Journal of Linear and Topological Algebra
Vol. 12*, No.* 03*,* 2023*,* 179- 194
DOR: 20.1001.1.22520201.2023.12.03.3.5
DOI: 10.30495/jlta.2023.1995617.1587

A variational approach to quasilinear elliptic systems with critical Hardy-Sobolev and sign-changing function exponents

A. Akhavan[a,b]

[a] *Department of Mathematics, Shahid Beheshti Higher Education Center of Tehran, Tehran, Iran.*
[b] *Education System of Shahriar, Shahriar, Iran.*

Received 31 August 2023; Revised 25 September 2023; Accepted 29 September 2023.
Communicated by Hamidreza Rahimi

Abstract. The main aim of the present work is to review and study a variational method in existence and multiplicity of positive solutions for quasilinear elliptic systems with critical Hardy-Sobolev and sign-changing function exponents.

Keywords: Multiple positive solutions, Nehari manifold, critical Hardy-Sobolev exponent, sign-changing function exponent.

2010 AMS Subject Classification: 35A15, 35B33, 35J70.

1. Introduction and preliminaries

Consider a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ with $0 \in \Omega$ and smooth boundary $\partial\Omega$. The problem we talk about is

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \dfrac{\alpha}{\alpha+\beta}\dfrac{|u|^{\alpha-2}u|v|^{\beta}}{|x|^{bp^*}} + \lambda f(x)\frac{|u|^{q-2}u}{|x|^s}, & \text{in } \Omega, \\ -\operatorname{div}(|x|^{-ap}|\nabla v|^{p-2}\nabla v) = \dfrac{\beta}{\alpha+\beta}\dfrac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^{bp^*}} + \mu f(x)\frac{|v|^{q-2}v}{|x|^s}, & \text{in } \Omega, \\ u > 0, v > 0, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

E-mail address: akramakhavan@yahoo.com (A. Akhavan).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

in which

$$\begin{cases} 1 < p < n, \quad -\infty < a < \frac{n-p}{p}, \quad a \leqslant b \leqslant a+1, \quad d = 1+a-b, \\ p^* = p(a,b) = \frac{pn}{n-pd}, \quad p(a,a) = \frac{2n}{n-2} = 2^*, \quad \alpha+\beta = p^*, \quad 1 < q < p^*, \\ s < (1+a)t + n\big(1 - \frac{t}{p}\big), \quad 1 < p_0 \leqslant \frac{np}{n-p}, \quad q < t < \frac{np}{n-p}, \quad \frac{1}{p_0} + \frac{q}{t} = 1, \\ \lambda, \mu \in \mathbb{R} \setminus \{0\}, \quad f(x) \in L^{p_0}(\Omega, |x|^{-s}), \quad f^{\pm} = \max\{\pm f, 0\} \neq 0 \qquad (\mathcal{H}), \end{cases}$$

where p^* and 2^* are the Hardy-Sobolev critical and the Sobolev critical exponents, respectively.

Using Caffarelli-Kohn-Nirenberg inequality [8, 17], we have

$$\Big(\int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*} dx\Big)^{\frac{p}{p^*}} \leqslant C_{a,p} \int_{\mathbb{R}^n} |x|^{-ap} |\nabla u|^p dx \quad \text{for all} \quad u \in C_0^{+\infty}(\mathbb{R}^n), \tag{2}$$

where $1 < p < n$, $-\infty < a < \frac{n-p}{p}$, $a \leqslant b \leqslant a+1$, $p^* = \frac{np}{n-pd}$, $d = 1+a-b$ and $C_{a,b} > 0$.

The completion of $C_0^{+\infty}(\Omega)$ is written by $W_0^{1,p}(\Omega, |x|^{-ap})$ regarding the norm

$$||u|| = \Big(\int_\Omega |x|^{-ap} |\nabla u|^p dx\Big)^{\frac{1}{p}}$$

for $1 < p < n$ and $-\infty < a < \frac{n-p}{p}$.

Using the inequality (2) and the boundedness of Ω, Xuan [17] showed that there exists $C > 0$ provided that

$$\Big(\int_\Omega \frac{|u|^t}{|x|^s} dx\Big)^{\frac{p}{t}} \leqslant C \int_\Omega \frac{|\nabla u|^p}{|x|^{ap}} dx, \quad \text{for all} \quad u \in W_0^{1,p}(\Omega, |x|^{-ap}) \tag{3}$$

in which $1 \leqslant t \leqslant \frac{np}{n-p}$, $s \leqslant (a+1)t + n[1-(t/p)]$, saying Caffarelli-Kohn-Nirenberg's inequality. On the other hand, the embedding $H_0^1(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-s})$ is continuous when $1 \leqslant t \leqslant \frac{np}{n-p}$ and $s \leqslant (a+1)t + n[1-(t/p)]$. Also, it is compact when $1 \leqslant t \leqslant \frac{np}{n-p}$ and $s \leqslant (a+1)t + n[1-(t/p)]$ (see [17, Theorem 2.1] for $\nu = 0$). Moreover, consider the space $W = \Big(W_0^{1,p}(\Omega, |x|^{-ap})\Big)^2$ with the norm

$$||(u,v)|| = \Big(\int_\Omega |x|^{-ap} |\nabla u|^p dx + \int_\Omega |x|^{-ap} |\nabla v|^p dx\Big)^{\frac{1}{p}}.$$

In addition, take the best constant Hardy-Sobolev constant $S_{a,b}$ as follows:

$$C^* = C^*_{a,p}(\Omega) = \inf_{u \in W_0^{1,p}(\Omega, |x|^{-ap}) \setminus \{0\}} \frac{\int_\Omega |x|^{-ap} |\nabla u|^p dx}{\Big(\int_\Omega |x|^{-bp^*} |u|^{p^*} dx\Big)^{\frac{p}{p^*}}}. \tag{4}$$

First, let's define some notations. Take Ω a domain in $\mathbb{R}^n$, $0 \in \Omega$, $1 < p < n$, $0 \leqslant a <$

$(n-p)/p$, $a \leqslant b < a+1$ and $p^* = \frac{pn}{n-pd}$, and set

$$S := \inf\Big\{ \frac{\int_\Omega |x|^{-ap}\big(|\nabla u|^p + |\nabla v|^p\big)dx}{\Big(\int_\Omega |x|^{-bp^*}|u|^\alpha |v|^\beta dx\Big)^{\frac{p}{p^*}}} \ : \ (u,v) \in W \setminus \{0\}\Big\}. \tag{5}$$

Then, we have

$$S := \Big[(\frac{\alpha}{\beta})^{\frac{\beta}{p^*}} + (\frac{\beta}{\alpha})^{\frac{\alpha}{p^*}}\Big] C^* = KC^*, \tag{6}$$

where $K = K(\alpha, \beta, p^*)$ ([1]). Moreover, we consider the space

$$W^{1,p}_{a,b}(\Omega) = \{u \in L^{p^*}(\Omega, |x|^{-bp^*}) \ : \ |\nabla u| \in L^p(\Omega, |x|^{-ap})\}$$

with the norm $||u||_{W^{1,p}_{a,b}(\Omega)} := ||u||_{L^{p^*}(\Omega,|x|^{-bp^*})} + ||\nabla u||_{L^p(\Omega,|x|^{-ap})}$. In addition, we take the constant $\widetilde{S}_{a,p}$ given by

$$\widetilde{S}_{a,p} := \inf\Big\{ \frac{\int_{\mathbb{R}^n} |x|^{-ap}|\nabla u|^p dx}{\Big(\int_{\mathbb{R}^n} |x|^{-bp^*}|u|^{p^*} dx\Big)^{\frac{p}{p^*}}} \ : \ u \in W^{1,p}_{a,b}(\mathbb{R}^n) \setminus \{0\}\Big\}.$$

