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Controlled *pg*-frames in Hilbert spaces

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Abstract. In this paper, for extending the concepts of p-frame and controlled frame for Hilbert spaces, we will introduce the concept of controlled pg-frames in Hilbert spaces. Then, we present characterizations of controlled pg-frames and some results of frames in the view of controlled pg-frames.

Keywords: Controlled frame, pg-frame, Hilbert space.

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1. Introduction and preliminaries

Frames as a generalization of the bases in Hilbert spaces were first introduced by Duffin and Schaeffer [9] to study some problems in the nonharmonic Fourier series in 1952. Various generalizations of frames for Hilbert spaces have been proposed recently. For example, frames of subspaces, wavelet frames, g-frames, weighted and controlled frames were developed, see [3, 8, 10, 12, 14, 15]. Today, frame theory has an abundance of applications in pure mathematics, applied mathematics, engineering, medicine and even quantum communication ([5–7]). Controlled frames have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces ([4]).

In the present paper, by using some ideas from [1], we will introduce controlled pg-frames in a Hilbert space $\mathcal{H}$ that allows every element $x \in \mathcal{H}$ to be represented by

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an unconditionally convergent series $\sum_{j\in J}(C'C)^{\frac{1}{2}}\Lambda_j^* y_j$, where $\{\Lambda_j\}_{j\in J}$ is a pg-frame, $\{y_j\}_{j\in J} \in \Big(\sum_{j\in J} \oplus \mathcal{H}_j\Big)_{l_q}$ and $\frac{1}{p}+\frac{1}{q}=1$.

Let $L(\mathcal{H}_1,\mathcal{H}_2)$ be the family of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces. As a special case, $L(\mathcal{H})$ is a collection of all bounded linear operators on $\mathcal{H}$. The operator Λ_j is in $L(\mathcal{H},\mathcal{H}_j)$ for any $j \in J$. $GL(\mathcal{H})$ respects the set of all bounded linear operators which have bounded inverse. If $S,T \in GL(\mathcal{H})$, then T^*, T^{-1} and ST are also in $GL(\mathcal{H})$. Let $GL^+(\mathcal{H})$ be the set of all positive operators in $GL(\mathcal{H})$. A bounded operator $T : \mathcal{H} \to \mathcal{H}$ is positive if $\langle Tf, f\rangle > 0$ for all $f \geqslant 0$. On complex Hilbert spaces, every bounded positive operator is self-adjoint, and any two bounded positive operators can commute with each other. In fact, if S,T are two positive operators On complex Hilbert space $\mathcal{H}$, then [13, Theorem 2.3.5] implies that S,T are self-adjoint and so we have $\langle STx, x\rangle = \langle Tx, Sx\rangle = \langle Tx, y\rangle \geqslant 0$. Therefore, ST is positive and then for all $x \in \mathcal{H}$, we give $0 \leqslant \langle STx, x\rangle = \langle x, TSx\rangle = \langle TSx, x\rangle$. Hence, $ST = TS$.

Throughout this paper, J is a subset of $\mathbb{N}$, $\mathcal{H}$ is a separable Hilbert space and $\{\mathcal{H}_j\}_{j\in J}$ is a sequence of separable Hilbert spaces. We also need the following lemma in the next section.

Lemma 1.1 [11] If $T : \mathcal{X} \to \mathcal{Y}$ is a bounded operator from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$, then its adjoint $T^* : \mathcal{Y}^* \to \mathcal{X}^*$ is surjective if and only if T has a bounded inverse on R_T.

2. Main results

In this section, we introduce controlled pg-frames in Hilbert spaces. We discuss characterizations of controlled pg-frames and give some results of frames in the view of controlled pg-frames.

Definition 2.1 Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a sequence in $B(\mathcal{H},\mathcal{H}_j)$ and $C, C' \in GL^+(\mathcal{H})$. We call $\{\Lambda_j\}_{j\in J}$ a (C,C')-controlled pg-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ if there exist $A, B > 0$ such that

$$A\|x\| \leqslant \Big(\sum_{j\in J} |\langle \Lambda_j Cx, \Lambda_j C'x\rangle|^p\Big)^{\frac{1}{p}} \leqslant B\|x\| \,, \quad (x \in \mathcal{H}). \tag{1}$$

A and B are called the (C,C')-controlled pg-frames bounds. If $C' = I$, then we call Λ a C-controlled pg-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$.

