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The triples of (v, u, ϕ)-contraction and (q, p, ϕ)-contraction in b-metric spaces and its application

E. L. Ghasab[a,*], H. A. Ebadizadeh[a], J. Sharafi[a]

[a] *Mathmatics Group, Faculty of Basic Sciences, Emam Ali University, Tehran, Iran.*

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Abstract. The aim of this work is to introduce the concepts of (v, u, ϕ)-contraction and (q, p, ϕ)-contraction, and to obtain new results in fixed point theory for four mappings in b-metric spaces. Finally, we have developed an example and an application for a system of integral equations that protects the main theorems.

Keywords: b-metric space, ϕ-function, (v, u, ϕ)-contraction, (q, p, ϕ)-contraction.

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1. Introduction and preliminaries

We start this research with the definition of a b-metric on a non-empty set $\mathcal{X}$, which is introduced by Bakhtin [2] and Czerwik [7].

Definition 1.1 [7] A mapping $d : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ is named a b-metric with a parameter $s \geqslant 1$ if, for all $x, y, z \in \mathcal{X}$, the following conditions are held:

(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

In this case, $(\mathcal{X}, d)$ is called a b-metric space.

Each metric space is a b-metric space with coefficient $s = 1$. Therefore, the class of b-metric spaces is larger than the class of metric spaces.

*Corresponding author.
E-mail address: e.l.ghasab@gmail.com (E. L. Ghasab); Ebadizadeh.h@gmail.com (H. A. Ebadizadeh); Javad_sharafi6@yahoo.com (J. Sharafi).

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***Example* 1.2** [1] For $p \in (0,1)$, take $X = l_p(\mathbb{R}) = \{x = \{x_n\} \subset \mathbb{R} : \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$. Define $d(x,y) = (\sum\limits_{n=1}^{\infty} |x_n - yn|^p)^{\frac{1}{p}}$. Then (X,d) is a b-metric space with $s = 2^{\frac{1}{p}}$.

Some of other definitions of convergent and Cauchy sequences, completeness, examples, applications and extensions of fixed point theory in this space are considered in [1, 3–5, 11, 14, 15] and references therein.

Definition 1.3 [10] Consider a b-metric space $(\mathcal{X},d)$ with a coefficient $s \geqslant 1$ and two self-mappings f and g on $\mathcal{X}$. Also, suppose that $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$. The pair $\{f,g\}$ is called compatible iff $\lim\limits_{n\to\infty} d(fgx_n, gfx_n) = 0$.

In this paper, we prove two new common fixed point theorems in b-metric spaces. Also, we support both main theorems with an example and an application of existence of a common solution for two systems of an integral equation.

2. Main results

Definition 2.1 The function $\phi : [0,\infty) \to [0,\infty)$ is named a ϕ-function if the following properties are held:

i) $\phi(t) = 0 \Leftrightarrow t = 0$;
ii) $\phi(t) < t$ for each $t \geqslant 0$.

The collection of all ϕ-functions will be denoted by Φ.

***Example* 2.2** Define a function $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = \frac{t}{2}$ if $t \in [0,\infty)$. Then it is clear that ϕ is a ϕ-function.

First, we define the concept of a (v,u,ϕ)-contraction.

Definition 2.3 Consider a b-metric space $(\mathcal{X},d)$ with a parameter $s \geqslant 1$ and four self-mappings f, g, A and B on $\mathcal{X}$. If there exist a function $\phi \in \Phi$ and two constants $v \in (0, \frac{1}{s})$ and $u \geqslant 0$ such that

$$\begin{aligned} d(fx,gy) \leqslant\ & v \max\{\phi(d(fx,Ax)), \phi(d(gy,By)), \phi(d(Ax,By))\} \\ & + u \min\{d(fy,gy), d(fx,gx)\} \end{aligned} \tag{1}$$

for each $x,y \in \mathcal{X}$, then (f,g,A,B) is called a (v,u,ϕ)-contraction.