Further, we define $R^{1,p}_{a,b}(\Omega) = \{u \in W^{1,p}_{a,b}(\Omega) : u(x) = u(|x|)\}$ with the norm $||u||_{R^{1,p}_{a,b}(\Omega)} = ||u||_{W^{1,p}_{a,b}(\Omega)}$. On the other hand, Horiuchi [10] proved that if $a \geqslant 0$, then

$$\widetilde{S}_{a,p,R} := \inf\Big\{ \frac{\int_{\mathbb{R}^n} |x|^{-ap}|\nabla u|^p dx}{\Big(\int_{\mathbb{R}^n} |x|^{-bp^*}|u|^{p^*} dx\Big)^{\frac{p}{p^*}}} \ : \ u \in R^{1,p}_{a,b}(\mathbb{R}^n) \setminus \{0\}\Big\} = \widetilde{S}_{a,p}, \tag{7}$$

and it is established by functions of the form $y_\epsilon(x) := k_{a,p}(\epsilon) U_{a,p,\epsilon}(x)$ for all $\epsilon > 0$, in which

$$k_{a,p}(\epsilon) = \widetilde{c}\epsilon^{\frac{n-pd}{p^2 d}}, \ \text{ and } \ U_{a,p,\epsilon}(x) = \Big(\epsilon + |x|^{\frac{pd(n-p-ap)}{(p-1)(n-pd)}}\Big)^{-\frac{n-pd}{pd}}.$$

It follows from the Caffarelli-Kohn-Nirenberg's inequality that $W^{1,p}_0(\Omega, |x|^{-ap})$ is a subset of $W^{1,p}_{a.e}(\mathbb{R}^n)$ and so $\widetilde{S}_{a,p} \leqslant C^*$.

Lemma 1.1 [13] Let R_1 and c_1 be positive constants, where $B(0, 3R_1) \subset \Omega$ and $\psi \in C^{+\infty}_0(B(0, 3R_1))$ with $\psi \geqslant 0$ in $B(0, 3R_1)$ and $\psi = 1$ in $B(0, 2R_1)$. Then the function given by

$$u_\epsilon(x) := \frac{\psi(x) U_{a,p,\epsilon}(x)}{||\psi U_{a,p,\epsilon}||_{L^{p^*}(\Omega,|x|^{-bp})}}$$

satisfies in the following conditions:

$$||u_\epsilon||^{p^*}_{L^{p^*}(\Omega,|x|^{-bp})} = 1 \quad \text{ and } \quad ||\nabla u_\epsilon||^p_{L^p(\Omega,|x|^{-ap})} \leqslant \widetilde{S}_{a,p,R} + O(\epsilon^{\frac{n-pd}{pd}}),$$

and

$$||f^{1/q}u_\epsilon||^q_{L^q(\Omega,|x|^{-s})} \geqslant \begin{cases} O(\epsilon^{\frac{(n-pd)q}{p^2d}}), & \text{if } q < \frac{(n-s)(p-1)}{n-p-ap}, \\ O(\epsilon^{\frac{(n-pd)q}{p^2d}}|\ln(\epsilon)|), & \text{if } q = \frac{(n-s)(p-1)}{n-p-ap}, \\ O(\epsilon^{\frac{(n-pd)(p-1)[(n-s)p-(n-p-ap)q]}{p^2d(n-p-ap)}}), & \text{if } q > \frac{(n-s)(p-1)}{n-p-ap}, \end{cases} \quad (8)$$

for all $f \in L^{p_0}(\Omega,|x|^{-s})$ with $f(x) \geqslant 0$ for $x\ inB(0,3R_1)$ and $\inf_{B(0,2R)} f > 0$ for some $0 < R \leqslant R_1$. Moreover, (8) is uniform in $f \in L^{p_0}(\Omega,|x|^{-s})$ satisfying $f(x) \geqslant 0$ with $x \in B(0,3R_1)$ and

$$\Big(1 + R^{\frac{pd(n-p-ap)}{(p-1)(n-pd)}}\Big)^{-\frac{(n-pd)q}{pd}} R^{n-s} \inf_{B(0,2R)} f \geqslant c_0 \text{ for some } R \in (0,R_0].$$

Furthermore, we put

$$\Theta_t = \big\{(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\} \mid 0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < t\big\},$$

where $||f||_s = ||f||_{L^{p_0}(\Omega,|x|^{-s})}$.

The main purpose of this paper is to prove two following theorems.

Theorem 1.2 Beside $(\mathcal{H})$, suppose that R_0 and c_0 are positive constants and $B(0,3R_0) \subset \Omega$. Then there exists $\Upsilon > 0$ provided that the problem (1) has a positive solution for each $(\lambda,\mu) \in \Theta_\Upsilon$ and for each $f \in L^{p_0}(\Omega,|x|^{-s})$ satisfying $f(x) \geqslant 0$ for all $x \in B(0,3R_0)$,

$$\Big(1 + R^{\frac{pd(n-p-ap)}{(p-1)(n-pd)}}\Big)^{-\frac{(n-pd)q}{pd}} R^{n-s} \inf_{B(0,2R)} f \geqslant c_0 \text{ for some } R \in (0,R_0].$$

Theorem 1.3 Beside $(\mathcal{H})$, suppose that R_0 and c_0 are positive constants and $B(0,3R_0) \subset \Omega$. Then there exists $\Upsilon_0 > 0$ provided that the problem (1) has at least two positive solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) for all $(\lambda,\mu) \in \Theta_{\Upsilon_0}$ and for each $f \in L^{p_0}(\Omega,|x|^{-s})$ satisfying $f(x) \geqslant 0$ for all $x \in B(0,3R_0)$,

$$\Big(1 + R^{\frac{pd(n-p-ap)}{(p-1)(n-pd)}}\Big)^{-\frac{(n-pd)q}{pd}} R^{n-s} \inf_{B(0,2R)} f \geqslant c_0 \text{ for some } R \in (0,R_0].$$

2. Nehari manifold

In the following, we introduce the corresponding energy functional of the problem (1) in W^*:

$$I_{\lambda,\mu}(u,v) = \frac{1}{p}||(u,v)||^p - \frac{1}{\alpha+\beta}\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^{bp^*}} - \frac{1}{q}K_{\lambda,\mu}(u,v),$$

for all $(u,v) \in W$, where

$$K_{\lambda,\mu}(u,v) = \lambda \int_\Omega f|x|^{-s}|u|^q dx + \mu \int_\Omega f|x|^{-s}|v|^q dx.$$

