Journal of Linear and Topological Algebra
Vol. 11*, No.* 03*,* 2022*,* 219- 223
DOR: 20.1001.1.22520201.2022.11.03.7.2
DOI: 10.30495/JLTA.2022.1962994.1504

Journal of Linear and Topological Algebra
JLTA

Expressions for the integer powers of the Min matrix

M. Beiranvand[a,*], M. Ghasemi Kamalvand[a]
[a] *Department of Mathematics, Lorestan University, Khorramabad, Iran.*

Received 10 July 2022; Revised 10 September 2022; Accepted 29 September 2022.
Communicated by Ghasem Soleimani Rad

Abstract. In this paper, we derive the general expression for the entries of the positive integer powers of the Min matrix $A = [\min\{i, j\}];\ i, j = 1, 2, \cdots, n$ of arbitrary order. Also, we give Maple 18 procedures in order to verify our calculations.

Keywords: Eigenvalues, eigenvectors, integer powers, Min matrix.

2010 AMS Subject Classification: 15B99, 15A18, 65F50.

1. Introduction and preliminaries

Arbitrary integer powers of a square matrix are used in order to solving some difference equations, differential equations and delay differential equations [1, 9, 13]. There have been several papers on computing the positive integer powers of various kinds of square matrices in recent years [2–4, 12, 14]. In this paper, we consider the Min matrix A of the following type

$$A = [\min\{i, j\}];\ i, j = 1, 2, \cdots, n; \tag{1}$$

that is,

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}.$$

*Corresponding author.
E-mail address: beiranvand.moh@fs.lu.ac.ir (M. Beiranvand); ghasemi.m@lu.ac.ir (M. Ghasemi Kamalvand).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

2. Main results

In this section, we present a formula for computing the (i, j) entry of the matrix A^m, where $m \in \mathbb{Z}$ and $\mathbb{Z}$ denotes the set of integer numbers.

Theorem 2.1 [7] The matrix (1) of order $n \geqslant 3$ has the eigenpairs (λ_k, v_k) given by

$$\lambda_k = \frac{1}{2}(1-\cos(r_k))^{-1} \quad , \quad v_k = [\sin(jr_k)]^T_{j=1,2,\cdots,n} \tag{2}$$

where $r_k = \frac{2k+1}{2n+1}\pi$ for $k = 0, 1, 2, \cdots, n-1$.

Theorem 2.2 [8] If $A \in M_n$ has n distinct eigenvalues, then A is diagonalizable.

Since $\theta \rightarrowtail \cos\theta$ is strictly decreasing on $[0,\pi]$, the eigenvalues of matrix (1), i.e. of A, are all distinct and hence, the matrix is diagonalizable by [8, Theorem 1.3.9]. The proof of [8, Theorem 1.3.7] shows that if we define the matrix $V = [v_0, v_1, \cdots, v_{n-1}]$, then $V^{-1}AV = \text{diag}(\lambda_0, \lambda_1, \cdots, \lambda_{n-1})$ will be a diagonal. Conversely we will have $A^m = V\text{diag}(\lambda_0^m, \lambda_1^m, \cdots, \lambda_{n-1}^m)V^{-1}$ for any integer m. We will use this to give explicit formulas for $[A^m]_{i,j}$. From (2), we can write the columns matrix V as

$$v_k = [\sin\frac{(2k+1)\pi}{2n+1}, \sin\frac{(2k+1)2\pi}{2n+1}, \sin\frac{(2k+1)3\pi}{2n+1}, \cdots, \sin\frac{(2k+1)n\pi}{2n+1}]^T, \tag{3}$$

for $k = 0, 1, 2, \cdots, n-1$. Hence,

$$V = [v_0, v_1, v_2, \cdots, v_{n-1}] = \begin{bmatrix} \sin\frac{\pi}{2n+1} & \sin\frac{3\pi}{2n+1} & \sin\frac{5\pi}{2n+1} & \cdots & \sin\frac{(2n-1)\pi}{2n+1} \\ \sin\frac{2\pi}{2n+1} & \sin\frac{6\pi}{2n+1} & \sin\frac{10\pi}{2n+1} & \cdots & \sin\frac{2(2n-1)\pi}{2n+1} \\ \sin\frac{3\pi}{2n+1} & \sin\frac{9\pi}{2n+1} & \sin\frac{15\pi}{2n+1} & \cdots & \sin\frac{3(2n-1)\pi}{2n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \sin\frac{(n-1)\pi}{2n+1} & \sin\frac{3(n-1)\pi}{2n+1} & \sin\frac{5(n-1)\pi}{2n+1} & \cdots & \sin\frac{(2n-1)(n-1)\pi}{2n+1} \\ \sin\frac{n\pi}{2n+1} & \sin\frac{3n\pi}{2n+1} & \sin\frac{5n\pi}{2n+1} & \cdots & \sin\frac{(2n-1)n\pi}{2n+1} \end{bmatrix}. \tag{4}$$