The following proposition shows that the image of a controlled pg-frame under a bounded operator is also a controlled pg-frame.

Proposition 2.2 Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a (C,C')-controlled pg-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. Let S be a bounded invertible operator such that commutes with C and C'. If $\Gamma_j = \Lambda_j S$, then $\{\Gamma_j\}_{j\in J}$ is a (C,C')-controlled pg-frame for $\mathcal{H}$.

Proof. Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a (C,C')-controlled pg-frame for $\mathcal{H}$ and S be a bounded invertible operator such that commutes with C and C'. Then, for each $x \in \mathcal{H}$, we have

$$A\|Sx\| \leqslant \Big(\sum_{j\in J} |\langle \Lambda_j CSx, \Lambda_j C'Sx\rangle|^p\Big)^{\frac{1}{p}} \leqslant B\|Sx\|.$$

Since S commutes with C and C', we have

$$A\|Sx\| \leqslant \Big(\sum_{j\in J}|\langle \Gamma_j Cx, \Gamma_j C'x\rangle|^p\Big)^{\frac{1}{p}} = \Big(\sum_{j\in J}|\langle \Lambda_j SCx, \Lambda_j SC'x\rangle|^p\Big)^{\frac{1}{p}} \leqslant B\|Sx\|.$$

Moreover, S is invertible, so

$$\|x\|^2 = \langle S^{-1}Sx, S^{-1}Sx\rangle \leqslant \|S^{-1}\|^2\|Sx\|^2$$

and we get

$$A\|S^{-1}\|^{-1}\|x\| \leqslant A\|Sx\| \leqslant \Big(\sum_{j\in J}|\langle \Gamma_j Cx, \Gamma_j C'x\rangle|^p\Big)^{\frac{1}{p}} \leqslant B\|Sx\| \leqslant B\|S\|\|x\|.$$

Therefore, $\{\Gamma_j\}_{j\in J}$ is a (C, C')-controlled pg-frame for $\mathcal{H}$ with pg-frame bounds $A\|S^{-1}\|^{-1}$ and $B\|S\|$. ■

If the operator S in Proposition 2.2 is an isometry, then we get the following corollary.

Corollary 2.3 Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a (C, C')-controlled pg-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ and S be an isometry such that commutes with C and C'. Then $\{\Lambda_j S\}_{j\in J}$ is a (C, C')-controlled pg-frame for $\mathcal{H}$ with the same pg-frame bounds.

Proposition 2.4 Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a (C, C')-controlled pg-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ and S be an operator such that commutes with C and C'. Then $\{\Gamma_j\}_{j\in J} = \{\Lambda_j S\}_{j\in J}$ is a (C, C')-controlled pg-frame for $\mathcal{H}$ if and only if S is bounded below.

Proof. Let $\{\Gamma_j\}_{j\in J} = \{\Lambda_j S\}_{j\in J}$ be a (C, C')-controlled pg-frame for $\mathcal{H}$ with pg-frame bounds M, N; that is, for each $f \in \mathcal{H}$, we have

$$M\|x\| \leqslant \Big(\sum_{j\in J}|\langle \Gamma_j Cx, \Gamma_j C'x\rangle|^p\Big)^{\frac{1}{p}} = \Big(\sum_{j\in J}|\langle \Lambda_j SCx, \Lambda_j SC'x\rangle|^p\Big)^{\frac{1}{p}} \leqslant N\|x\|.$$

Assume that A, B are pg-frame bounds of $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$. Since

$$A\|Sx\| \leqslant \Big(\sum_{j\in J}|\langle \Lambda_j CSx, \Lambda_j C'Sx\rangle|^p\Big)^{\frac{1}{p}} \leqslant B\|Sx\|,$$

we have $M\|x\| \leqslant B\|Sx\|$. Therefore, $\|Sx\| \geqslant \frac{M}{B}\|x\|$ and hence, S is bounded below. Conversely, suppose that there exists $\delta > 0$ such that $\|Sx\| \geqslant \delta\|x\|$. Since

$$A\delta\|x\| \leqslant A\|Sx\| \leqslant \Big(\sum_{j\in J}|\langle \Lambda_j SCx, \Lambda_j SC'x\rangle|^p\Big)^{\frac{1}{p}} \leqslant B\|Sx\| \leqslant B\|S\|\|x\|,$$

$\{\Gamma_j\}_{j\in J} = \{\Lambda_j S\}_{j\in J}$ is a pg-frame with bounds $A\delta$ and $B\|S\|$. ■