Using the weighted Hardy-Sobolev inequality, $I_{\lambda,\mu} \in C^1(W,\mathbb{R})$. Since the energy functional $I_{\lambda,\mu}$ isn't bounded below on W, it's useful to take the functional on the Nehari manifold. Also, the solutions of system (1) are the critical points of the energy functional $I_{\lambda,\mu}$. If $I_{\lambda,\mu}$ is bounded below and has a minimizer on W, then this minimizer is a critical point of $I_{\lambda,\mu}$. Hence, it's a solution of the corresponding elliptic equation. However, this energy functional isn't bounded below on the whole space W, but it's bounded on an appropriate subset, called Nehari manifold.

$$N_{\lambda,\mu} = \big\{(u,v) \in W \setminus \{(0,0)\} | \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\big\},$$

where

$$\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = ||(u,v)||^p - \int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx - K_{\lambda,\mu}(u,v).$$

Note that $N_{\lambda,\mu}$ contains each nonzero solution of (1). If we define $\Phi_{\lambda,\mu}(u,v) = \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle$, then

$$\begin{aligned}\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle &= p||(u,v)||^p - p^* \int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx - qK_{\lambda,\mu}(u,v)\\ &= (p-q)||(u,v)||^p - (p^*-q)\int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx\\ &= (p-p^*)||(u,v)||^p - (q-p^*)K_{\lambda,\mu}(u,v)\\ &= (p-p^*)\int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx - (q-p)K_{\lambda,\mu}(u,v). \qquad (9)\end{aligned}$$

for $(u,v) \in N_{\lambda,\mu}$. Now, we break $N_{\lambda,\mu}$ in three parts:

$$\begin{aligned}N^+_{\lambda,\mu} &= \big\{(u,v),(u,v) \in N_{\lambda,\mu} : \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle > 0\big\},\\ N^0_{\lambda,\mu} &= \big\{(u,v) \in N_{\lambda,\mu} : \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\big\},\\ N^-_{\lambda,\mu} &= \big\{(u,v) \in n_{\lambda,\mu} : \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle < 0\big\}.\end{aligned}$$

To prove our main result, we now state some important properties of $N^+_{\lambda,\mu}$, $N^0_{\lambda,\mu}$ and $N^-_{\lambda,\mu}$.

Lemma 2.1 There exists a positive number $\Upsilon = \Upsilon(q,n,,K,C,C^*) > 0$ so that $(\lambda,\mu) \in \Theta_\Upsilon$ implies that $N^0_{\lambda,\mu} = \emptyset$.

Proof. Assume that

$$\Upsilon = \Big(\frac{p-q}{(p^*-q)}\Big)^{\frac{p}{p^*-p}}\Big(\frac{p^*-p}{p^*-q}\Big)^{\frac{p}{p-q}}(KC^*)^{-\frac{p^*}{p^*-p}}C^{-\frac{q}{p-q}}.$$

Then there exists (λ, μ) with

$$0 < \left(|\lambda|||f||_s\right)^{\frac{p}{p-q}} + \left(|\mu|||f||_s\right)^{\frac{p}{p-q}} < \Upsilon$$

such that $N^0_{\lambda,\mu} \neq \emptyset$. Then, for $(u,v) \in N^0_{\lambda,\mu}$ and by (9), we get

$$\begin{aligned} 0 &= \langle \Phi'_{\lambda,\mu}(u,v), (u,v)\rangle \\ &= (p-q)||(u,v)||^p - (p^*-q)\int_\Omega |x|^{-bp^*}|u|^\alpha |v|^\beta dx \\ &= (p-p^*)||(u,v)||^p - (q-p^*)K_{\lambda,\mu}(u,v). \end{aligned} \tag{10}$$

It follows from (5) and (10) that

$$\frac{p-q}{p^*-q}||(u,v)||^p = \int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx \leqslant (KC^*)^{\frac{p^*}{p}}||(u,v)||^{p^*}.$$

Thus,

$$||(u,v)|| \geqslant \left(\frac{p-q}{p^*-q}(KC^*)^{-\frac{p^*}{p}}\right)^{\frac{1}{p^*-p}}. \tag{11}$$

Also, using (10), we have

$$\begin{aligned} \frac{p^*-p}{p^*-q}||(u,v)||^p &= K_{\lambda,\mu}(u,v) \\ &= \int_\Omega \lambda f|x|^{-s}|u|^q dx + \int_\Omega \mu f|x|^{-s}|v|^q dx \\ &\leqslant C^{\frac{q}{p}}(|\lambda|||f||_s||u||^q + |\mu|||f||_s||v||^q) \\ &\leqslant C^{\frac{q}{p}}\left((|\lambda|||f||_s)^{\frac{p}{p-q}} + (|\mu|||f||_s)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}||(u,v)||^q, \end{aligned}$$

implying that

$$||(u,v)|| \leqslant \left(\frac{p^*-q}{p^*-p}C^{\frac{q}{p}}\right)^{\frac{1}{p-q}}\left[(|\lambda|||f||_s)^{\frac{p}{p-q}} + (|\mu|||f||_s)^{\frac{p}{p-q}}\right]^{\frac{1}{p}}. \tag{12}$$

Using (11) and (12), we deduce that $\left(|\lambda|||f||_s\right)^{\frac{p}{p-q}} + \left(|\mu|||f||_s\right)^{\frac{p}{p-q}} \geqslant \Upsilon$, which is contradiction. Hence, there exists $\Upsilon > 0$ so that for $0 < \left(|\lambda|||f||_s\right)^{\frac{p}{p-q}} + \left(|\mu|||f||_s\right)^{\frac{p}{p-q}} < \Upsilon$ and we have $N^0_{\lambda,\mu} = \emptyset$. ■

Lemma 2.2 The energy functional $I_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$.

Proof. Let $(u,v) \in n_{\lambda,\mu}$. Using Hölder inequality and Caffarelli-Kohn-Nirenberg's inequality, we obtain

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \frac{p^*-p}{pp^*}||(u,v)||^p - \frac{p^*-q}{qp^*}K_{\lambda,\mu}(u,v) \\ &\geqslant \frac{p^*-p}{pp^*}||(u,v)||^p - \frac{p^*-q}{qp^*}C^{\frac{q}{p}}\Big[\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}\Big]^{\frac{p-q}{p}}||(u,v)||^q. \end{aligned}$$

Since $1 < q < p$, $I_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$. ■

Further, similar to the argument in Brown and Zhang [2, Theorem 2.3], we will have following lemma.

Lemma 2.3 Let $(u_0, v_0) \in N_{\lambda,\mu}$ be a local minimizer of $I_{\lambda,\mu}$ such that $(u_0, v_0) \notin N^0_{\lambda,\mu}$. Then $I'_{\lambda,\mu}(u_0, v_0) = 0$ in W^{-1}, where W^{-1} is the dual space of W.

Also, take $\Upsilon_0 = \big(\frac{q}{p}\big)^{\frac{p}{p-q}}\Upsilon < \Upsilon$. If $(\lambda,\mu) \in \Theta_{\Upsilon_0}$, then we gain $N_{\lambda,\mu} = N^+_{\lambda,\mu} \cup N^-_{\lambda,\mu}$. If we define

$$\begin{aligned} \theta_{\lambda,\mu} &= \inf_{(u,v)\in N_{\lambda,\mu}} I_{\lambda,\mu}(u,v), \\ \theta^+_{\lambda,\mu} &= \inf_{(u,v)\in N^+_{\lambda,\mu}} I_{\lambda,\mu}(u,v), \\ \theta^-_{\lambda,\mu} &= \inf_{(u,v)\in N^-_{\lambda,\mu}} I_{\lambda,\mu}(u,v), \end{aligned}$$

then we will have the following lemma.