Theorem 2.3 Suppose V is defined as above. Then

$$V^{-1} = \frac{4}{2n+1}V^T. \tag{5}$$

Proof. To prove, it is enough to show that $v_i{}^T v_j = \begin{cases} \frac{2n+1}{4}, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$ for $i, j = 0, 1, 2, \cdots, n-1$. By using formulas, $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$ and $\sum_{k=1}^{n}\cos k\theta = \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} - \frac{1}{2}$ [11], we can conclude

$$v_i{}^T v_j = \sum_{k=1}^{n} \sin\frac{(2i+1)k\pi}{2n+1}\sin\frac{(2j+1)k\pi}{2n+1} = \frac{1}{2}\sum_{k=1}^{n}(\cos\frac{2k\pi(i-j)}{2n+1} - \cos\frac{2k\pi(i+j+1)}{2n+1}).$$

If $i = j$, then

$$
\begin{aligned}
{v_i}^T v_i &= \frac{n}{2} - \frac{1}{2}\sum_{k=1}^{n} \cos\frac{2k\pi(2i+1)}{2n+1} = \frac{n}{2} - \frac{1}{2}\Big(\frac{\sin((n+\frac{1}{2})\frac{2\pi(2i+1)}{2n+1})}{2\sin\frac{1}{2}\frac{2\pi(2i+1)}{2n+1}} - \frac{1}{2}\Big) \\
&= \frac{n}{2} - \frac{1}{2}\Big(\frac{\sin(2i+1)\pi}{2\sin\frac{(2i+1)\pi}{2n+1}} - \frac{1}{2}\Big) = \frac{n}{2} - \frac{1}{2}\Big(0 - \frac{1}{2}\Big) = \frac{2n+1}{4}.
\end{aligned}
$$

If $i \neq j$, then

$$
\begin{aligned}
{v_i}^T v_j &= \frac{1}{2}\sum_{k=1}^{n}\Big(\cos\frac{2k\pi(i-j)}{2n+1} - \cos\frac{2k\pi(i+j+1)}{2n+1}\Big) \\
&= \frac{1}{2}\Big(\frac{\sin((n+\frac{1}{2})\frac{2(i-j)\pi}{2n+1})}{2\sin\frac{1}{2}\frac{2(i-j)\pi}{2n+1}} - \frac{\sin((n+\frac{1}{2})\frac{2(i+j+1)\pi}{2n+1})}{2\sin\frac{1}{2}\frac{2(i+j+1)\pi}{2n+1}}\Big) \\
&= \frac{1}{2}\Big(\frac{\sin(i-j)\pi}{2\sin\frac{(i-j)\pi}{2n+1}} - \frac{\sin(i+j+1)\pi}{2\sin\frac{(i+j+1)\pi}{2n+1}}\Big) = 0.
\end{aligned}
$$

■

Theorem 2.4 Let A be the Min matrix given in (1). Then for any integer m, the (i, j) entry of the matrix A^m is given by

$$
[A^m]_{i,j} = \frac{4}{2n+1}\sum_{k=0}^{n-1} \lambda_k^m \sin\frac{(2k+1)i\pi}{2n+1} \sin\frac{(2k+1)j\pi}{2n+1} \qquad (6)
$$

for $i, j = 1, 2, \cdots, n$, where $\lambda_k = \frac{1}{2}(1 - \cos\frac{(2k+1)\pi}{2n+1})^{-1}$ for $k = 0, 1, 2, \cdots, n-1$.

Proof. By substituting (2), (4) and (5) in $A^m = VJ^mV^{-1}$, where $J = \mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_{n-1})$ and doing necessary computation the desired relation is obtained. ■

By using following theorem, we can compute inverse of the matrix (1).