Definition 2.5 Let $\{\mathcal{H}_j\}_{j\in J}$ be a sequence of Hilbert spaces and $p > 1$. Consider

$$\Big(\sum_{j\in J}\oplus\mathcal{H}_j\Big)_{l_p} = \Big\{\{x_j\}_{j\in J} : x_j \in \mathcal{H}_j, \Big(\sum_{j\in J}|\langle x_j, x_j\rangle|^p\Big)^{\frac{1}{p}} < \infty\Big\}.$$

Then $\left(\sum_{j\in J}\oplus\mathcal{H}_j\right)_{l_p}$ is a Hilbert space with the inner product and the norm given by

$$\langle\{x_j\},\{y_j\}\rangle=\sum_{j\in J}\langle x_j,y_j\rangle_{\mathcal{H}_j}\quad,\quad \|\{x_j\}_{j\in J}\|_p=\left(\sum_{j\in J}|\langle x_j,x_j\rangle|^p\right)^{\frac{1}{p}},$$

respectively.

Let $1<p,q<\infty$ be conjugate exponents, i.e. $\frac{1}{p}+\frac{1}{q}=1$. By [1, Lemma 3.6] and the Riesz representation theorem for Hilbert spaces, we have the following lemma.

Lemma 2.6 [2] Let $1<p,q<\infty$ be conjugate exponents. Then

$$\left(\sum_{j\in J}\oplus\mathcal{H}_j\right)^*_{l_p}=\left(\sum_{j\in J}\oplus\mathcal{H}_j\right)_{l_q}.$$

Definition 2.7 Let $\Lambda=\{\Lambda_j\}_{j\in J}$ be a (C,C')-controlled pg-frame. We define the bounded linear operator $T_{CC'}$ by

$$T_{CC'}:\left(\sum_{j\in J}\oplus\mathcal{H}_j\right)_{l_q}\to\mathcal{H}\ ,\qquad T_{CC'}\Big(\{y_j\}_{j\in J}\Big)=\sum_{j\in J}(C'C)^{\frac{1}{2}}\Lambda_j^*y_j,$$

and the operator

$$T^*_{CC'}:\ \mathcal{H}\to\left(\sum_{j\in J}\oplus\mathcal{H}_j\right)_{l_p},\qquad T^*_{CC'}(x)=\{\Lambda_j(C'C)^{\frac{1}{2}}x\}_{j\in J}.$$

Based on the above linear operators, we introduce the following linear operator $S_{CC'}:\mathcal{H}\to\mathcal{H}$ by

$$S_{CC'}x=T_{CC'}T^*_{CC'}x=\sum_{j\in J}C'\Lambda_j^*\Lambda_jCx,\quad (x\in\mathcal{H}).$$

The operators $T_{CC'}$, $T^*_{CC'}$ and $S_{CC'}$ are called the synthesis operator, analysis operator and frame operator of $\{\Lambda_\omega\}_{\omega\in\Omega}$.

Now, we characterize the pg-Bessel sequence and the pg-frame by the operator $T_{CC'}$.

Proposition 2.8 Let $C,C'\in GL(\mathcal{H})$. $\{\Lambda_j\}_{j\in J}$ is a (C,C')-controlled pg-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ if and only if the operator $T_{CC'}$ is well-defined and bounded operator.

Proof. Assume that $\{\Lambda_j\}_{j\in J}$ is a (C,C')-controlled pg-Bessel sequence with bound B. We show that for each the series $\{y_j\}_{j\in J}$ in $\left(\sum_{j\in J}\oplus\mathcal{H}_j\right)_{l_q}$, the series $\{\Lambda_j(C'C)^{\frac{1}{2}}x\}_{j\in J}$

is convergent unconditionally. For finite subsets J_1, J_2 of J that $J_2 \nsubseteq J_1$, we have

$$\begin{aligned}\|\sum_{j\in J_1\setminus J_2} y_j\Lambda_j(C'C)^{\frac{1}{2}}\| &= \sup_{\|x\|=1}\|\sum_{j\in J_1\setminus J_2} y_j\Lambda_j(C'C)^{\frac{1}{2}}x\| \\ &\leqslant \sup_{\|x\|=1}\sum_{j\in J_1\setminus J_2}\|y_j\|\|\Lambda_j(C'C)^{\frac{1}{2}}x\| \\ &\leqslant \Big(\sup_{\|x\|=1}\sum_{j\in J_1\setminus J_2}\|y_j\|^q\Big)^{\frac{1}{q}}\sup_{\|x\|=1}\Big(\sum_{j\in J_1\setminus J_2}\|\Lambda_j(C'C)^{\frac{1}{2}}x\|^p\Big)^{\frac{1}{p}} \\ &\leqslant B\Big(\sup_{\|x\|=1}\sum_{j\in J_1\setminus J_2}\|y_j\|^q\Big)^{\frac{1}{q}}.\end{aligned}$$