Lemma 2.4 For each $(\lambda,\mu) \in \Theta_{\Upsilon_0}$ there exists a positive number Υ_0 such that
(i) $\theta_{\lambda,\mu} < \theta^+_{\lambda,\mu} < 0$;
(ii) $\theta^-_{\lambda,\mu} > \delta$, for some $\delta = \delta(p, q, n, \lambda, \mu, K, C^*) > 0$

Proof. (i) Let $(u, v) \in N^+_{\lambda,\mu}$. Using (9), we obtain

$$K_{\lambda,\mu}(u,v) \geqslant \frac{p^*-p}{p^*-q}||(u,v)||^p,$$

implying that

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= (\frac{1}{p} - \frac{1}{p^*})||(u,v)||^p - (\frac{1}{q} - \frac{1}{p^*})K_{\lambda,\mu}(u,v) \\ &\leqslant (\frac{1}{p} - \frac{1}{p^*})||(u,v)||^p - (\frac{1}{q} - \frac{1}{p^*})\frac{p^*-p}{p^*-q}||(u,v)||^p \\ &\leqslant \frac{p^*-p}{p^*}(\frac{1}{p} - \frac{1}{q})||(u,v)||^p < 0. \end{aligned}$$

Hence, it follows from the definition of $\theta_{\lambda,\mu}$ and $\theta^+_{\lambda,\mu}$ that $\theta_{\lambda,\mu} < \theta^+_{\lambda,\mu} < 0$.
(ii) Let $(u, v) \in N^-_{\lambda,\mu}$ and apply Lemma 2.1. Then we have

$$||(u,v)|| \geqslant \Big(\frac{p-q}{(p^*-q)}\Big)^{\frac{1}{p^*-p}}(KC^*)^{-\frac{p^*}{p(p^*-p)}}.$$

Moreover, by Lemma 2.2, we get

$$
\begin{aligned}
I_{\lambda,\mu}(u,v) &\geqslant \frac{p^*-p}{pp^*}||(u,v)||^p - \frac{p^*-q}{qp^*}C^{\frac{q}{p}}\Big[\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}\Big]^{\frac{p-q}{p}}||(u,v)||^q \\
&= ||(u,v)||^q\Big[\frac{p^*-p}{pp^*}||(u,v)||^{p-q} - \frac{p^*-q}{qp^*}C^{\frac{q}{p}}\Big(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\Big] \\
&\geqslant \Big(\frac{p-q}{(p^*-q)}\Big)^{\frac{q}{p^*-p}}(KC^*)^{-\frac{qp^*}{p(p^*-p)}}\Big[\frac{p^*-p}{pp^*}||(u,v)||^{p-q} - \frac{p^*-q}{qp^*}C^{\frac{q}{p}}\Big(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} \\
&\qquad + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\Big].
\end{aligned}
$$

Thus, if $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < \Upsilon_0$, then we obtain $I_{\lambda,\mu}(u,v) \geqslant \delta = \delta(p,q,n,K,C,\lambda,\mu) > 0$ for each $(u,v) \in N^-_{\lambda,\mu}$. ■

Now, set

$$t_{\max} = \Big[\big(\frac{p-q}{p^*-q}\big)\frac{||(u,v)||^p}{\int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx}\Big]^{\frac{1}{p^*-p}}$$

for each $(u,v) \in W \setminus \{(0,0)\}$. Then we have the following lemma.

Lemma 2.5 Let $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < \Upsilon_0$. Then, for each $(u,v) \in W$, there exists $t_{\max} > 0$ provided that

(i) If $K_{\lambda,\mu}(u,v) \leqslant 0$, then there is a unique $t^- > t_{\max}$ so that $(t^-u, t^-v) \in N^-_{\lambda,\mu}$ and

$$I_{\lambda,\mu}(t^-u,t^-v) = \sup_{t\geqslant 0} I_{\lambda,\mu}(tu,tv);$$

(ii) If $K_{\lambda,\mu}(u,v) > 0$, then there are unique t^+ and t^- with $0 < t^+ < t_{\max} < t^-$ so that $(t^+u, t^+v) \in N^+_{\lambda,\mu}$, $(t^-u,t^-v) \in N^-_{\lambda,\mu}$ and

$$I_{\lambda,\mu}(t^+u,t^+v) = \inf_{0\leqslant t\leqslant t_{\max}} I_{\lambda,\mu}(tu,tv) \quad \text{and} \quad I_{\lambda,\mu}(t^-u,t^-v) = \sup_{t\geqslant t_{\max}} I_{\lambda,\mu}(tu,tv).$$

Proof. Fix $(u,v) \in W$ and for $t \geqslant 0$, set

$$g(t) = t^{p-q}||(u,v)||^p - t^{p^*-q}\int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx.$$

Clearly, $g(0) = 0$ and $\lim_{t\to+\infty} g(t) = -\infty$. As

$$g'(t) = (p-q)t^{p-q-1}||(u,v)||^p - (p^*-q)t^{p^*-q-1}\int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx,$$

we have $g'(t) = 0$ at a unique number $t = t_{\max} > 0$, $g'(t) > 0$ for $t \in [0, t_{\max})$ and $g'(t) < 0$ for $t \in (t_{\max}, +\infty)$. Hence, $g(t)$ take its maximum at $t_{\max}$, increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, +\infty)$. It's clear that $(tu, tv) \in N^+_{\lambda,\mu}$ (or $(tu,tv) \in N^-_{\lambda,\mu}$) iff

$g'(t) > 0$ (or $g' < 0$). Additionally,

$$\begin{aligned}
g(t_{\max}) &= \Big[\big(\frac{p-q}{p^*-q}\big)\frac{||(u,v)||^p}{\int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx}\Big]^{\frac{p-q}{p^*-p}}||(u,v)||^p \\
&\quad - \Big[\big(\frac{p-q}{p^*-q}\big)\frac{||(u,v)||^p}{\int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx}\Big]^{\frac{p*-q}{p^*-p}}\int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx \\
&= ||(u,v)||^q\Big[\Big(\frac{p-q}{p^*-q}\Big)^{\frac{p-q}{p^*-p}} - \Big(\frac{p-q}{p^*-q}\Big)^{\frac{p^*-q}{p^*-p}}\Big]\Big(\frac{||(u,v)||^{p^*}}{\int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta dx}\Big)^{\frac{p-q}{p^*-p}} \\
&\geqslant \Big(\frac{p-q}{p^*-q}\Big)^{\frac{p-q}{p^*-p}}\Big(\frac{p^*-p}{p^*-q}\Big)\Big(\frac{1}{KC^*}\Big)^{\frac{p^*(p-q)}{p(p^*-p)}}||(u,v)||^q.
\end{aligned}$$

(i) If $K_{\lambda,\mu}(u,v) \leqslant 0$, then there is a unique $t^- > t_{\max}$ provided that $g(t^-) = K_{\lambda,\mu}(u,v)$ and $g'(t^-) < 0$. Now, we have

$$(p-q)(t^-)^p||(u,v)||^p - (p^*-q)(t^-)^{p^*}\int_\Omega |x|^{-bp^*}|u|^\alpha|v|^\beta)dx = (t^-)^{q+1}g(t^-) < 0$$

and

$$\langle I'_{\lambda,\mu}(t^-u,t^-v),(t^-u,t^-v)\rangle = (t^-)^q\Big[g(t^-) - K_{\lambda,\mu}(u,v)\Big] = 0.$$