Theorem 2.5 [5]. Let A be Min matrix given in (1). Then it's inverse matrix is the $n \times n$ tridiagonal matrix as follow:

$$
A^{-1} = \begin{bmatrix}
2 & -1 & & & & \\
-1 & 2 & -1 & & 0 & \\
 & -1 & 2 & -1 & & \\
 & & \ddots & \ddots & \ddots & \\
 & 0 & & -1 & 2 & -1 \\
 & & & & -1 & 1
\end{bmatrix}.
$$

Since all eigenvalues of matrix A are nonzero, we can compute the integer powers of the above tridiagonal matrix by using the formula (6).

3. Future work

In the same way that presented in this paper or other ways, the powers of the Max matrix $B = [\max\{i, j\}]; i, j = 1, 2, \cdots, n$ of arbitrary order can be calculated. By using

Theorem 7.1 [10], we can compute inverse matrix B as follow:

$$B^{-1} = \begin{bmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & -2 & 1 \\ & & & & 1 & -\frac{n-1}{n} \end{bmatrix}.$$

The reader can check the inverse correctness of the above tridiagonal matrix using the method presented in the paper [6]. Following Maple 18 procedure calculates the mth power of $n \times n$ Min matrix in (1).

```
> restart :
  with(ListTools) :
  power :=proc( n, m)
   local k, λ, i, j, A, power;
  for k  from 0 to n − 1
  do
  λ[k] := ( 1 / (2(1 - cos((2·k + 1)·Pi / (2·n + 1)))) );
  end do;
  power := [ ] :
  for i from 1 to n
  do
  for j from 1 to n
  do
  A[m, i, j] := (4 / (2·n + 1)) · sum( (λ[κ])^m ·sin((2·κ + 1)·i·Pi / (2·n + 1)) ·sin((2·κ + 1)·j·Pi / (2·n + 1)), κ
      = 0 .. n − 1);
  power := FlattenOnce([power, A[m, i, j]]);
  od;
  od;
  print(simplify(Matrix(n, n, power)));
  end proc:
```

power(4, 1)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad \mathbf{(1)}$$

power(4, 2)

$$\begin{bmatrix} 4 & 7 & 9 & 10 \\ 7 & 13 & 17 & 19 \\ 9 & 17 & 23 & 26 \\ 10 & 19 & 26 & 30 \end{bmatrix} \quad \mathbf{(2)}$$

power(4, -1)

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \mathbf{(3)}$$

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Marcel-Dekker, New York. 1992.
[2] M. Beiranvand, M. Ghasemi Kamalvand, Explicit expression for arbitrary positive powers of special tridiagonal matrices, J. Appl. Math. (2020), 2020:7290403.
[3] M. Beiranvand, M. Ghasemi Kamalvand, On computing of integer positive powers for one type of tridiagonal and antitridiagonal matrices of even order, J. Linear. Topological. Algebra. 10 (1) (2021), 59-69.
[4] M. Beiranvand, M. Ghasemi Kamalvand, Positive integer powers of certain tridiagonal matrices and corresponding anti-tridiagonal matrices, Adv. Math. Physics. (2022), 2022:8445721.
[5] R. Bhatia, Min matrices and mean matrices, Math. Intelligencer. 33 (2011), 2228.
[6] D. Caratelli, P. Emilio Ricci, Inversion of tridiagonal matrices using the Dunford-Taylors integral, Symmetry 13 (2021), 5:870.
[7] C. M. Da Fonseca, On the eigenvalues of some tridiagonal matrices, J. Comput. Appl. Math. 200 (2007), 283-286.
[8] R. A. Horn, C. R. Johnson, Matrix Analysis, Cambridge University Press, U.K., 2012.
[9] G. James, Advanced Modern Engineering Mathematics, Addison Wesley, 1994.
[10] M. Mattila, P. Haukkaneni, Studying the various properties of MIN and MAX matrices-elementary vs. more advanced methods, Spec. Matrices. 4 (2016), 101-109.
[11] A. Milton, S. Irene, A Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications, New York, 1972.
[12] A. Oteles, M. Akbulak, Positive integer powers of one type of complex tridiagonal matrix, Bull. Malays. Math. Sci. Soc. 37 (4) (2014), 971-981.
[13] J. Rimas, Investigation of dynamics of mutually synchronized system, Telecommunications and Radio Engineering. 32 (2) (1977), 68-79.
[14] H. Wu, On computing of arbitrary positive powers for one type of anti-tridiagonal matrices of even order, Appl. Math. Comput. 217 (2010), 2750-2756.