Hence, $\{\Lambda_j(C'C)^{\frac{1}{2}}x\}_{j\in J}$ is unconditionally convergent. By the same argument,

$$\|\sum_{j\in J} y_j\Lambda_j(C'C)^{\frac{1}{2}}\| \leqslant B\Big(\sup_{\|x\|=1}\sum_{j\in J_1\setminus J_2}\|y_j\|^q\Big)^{\frac{1}{q}}.$$

Therefore,

$$\|T_{CC'}\{y_j\}_{j\in J}\| \leqslant B\Big(\sup_{\|x\|=1}\sum_{j\in J_1\setminus J_2}\|y_j\|^q\Big)^{\frac{1}{q}} = B\|\{y_j\}_{j\in J}\|_q.$$

This implies that $T_{CC'}$ is bounded and $\|T_{CC'}\| \leqslant B$.

Conversely, assume that $T_{CC'}$ is well defined and bounded. For $x \in \mathcal{H}$, define

$$\begin{aligned}&F_x : \Big(\sum_{j\in J}\oplus\mathcal{H}_j\Big)_{l_q} \to \mathbb{C} \\ &F_x(\{y_j\}) = \langle T_{CC'}\{y_j\}, x\rangle = \sum_{j\in J}\langle (C'C)^{\frac{1}{2}}\Lambda_j^* y_j, x\rangle.\end{aligned}$$

Then $\|F_x\} \leqslant \|T^*_{CC'}\|\|\{y_j\}\|_q\|x\|$. Therefore, $F_x \in \Big(\sum_{j\in J}\oplus\mathcal{H}_j\Big)^*_{l_q}$ and $(CC')^{\frac{1}{2}}\Lambda_j x \in \Big(\sum_{j\in J}\oplus\mathcal{H}_j\Big)_{l_p}$. By the Hahn-Banach theorem, there exists $\{y_j\} \in \Big(\sum_{j\in J}\oplus\mathcal{H}_j\Big)_{l_q}$ such that $\|(CC')^{\frac{1}{2}}\Lambda_j x\|_p = |F_x|$. Hence,

$$\begin{aligned}\Big(\sum_{j\in J}|\langle\Lambda_j Cx, \Lambda_j C'x\rangle|^p\Big)^{\frac{1}{p}} &= \|(CC')^{\frac{1}{2}}\Lambda_j x\|_p = \|F_x\| \\ &\leqslant \sup_{\|\{y_j\}\|_q\leqslant 1}|\langle T^*_{CC'}\{y_j\}, x\rangle| \\ &\leqslant \|T_{CC'}\|\|x\|.\end{aligned}$$

This completes the proof. ■

Lemma 2.9 Let $\{\Lambda_j\}_{j\in J}$ be a (C, C')-controlled pg-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$

and $C, C' \in GL^+(\mathcal{H})$ such that each of their commutes with $\Lambda_j^*\Lambda_j$. Then, the operator $T_{CC'}^*$ has closed range.

Proof. If $\{\Lambda_j\}_{j\in J}$ is a (C, C')-controlled pg-frame, then there exist $A, B > 0$ such that

$$A\|x\| \leqslant \Big(\sum_{j\in J} |\langle \Lambda_j Cx, \Lambda_j C'x\rangle|^p\Big)^{\frac{1}{p}} \leqslant B\|x\| \ , \quad (x \in \mathcal{H}).$$

Moreover,

$$\begin{aligned}\|T_{CC'}^*(x)\|_p &= \Big(\sum_{j\in J} |\langle \Lambda_j (C'C)^{\frac{1}{2}}x, \Lambda_j (C'C)^{\frac{1}{2}}x\rangle|^p\Big)^{\frac{1}{p}} \\ &= \Big(\sum_{j\in J} |\langle (C'C)^{\frac{1}{2}}\Lambda_j^*\Lambda_j (C'C)^{\frac{1}{2}}x, x\rangle|^p\Big)^{\frac{1}{p}} \\ &= \Big(\sum_{j\in J} |\langle \Lambda_j Cx, \Lambda_j C'x\rangle|^p\Big)^{\frac{1}{p}}.\end{aligned}$$