Thus, $(t^-u,t^-v) \in N^-_{\lambda,\mu}$. Since we have $g'(t) < 0$ and $g''(t) < 0$ for $t > t_{\max}$, then

$$I_{\lambda,\mu}(t^-u,t^-v) = \sup_{t\geqslant 0} I_{\lambda,\mu}(tu,tv).$$

(ii) Assume $K_{\lambda,\mu}(u,v) > 0$. For $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_{L^{p_0}(\Omega,|x|^{-\beta})}\big)^{\frac{p}{p-q}} < \Upsilon_0 < \Upsilon$, we obtain

$$\begin{aligned}
g(0) = 0 &< K_{\lambda,\mu}(u,v) \\
&\leqslant C^{\frac{q}{p}}\Big(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}||(u,v)||^q \\
&\leqslant \Big(\frac{p-q}{p^*-q}\Big)^{\frac{p-q}{p^*-p}}\Big(\frac{p^*-p}{p^*-q}\Big)\Big(\frac{1}{KC^*}\Big)^{\frac{p^*(p-q)}{p(p^*-p)}}||(u,v)||^q \leqslant g(t_{\max}).
\end{aligned}$$

There are unique t^+ and t^- so that $0 < t^+ < t_{\max} < t^-$, $g(t^+) = K_{\lambda,\mu}(u,v) = g(t^-)$ and $g'(t^+) > 0 > g'(t^-)$. Now, we have $(t^+u,t^+v) \in N^+_{\lambda,\mu}$, $(t^-u,t^-v) \in N^-_{\lambda,\mu}$ and

$$I_{\lambda,\mu}(t^-u,t^-v) \geqslant I_{\lambda,\mu}(tu,tv) \geqslant I_{\lambda,\mu}(t^+u,t^+v)$$

for all $t \in [t^+,t^-]$ and $I_{\lambda,\mu}(t^+u,t^+v) \leqslant I_{\lambda,\mu}(tu,tv)$ for all $t \in [0,t_{\max}]$. Thus, we have

$$I_{\lambda,\mu}(t^+u,t^+v) = \inf_{0\leqslant t\leqslant t_{\max}} I_{\lambda,\mu}(tu,tv) \quad \text{and} \quad I_{\lambda,\mu}(t^-u,t^-v) = \sup_{t\geqslant t_{\max}} I_{\lambda,\mu}(tu,tv).$$

∎

3. Proof of main results

Before the proof of Theorem 1.2 and Theorem 1.3, we need the following results.

Lemma 3.1 [16]

(i) Let $(\lambda,\mu) \in \Theta_\Upsilon$. Then there exists a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset N_{\lambda,\mu}$ in W for $I_{\lambda,\mu}$;

(ii) Let $(\lambda,\mu) \in \Theta_{\Upsilon_0}$. Then there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset N^-_{\lambda,\mu}$ in W for $I_{\lambda,\mu}$.

Theorem 3.2 Let $(\lambda,\mu) \in \Theta_\Upsilon$ and $(\mathcal{H})$ hold. Then $I_{\lambda,\mu}$ has a minimizer (u_0^+, v_0^+) in $N^+_{\lambda,\mu}$ and satisfies the following conditions:

(i) $I_{\lambda,\mu}(u_0^+, v_0^+) = \theta^+_{\lambda,\mu}$,

(ii) (u_0^+, v_0^+) is a solution of problem (1) provided that $u_0^+ \geqslant 0$ and $v_0^+ \geqslant 0$ in Ω.

Proof. Using Lemma 3.1(i), there exists a minimizing sequence $\{(u_n, v_n)\}$ for $I_{\lambda,\mu}$ on $N_{\lambda,\mu}$ provided that

$$I_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu} + o(1) \text{ and } I'_{\lambda,\mu}(u_n, v_n) = o(1) \text{ in } W^{-1}. \tag{13}$$

Then, by Lemma 2.2 and the continuity of embedding theorem, there exists a subsequence $\{(u_n, v_n)\}$ and $(u_0^+, v_0^+)) \in W$ provided that

$$\begin{cases} u_n \rightharpoonup u_0^+, \ \ v_n \rightharpoonup v_0^+, & \text{weakly in } W_0^{1,p}(\Omega, |x|^{-ap}),\\ u_n \to u_0^+, \ \ v_n \to v_0^+, & \text{strongly in } L^q(\Omega, |x|^{-s}),\\ u_n \to u_0^+, \ \ v_n \to v_0^+, & \text{a.e in } \Omega, \end{cases} \tag{14}$$

as $n \to +\infty$, which implies that $K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u_0^+, v_0^+)$ as $n \to +\infty$. By (13) and (14), it's easy to show that (u_0^+, v_0^+) is a weak solution of the problem (1). As

$$I_{\lambda,\mu}(u_n, v_n) = \Big(\frac{1}{p} - \frac{1}{p^*}\Big)||(u_n, v_n)||^p - \frac{p^* - q}{qp^*} K_{\lambda,\mu}(u_n, v_n) \geqslant -\frac{p^* - q}{qp^*} K_{\lambda,\mu}(u_n, v_n),$$

and by Lemma 2.2(i), $I_{\lambda,\mu}(u_n, v_n) \to \theta_{\lambda,\mu} < 0$ as $n \to +\infty$. Letting $n \to +\infty$, we have $K_{\lambda,\mu}(u_0^+, v_0^+) > 0$. Now, we show that

$$\begin{cases} u_n \to u_0^+, & \text{strongly in } W_0^{1,p}(\Omega, |x|^{-ap}),\\ v_n \to v_0^+, & \text{strongly in } W_0^{1,p}(\Omega, |x|^{-ap}), \end{cases}$$

and $I_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu}$. Applying Fatou's lemma and $(u_0^+, v_0^+) \in N_{\lambda,\mu}$, we obtain

$$\begin{aligned} \theta_{\lambda,\mu} &\leqslant I_{\lambda,\mu}(u_0^+, v_0^+) = \Big(\frac{1}{p} - \frac{1}{p^*}\Big)||(u_0^+, v_0^+)||^p - \frac{p^* - q}{qp^*} K_{\lambda,\mu}(u_0^+, v_0^+)\\ &\leqslant \liminf_{n\to+\infty} \Big[\Big(\frac{1}{p} - \frac{1}{p^*}\Big)||(u_n, v_n)||^p - \frac{p^* - q}{qp^*} K_{\lambda,\mu}(u_n, v_n)\Big]\\ &\leqslant \liminf_{n\to+\infty} I_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu}, \end{aligned}$$

implying that

$$I_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu} \quad \text{and} \quad \lim_{n\to+\infty} ||(u_n, v_n)||^p = ||(u_0^+, v_0^+)||^p.$$

