\mathsf{G}(\Im,\zeta)$ and $B^*_\mathsf{G}(\Im,\zeta)$.

Proof. Fix an $\Im$-$\mathcal{V.I.P.}$ $\langle\cdot,\cdot\rangle$ on $\mathcal{Z}_\mathsf{G}$. By Corollary 15.4.8 of [22] and by the definition of G-projective G-Hilbert $\Im$-modules, $\mathcal{Z}_\mathsf{G}$ should be finitely generated, and each finitely generated G-Hilbert $\Im$-module, in the purely algebraic meaning, is G-projective. Again, consider the diagram (2.1). By assumption there is an $\Im$-linear G-equivariant $\mho : \mathcal{Z}_\mathsf{G} \longrightarrow \mathcal{Y}_\mathsf{G}$ such that $\Omega = \Delta\circ\mho$. We show that $\mho$ is bounded. By [7], there is a set of finite algebraic of generators $\{\iota_1, ..., \iota_\mathfrak{n}\}$ of $\mathcal{Z}_\mathsf{G}$ so that the reconstruction formula $\iota = \sum_{\alpha=1}^{\mathfrak{n}}\langle\iota,\iota_\alpha\rangle\iota_\alpha$ is valid for any $\iota \in \mathcal{Z}_\mathsf{G}$. This $\{\iota_1, ..., \iota_\mathfrak{n}\}$ of generators is said that a normalized tight frame of $\mathcal{Z}_\mathsf{G}$ than the fixed $\Im$-$\mathcal{V.I.P.}$ $\langle\cdot,\cdot\rangle$. Thus, $\mho(\iota) = \sum_{\alpha=1}^{\mathfrak{n}}\langle\iota,\iota_\alpha\rangle\mho(\iota_\alpha)$ for every $\iota \in \mathcal{Z}_\mathsf{G}$. By the Cauchy-Schwarz inequality for G-Hilbert $\Im$-modules, we get the following inequality (e.g. Proposition 1.1 of [13])

$$\begin{aligned}
\|\mho\| &= \left\| \left\langle \sum_{\alpha=1}^{n} \langle \iota, \iota_\alpha \rangle \mho(\iota_\alpha), \mho(\iota) \right\rangle_{\mathcal{Y}_{\mathsf{G}}} \right\| \\
&= \left\| \sum_{\alpha=1}^{n} \langle \iota, \iota_\alpha \rangle \langle \mho(\iota_\alpha), \mho(\iota) \rangle_{\mathcal{Y}_{\mathsf{G}}} \right\| \\
&\leqslant \sum_{\alpha=1}^{n} \|\iota\|^{\frac{1}{2}} \|\iota_\alpha\|^{\frac{1}{2}} \|\mho(\iota_i)\|_{\mathcal{Y}_{\mathsf{G}}}^{\frac{1}{2}} \|\mho(\iota)\|_{\mathcal{Y}_{\mathsf{G}}}^{\frac{1}{2}} \\
&= \left(\sum_{\alpha=1}^{n} \|\iota_\alpha\|^{\frac{1}{2}} \|\mho(\iota_\alpha)_{\mathcal{Y}_{\mathsf{G}}}^{\frac{1}{2}} \right) \|\iota\|^{\frac{1}{2}} \|\mho(\iota)\|_{\mathcal{Y}_{\mathsf{G}}}^{\frac{1}{2}}.
\end{aligned}$$

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