Hence, $A\|x\| \leqslant \|T_{CC'}^*(x)\|_p \leqslant B\|x\|$ for $x \in \mathcal{H}$. If $T_{CC'}^*(x) = 0$, then $\|x\| = 0$ and so $x = 0$. This implies that $T_{CC'}^*$ is one-to-one and $\mathcal{H} \simeq R_{T_{CC'}^*}$. Therefore, $T_{CC'}^*$ has closed range. ■

In the following, we show that the frame operator is bounded.

Proposition 2.10 Let $C, C' \in GL^+(\mathcal{H})$ and each of their commutes with $\Lambda_j^*\Lambda_j$. If $\{\Lambda_j\}_{j\in J}$ is a (C, C')-controlled pg-frame for $\mathcal{H}$, then $S_{CC'}$ is bounded.

Proof. Let $\{\Lambda_j\}_{j\in J}$ be a (C, C')-controlled pg-frame for $\mathcal{H}$. We have

$$\begin{aligned}|\langle S_{CC'}x, x\rangle^p| &= |\langle \sum_{j\in J} C'\Lambda_j^*\Lambda_j Cx, x\rangle^p| \\ &\leqslant |\sum_{j\in J}\langle \Lambda_j Cx, \Lambda_j C'x\rangle|^p \\ &= |\langle C'S_{CC'}Cx, x\rangle|^p \\ &= |\langle S_{CC'}C'Cx, x\rangle|^p \\ &= |\langle \sum_{j\in J} \Lambda_j^*\Lambda_j C'Cx, x\rangle|^p \\ &= |\langle \sum_{j\in J} \Lambda_j Cx, \Lambda_j C'x\rangle|^p \\ &\leqslant B\|x\|.\end{aligned}$$

Therefore, $S_{CC'}$ is bounded. ■

Theorem 2.11 Let $C, C' \in GL^+(\mathcal{H})$ and each of their commutes with $\Lambda_j^*\Lambda_j$. Then $\{\Lambda_j\}_{j\in J}$ is a (C, C')-controlled pg-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ if and only if the operator $T_{CC'}$ is a surjective bounded operator.

Proof. If $\{\Lambda_j\}_{j\in J}$ is a (C, C')-controlled pg-frame, by Proposition 2.8, $T_{CC'}$ is a well-defined and bounded. The proof of Lemma 2.9 shows that $T^*_{CC'}$ is injective, so by Lemma 1.1, $T_{CC'}$ is onto.

Conversely, suppose that $T_{CC'}$ is bounded and onto. Then, by Proposition 2.8, $\{\Lambda_j\}_{j\in J}$ is a pg-Bessel sequence. Since $T_{CC'}$ is onto, Lemma 1.1 implies that $T^*_{CC'}$ has a bounded inverse. Hence, there exists $A > 0$ such that $\|T^*_{CC'}x\| \geqslant A\|x\|$ for all $x \in H$. In other words, $\{\Lambda_j\}_{j\in J}$ satisfies the lower pg-frame condition. ■

Finally, we get the following characterization for elements of a Hilbert space.

Corollary 2.12 Let $C, C' \in GL^+(\mathcal{H})$ and each of their commutes with $\Lambda_j^*\Lambda_j$. If $\{\Lambda_j\}_{j\in J}$ is a (C, C')-controlled pg-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$, then for each $x \in \mathcal{H}$, there exists a $\{y_j\}_{j\in J} \in \left(\sum_{j\in J} \oplus \mathcal{H}_j\right)_{l_q}$ such that $x = \sum\limits_{j\in J} (C'C)^{\frac{1}{2}}\Lambda_j^* y_j$.

3. Conclusion

In this paper, we have proposed the concept of controlled pg-frames in Hilbert spaces, which is an extension of p-frames and controlled frames. We have shown the image of a controlled pg-frame under a bounded operator is also a controlled pg-frame. Then, we have characterized the pg-Bessel sequence and the pg-frame by the synthesis operator, and we have proved the frame operator is bounded. Finally, We have given a characterization of elements of a Hilbert space as a series.

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