Then, $u_n \to u_0^+$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$ and $v_n \to v_0^+$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$. In addition, we get $(u_0^+, v_0^+) \in N_{\lambda,\mu}^+$. Indeed, if $(u_0^+, v_0^+) \in N_{\lambda,\mu}^-$, by Lemma 2.5, there are unique t_0^+ and t_0^- provided that $(t_0^+ u_0^+, t_0^+ v_0^+) \in N_{\lambda,\mu}^+$, $(t_0^- u_0^+, t_0^- v_0^+) \in N_{\lambda,\mu}^-$ and $t_0^+ < t_0^- = 1$. As

$$\frac{d}{dt} I_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) = 0 \text{ and } \frac{d^2}{dt^2} I_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \leqslant t_0^-$ provided that $I_{\lambda,\mu}(t_0^+ u_.^+, t_0^+ v_0^+) < I_{\lambda,\mu}(\bar{t}_0 u_0^+, \bar{t}_0 v_0^+)$. Using Lemma 2.5, we obtain

$$I_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ u_0^+) < I_{\lambda,\mu}(\bar{t}_0 u_0^+, \bar{t}_0 u_0^+) \leqslant I_{\lambda,\mu}(t_0^- u_0^+, t_0^- v_0^+) = I_{\lambda,\mu}(u_0^+, v_0^+)$$

which contradicts $I_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu}^+$. As $I_{\lambda,\mu}(u_0^+, v_0^+) = I_{\lambda,\mu}(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in N_{\lambda,\mu}^+$ and by Lemma 2.2, (u_0^+, v_0^+) is non-negative solution of problem (1). ■

The following two lemmas are similar to that are proved by Hsu [11].

Lemma 3.3 If $\{(u_n, v_n)\} \in W$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$ with $(u_n, v_n) \rightharpoonup (u, v)$ in W, then $I'_{\lambda,\mu}(u, v) = 0$ and there exists a positive constant Υ depending on p, q, n, C and C^* so that $I_{\lambda,\mu}(u, v) \geqslant -\Big(\big(|\lambda| ||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu| ||f||_s\big)^{\frac{p}{p-q}}\Big)\Upsilon$.

Lemma 3.4 If $\{(u_n, v_n)\} \in W$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$, then $\{(u_n, v_n)\}$ is bounded in W.

Lemma 3.5 $I_{\lambda,\mu}$ satisfies the $(PS)_{c^*}$ condition with c^* satisfying

$$-\infty < c^* < c_\infty = \Big(\frac{1}{p} - \frac{1}{p^*}\Big)(KC^*)^{\frac{p^*}{p^*-p}} - \Upsilon\Big(\big(|\lambda| ||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu| ||f||_s\big)^{\frac{p}{p-q}}\Big).$$

Proof. Let $\{(u_n, v_n)\} \in W$ be a $(PS)_{c^*}$-sequence for $I_{\lambda,\mu}$ with $c^* \in (-\infty, c_\infty)$. It follows from Lemma 3.4 that $\{(u_n, v_n)\}$ is bounded in W, and then $(u_n, v_n) \rightharpoonup (u, v)$ up to a subsequence, (u, v) is a critical point of $I_{\lambda,\mu}$. Also, we may assume

$$\begin{cases} u_n \rightharpoonup u, \;\; v_n \rightharpoonup v, & \text{weakly in } W_0^{1,p}(\Omega, |x|^{-ap}), \\ u_n \to u, \;\; u_n \to u, & \text{strongly in } L^q(\Omega, |x|^{-s})), \\ u_n \to u, \;\; u_n \to u, & \text{a.e. on } \Omega. \end{cases}$$

Hence, we have $I'_{\lambda,\mu}(u, v) = 0$ and

$$K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u, v) \;\text{ as }\; n \to +\infty. \tag{15}$$

Let $\widetilde{u}_n = u_n - u$ and $\widetilde{v}_n = v_n - v$. Then by Brèzis-Lieb lemma [3], we obtain

$$||(\widetilde{u}_n, \widetilde{v}_n)||^p \to ||(u_n, v_n)||^p - ||(u, v)||^p \text{ as } n \to +\infty \tag{16}$$

and

$$\int_\Omega |x|^{-bp^*} |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx \to \int_\Omega |x|^{-bp^*} |u_n|^\alpha, |v_n|^\beta dx - \int_\Omega |x|^{-bp^*} |u|^\alpha |v|^\beta dx, \quad \text{as} \quad n \to +\infty. \tag{17}$$

As $I_{\lambda,\mu}(u_n, v_n) = c^* + o(1)$, $I'_{\lambda,\mu}(u_n, v_n) = o(1)$ and (15)-(17), we conlude that

$$\frac{1}{p}||(\widetilde{u}_n, \widetilde{v}_n)||^p - \frac{1}{p^*} \int_\Omega |x|^{-bp^*} |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx = c^* - I_{\lambda,\mu}(u, v) + o(1), \tag{18}$$

and

$$||(\widetilde{u}_n, \widetilde{v}_n)||^p - \int_\Omega |x|^{-bp^*} |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx = o(1).$$

Thus, we can take

$$||(\widetilde{u}_n, \widetilde{v}_n)||^p \to l \quad \text{and} \quad \int_\Omega |x|^{-bp^*} |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx \to l. \tag{19}$$

If $l = 0$, the proof is complete. Let $l > 0$. It follows from (19) that

$$(KC^*)^{\frac{p}{p^*}} = (KC^*) \lim_{n \to +\infty} \Big[\int_\Omega |x|^{-bp^*} |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx \Big]^{\frac{p}{p^*}} \leqslant \lim_{n \to +\infty} ||(\widetilde{u}_n, \widetilde{v}_n)||^p = l,$$

implying that $l \geqslant (KC^*)^{\frac{p^*}{p^*-p}}$. Additionally, from Lemma 3.3, (18) and (19), we obtain

$$c^* = (\frac{1}{p} - \frac{1}{p^*}) l + I_{\lambda,\mu}(u, v) \geqslant (\frac{1}{p} - \frac{1}{p^*}) (KC^*)^{\frac{p^*}{p^*-p}} - \Upsilon \Big(\big(|\lambda| ||f||_s \big)^{\frac{p}{p-q}} + \big(|\mu| ||f||_s \big)^{\frac{p}{p-q}} \Big),$$

which contradicts $c^* < \big(\frac{1}{p} - \frac{1}{p^*}\big)(KC^*)^{\frac{p^*}{p^*-p}} - \Upsilon \Big(\big(|\lambda| ||f||_s \big)^{\frac{p}{p-q}} + \big(|\mu| ||f||_s \big)^{\frac{p}{p-q}} \Big)$. ■

Lemma 3.6 There exist a non-negative function $(u, v) \in W \setminus \{(0, 0)\}$ and $c_* > 0$ so that $\sup_{t \geqslant 0} I_{\lambda,\mu}(tu, tv) < KC^*$ for $0 < \big(|\lambda| ||f||_s \big)^{\frac{p}{p-q}} + \big(|\mu| ||f||_s \big)^{\frac{p}{p-q}} < c_*$. Particularly, $\theta^-_{\lambda,\mu} < c_\infty$ for all $0 < \big(|\lambda| ||f||_s \big)^{\frac{p}{p-q}} + \big(|\mu| ||f||_{L^{p_0}(\Omega, |x|^{-\beta})} \big)^{\frac{p}{p-q}} < c_*$.

Proof. Fix the constants $R_1 = R_0$ and $c_1 = c_0$ in Lemma 1.1 and define the functional $I : W \to \mathbb{R}$ by

$$I(u, v) = \frac{1}{p}||(u, v)||^p - \frac{1}{p^*} \int_\Omega |x|^{-bp^*} |u|^\alpha |v|^\beta dx \text{ for all } (u, v) \in W.$$

Set $u_0 = \alpha^{1/p} u_\epsilon$, $v_0 = \beta^{1/p} u_\epsilon$ for each $(u_0, v_0) \in W$. Then by Lemma 1.1, we obtain

$$\begin{aligned}
\sup_{t\geqslant 0} I(te_1u_\epsilon, te_2u_\epsilon) &\leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)\Big[\frac{(\alpha+\beta)\int_\Omega |x|^{-ap}|\nabla u_\epsilon|^p dx}{\big(\alpha^{\alpha/p}\beta^{\beta/p}\int_\Omega |x|^{-bp^*}|\nabla u_\epsilon|^p dx\big)^{\frac{p}{p^*}}}\Big]^{\frac{p^*}{p^*-p}} \\
&\leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)\Big[\Big(\frac{\alpha}{\beta}\Big)^{\beta/\alpha}+\Big(\frac{\beta}{\alpha}\Big)^{\alpha/\beta}\Big]^{\frac{p^*}{p^*-p}}\Big[\frac{\int_\Omega |x|^{-ap}|\nabla u_\epsilon|^p dx}{\big(\int_\Omega |x|^{-bp^*}|u_\epsilon|^{p^*}dx\big)^{\frac{p}{p^*}}}\Big]^{\frac{p^*}{p^*-p}} \\
&\leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)\Big[\Big(\frac{\alpha}{\beta}\Big)^{\beta/\alpha}+\Big(\frac{\beta}{\alpha}\Big)^{\alpha/\beta}\Big]^{\frac{p^*}{p^*-p}}(\widetilde{S}_{a,p,R}+O(\epsilon^{\frac{n-pd}{pd}}))^{\frac{p^*}{p^*-p}} \\
&\leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)\Big[\Big(\frac{\alpha}{\beta}\Big)^{\beta/\alpha}+\Big(\frac{\beta}{\alpha}\Big)^{\alpha/\beta}\Big]^{\frac{p^*}{p^*-p}}(C^*+O(\epsilon^{\frac{n-pd}{pd}}))^{\frac{p^*}{p^*-p}} \\
&\leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)(KC^*)^{\frac{p^*}{p^*-p}}+O(\epsilon^{\frac{n-pd}{pd}}), \qquad (20)
\end{aligned}$$

where the following fact has been used:

$$\sup_{t\geqslant 0}\Big(\frac{t^p}{p}A-\frac{t^{p^*}}{p^*}B\Big)=\Big(\frac{1}{p}-\frac{1}{p^*}\Big)\Big(\frac{A}{B^{\frac{p}{p^*}}}\Big)^{\frac{p^*}{p^*-p}}, \quad A,B>0.$$

We can choose $\delta_1 > 0$ so that

$$c_\infty = \Big(\frac{1}{p}-\frac{1}{p^*}\Big)(KC^*)^{\frac{p^*}{p^*-p}} - \Big(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}\Big)\Upsilon > 0.$$

for all $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < \delta_1$. Using the definitions $I(u,v)$ and (u_0,v_0), we have $I_{\lambda,\mu}(tu_0,tv_0) \leqslant \frac{t^p}{p}||(u_0,u_0)||^p$ for all $t \geqslant 0$ and $\lambda,\mu > 0$, implying there exists $t_0 \in (0,1)$ satisfying

$$\sup_{0\leqslant t\leqslant t_0} I_{\lambda,\mu}(t_0u_0,t_0u_0) < c_\infty \text{ for all } 0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < \delta_1.$$

Using (20) and applying the definitions $I(u,v)$ and (u_0,v_0), we have for $\alpha,\beta > 1$ that

$$\begin{aligned}
\sup_{t\geqslant t_0} I_{\lambda,\mu}(t_0u_0,t_0u_0) &= \sup_{t\geqslant t_0}\big(I(t_0u_0,t_0u_0)-\frac{t^q}{q}K_{\lambda,\mu}(u_0,v_0)\big) \\
&\leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)(KC^*)^{\frac{p^*}{p^*-p}}+O(\epsilon^{\frac{n-pd}{pd}}) \\
&\quad -\frac{t_0^q}{q}(\alpha^{q/p}\lambda+\beta^{q/p}\mu)\int_{B(x_0,R_0)}|x|^{-s}|u_\epsilon|^q dx \\
&\leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)(KC^*)^{\frac{p^*}{p^*-p}}+O(\epsilon^{\frac{n-pd}{pd}}) \\
&\quad -\frac{t_0^q}{q}(\lambda+\mu)\int_{B(x_0,R_0)}|x|^{-s}|u_\epsilon|^q dx. \qquad (21)
\end{aligned}$$

Also,

$$\frac{(n-pd)q}{p^2d} < \frac{n-pd}{pd}. \tag{22}$$

Now, let $q < \frac{(n-s)(p-1)}{n-p-ap}$. From (21), (22) and Lemma 1.1, we have

$$\sup_{t\geqslant t_0} I_{\lambda,\mu}(t_0u_0,t_0u_0) \leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)(KC^*)^{\frac{p^*}{p^*-p}} + O(\epsilon^{\frac{n-pd}{pd}}) - \frac{t_0^q}{q}(\lambda+\mu)O(\epsilon^{\frac{(n-pd)q}{p^2d}}) \tag{23}$$

Now, for all $\epsilon = \Big(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}\Big)^{\frac{pd}{n-pd}} \in (0,R_0)$, we obtain

$$\begin{aligned}\sup_{t\geqslant t_0} I_{\lambda,\mu}(t_0u_0,t_0v_0) \leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)(KC^*)^{\frac{p^*}{p^*-p}} + O(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}})\\ - \frac{t_0^q}{q}(\lambda+\mu)\Big(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}\Big)^{\frac{q}{p}}.\end{aligned}$$

Thus, we can choose $\delta_2 > 0$ so that

$$\begin{aligned}O(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}) - \frac{t_0^q}{q}(\lambda+\mu)\Big(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}}\\ + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}\Big)^{\frac{q}{p}} \leqslant -\Upsilon(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}).\end{aligned}$$

for all $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < \delta_2$. If we set $c_* = \min\{\delta_1, R_0, \delta_2\}$ and $\epsilon = \Big(\big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}}\Big)^{\frac{pd}{n-pd}}$, then

$$\sup_{t\geqslant t_0} I_{\lambda,\mu}(t_0u_0,t_0u_0) \leqslant c_\infty. \tag{24}$$

for $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < c_*$. Similarly, let $q = \frac{(n-s)(p-1)}{n-p-ap}$. It follows from (21), (22) and Lemma 1.1 that

$$\sup_{t\geqslant t_0} I_{\lambda,\mu}(t_0u_0,t_0u_0) \leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)(KC^*)^{\frac{p^*}{p^*-p}} + O(\epsilon^{\frac{n-pd}{pd}}) - \frac{t_0^q}{q}(\lambda+\mu)O(\epsilon^{\frac{(n-pd)q}{p^2d}}|\ln\epsilon|). \tag{25}$$

If $q > \frac{(n-s)(p-1)}{n-p-ap}$, then

$$\begin{aligned}\sup_{t\geqslant t_0} I_{\lambda,\mu}(t_0u_0,t_0u_0) \leqslant \Big(\frac{1}{p}-\frac{1}{p^*}\Big)(KC^*)^{\frac{p^*}{p^*-p}} + O(\epsilon^{\frac{n-pd}{pd}})\\ - \frac{t_0^q}{q}(\lambda+\mu)O(\epsilon^{\frac{(n-pd)(p-1)[(n-\beta)p-(n-p-ap)q]}{p^2d(n-p-ap)}}).\end{aligned} \tag{26}$$

Now, by (25) and (26), we have

$$\sup_{t\geqslant t_0} I_{\lambda,\mu}(t_0u_0, t_0u_0) \leqslant c_\infty. \tag{27}$$

Ultimately, we prove $\theta^-_{\lambda,\mu} < c_\infty$ for all $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < c_*$. Note that $(u_0, v_0) = (\alpha^{1/p}u_\epsilon, \beta^{1/p}u_\epsilon)$. It is easy to see that

$$\int_\Omega |x|^{-bp^*}|u_0|^\alpha|v_0|^\beta dx > 0.$$

Now, using Lemma 2.5, definition $\theta^-_{\lambda,\mu}$, (24) and (27), we there exists $t_0 > 0$ so that $(t_0u_0, t_0v_0) \in N^-_{\lambda,\mu}$ and

$$\theta^-_{\lambda,\mu} \leqslant I_{\lambda,\mu}(t_0u_0, t_0v_0) \leqslant \sup_{t\geqslant t_0} I_{\lambda,\mu}(t_0u_0, t_0u_0) < c_\infty,$$

for all $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < c_*$. ■

Theorem 3.7 Let $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < c'_*$, where $c'_* = \min\{c_*, \Upsilon_0\}$ and $(\mathcal{H})$ holds. Then $I_{\lambda,\mu}$ has a minimizer (u^-_0, v^-_0) in $N^-_{\lambda,\mu}$ and satisfies

(i) $I_{\lambda,\mu}(u^-_0, v^-_0) = \theta^-_{\lambda,\mu}$,
(ii) (u^-_0, v^-_0) is a solution of the problem (1) so that $u^-_0 \geqslant 0$ and $v^-_0 \geqslant 0$ in Ω.

Proof. Using Lemma 3.1(ii), there exists a minimizing sequence $\{(u_n, v_n)\} \subset N^-_{\lambda,\mu}$ in W for $I_{\lambda,\mu}$ and for all $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < \Upsilon_0$. It follows from Lemmas 3.5, 3.6 and 2.3(ii) that $I_{\lambda,\mu}$ satisfies $(PS)_{\theta^-_{\lambda,\mu}}$ condition and $\theta^-_{\lambda,\mu} > 0$ for $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < c_*$. Since $I_{\lambda,\mu}$ is coercive on $N_{\lambda,\mu}$, we conclude that (u_n, v_n) is bounded in W. Hence, there exist a subsequence still denote by (u_n, v_n) and $(u^-_0, v^-_0) \in N^-_{\lambda,\mu}$ so that $(u_n, v_n) \to (u^-_0, v^-_0)$ strongly in W and $I_{\lambda,\mu}(u^-_0, v^-_0) = \theta^-_{\lambda,\mu} > 0$ for all $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < c'_*$. Now, similar to the same arguments in proof of Theorem 3.2, (u^-_0, v^-_0) is a positive solution of problem (1) for all $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < c'_*$. ■

Now, we are ready to complete the proof of Theorem 1.2 and Theorem 1.3. Applying Theorem 3.2, we conclude that problem (1) has a positive solution $(u^+_0, v^+_0) \in N^+_{\lambda,\mu}$ for all $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < \Upsilon$. On the other hand, from Theorem 3.7, we obtain the second positive solution $(u^-_0, v^-_0) \in N^-_{\lambda,\mu}$ for all $0 < \big(|\lambda|||f||_s\big)^{\frac{p}{p-q}} + \big(|\mu|||f||_s\big)^{\frac{p}{p-q}} < c'_* < \Upsilon_0$. Since $N^+_{\lambda,\mu} \cap N^-_{\lambda,\mu} = \emptyset$, (u^+_0, v^+_0) and (u^-_0, v^-_0) are distinct. This completes the proof of Theorem 1.2 and Theorem 1.3.

4. Conclusion

Although, a system of nonlinear, quasilinear, sublinear or semi-linear of elliptic equations are solved by several authors [1–13, 15, 16], a variational approach has been used to solve quasilinear elliptic systems with critical Hardy-Sobolev and sign-changing function exponents.

References

[1] C. O. Alves, D. C. De Morais Filho, M. A. S. Souto, On systems of elliptic equations involving subcritical or critical Sobolev exponents, Nonlinear Anal. 42 (2000), 771-787.
[2] P. A. Binding, P. Drabek, Y. X. Huang, On Neumann boundary value problems for some quasilinear elliptic equations, Electron. J. Differential Equations. 5 (1997), 1-11.
[3] H. Brèzis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983), 486-490.
[4] K. J. Brown, The Nehari manifold for a semilinear elliptic equation involving a sublinear term, Calc. Var. 22 (2005), 483-494.
[5] K. J. Brown, T. F. Wu, A semilinear elliptic system involving nonlinear boundary condition and sign changing weight function, J. Math. Anal. Appl. 337 (2008), 1326-1336.
[6] K. J. Brown, T. F. Wu, On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function, J. Math. Anal. Appl. 318 (2006), 253-270.
[7] K. J. Brown, Y. Zhang, The Nehari manifold for a semilinear elliptic problem with a sign changing weight function, J. Differential Equations. 193 (2003), 481-499.
[8] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights, Compos. Math. 53 (1984), 259-275.
[9] P. Drabek, S. I. Pohozaev, Positive solutions for the p-Laplacian: application of the fibering method, Proc. Roy. Soc. Edinburgh Sect. A. 127 (1997), 721-747.
[10] T. Horiuchi, Best constant in weighted Sobolev inequality with weights being powers of distance from origin, J. Inequal. Appl. 1 (1997), 275-292.
[11] T. S. Hsu, Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities, Nonlinear Anal. 71 (2009), 2688-2698.
[12] D. Kang, Nontrivial solutions to semilinear elliptic problems involving two critical Hardy-Sobolev exponents, Nonlinear Anal. 72 (2010), 4230-4243.
[13] O. H. Miyagaki, R. S. Rodrigues, On multiple solutions for a singular quasilinear elliptic system involving critical Hardy-Sobolev exponents, Houston J. Math. 34 (2008), 1271-1293.
[14] Z. Nehari, On a class of nonlinear second-order differential equations, Trans. Amer. Math. Soc. 95 (1960), 101-123.
[15] V. Todorčević, Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics, Springer Cham, 2019.
[16] T. F. Wu, On semilinear elliptic equations involving concave-convex nonlinearities and sign changing weight function, J. Math. Anal. Appl. 318 (2006), 253-270.
[17] B. Xuan, The solvability of quasilinear Brezis-Nirenberg-type problems with singular weights, Nonlinear Anal. 62 (2005), 703-725.