Let $x_0 \in \mathcal{X}$ be an optional point and f, g, A and B be four self-mappings so that $f(\mathcal{X}) \subseteq B(\mathcal{X})$, $g(\mathcal{X}) \subseteq A(\mathcal{X})$. Choose $x_1 \in \mathcal{X}$ so that $fx_0 = Bx_1$ and $x_2 \in \mathcal{X}$ so that $gx_1 = Ax_2$. This can be accomplished as $f(\mathcal{X}) \subseteq B(\mathcal{X})$ and $g(\mathcal{X}) \subseteq A(\mathcal{X})$. By continuing this process, we obtain a sequence $\{z_n\}$ introduced by $z_{2n} = fx_{2n} = Bx_{2n+1}$ and $z_{2n+1} = gx_{2n+1} = Ax_{2n+2}$ for all $n \geqslant 0$. The sequence $\{z_n\}$ is named a Jungck type iterative sequence with initial guess x_0.

Theorem 2.4 Assume that f, g, A and B are four self-mappings on a complete b-metric space $\mathcal{X}$ with a parameter $s \geqslant 1$ provided that the pairs $\{f,A\}$ and $\{g,B\}$ are compatible, $f(\mathcal{X}) \subset B(\mathcal{X})$ and $g(\mathcal{X}) \subset A(\mathcal{X})$. If (f,g,A,B) is a (v,u,ϕ)-contraction, then f, g, A and B have a common fixed point in $\mathcal{X}$ so that A and B are continuous.

Proof. Suppose x_0 is an arbitrary point of $\mathcal{X}$. Construct Jungck type iterative sequence $\{z_n\}$ in $\mathcal{X}$ with initial guess x_0. Now, we show that $\{z_n\}$ is a Cauchy sequence. From (1), we have

$$\begin{aligned} d(z_{2n}, z_{2n+1}) &= \phi(d(fx_{2n}, gx_{2n+1})) \qquad (2)\\ &\leqslant v \max\{\phi(d(fx_{2n}, Ax_{2n})), \phi(d(gx_{2n+1}, Bx_{2n+1})), \phi(d(Ax_{2n}, Bx_{2n+1}))\}\\ &\quad + u \min\{d(fx_{2n+1}, gx_{2n+1}), d(fx_{2n}, gx_{2n})\}\\ &= v \max\{\phi(d(z_{2n}, z_{2n-1})), \phi(d(z_{2n+1}, z_{2n})), \phi(d(z_{2n-1}, z_{2n}))\}\\ &\quad + u \min\{d(z_{2n+1}, z_{2n+1}), d(z_{2n}, z_{2n})\}\\ &= v \max\{\phi(d(z_{2n}, z_{2n-1})), \phi(d(z_{2n+1}, z_{2n}))\}. \end{aligned}$$

Now, let $\phi(d(z_{2n}, z_{2n+1})) > \phi(d(z_{2n-1}, z_{2n}))$. Then, by (2), we have $d(z_{2n}, z_{2n+1}) < v\phi(d(z_{2n}, z_{2n+1}))$, which is a contradiction. Hence, $\phi(d(z_{2n}, z_{2n+1})) \leqslant \phi(d(z_{2n-1}, z_{2n}))$, which implies by (2) that

$$d(z_{2n}, z_{2n+1}) \leqslant v\phi(d(z_{2n-1}, z_{2n})) < vd(z_{2n-1}, z_{2n}). \qquad (3)$$

By a similar argument, we have

$$d(z_{2n-1}, z_{2n}) \leqslant v\phi(d(z_{2n-2}, z_{2n-1})) < vd(z_{2n-2}, z_{2n-1}). \qquad (4)$$

Now, from (3) and (4), we get

$$d(z_n, z_{n-1})) \leqslant v\phi(d(z_{n-1}, z_{n-2})) < vd(z_{n-1}, z_{n-2})$$

for $n \geqslant 2$, where $0 < v < \frac{1}{s}$. By induction, we have

$$d(z_n, z_{n-1}) \leqslant v^{n-1} d(z_1, z_0) \qquad (5)$$

for all $n \geqslant 2$. Now, we prove that $\{z_n\}$ is a Cauchy sequence. First we show that $\lim_{m,n\to\infty} d(z_m, z_n) = 0$ for each $m, n \in \mathbb{N}$ with $m > n > 1$. Then, by $(b3)$, we get

$$\begin{aligned} d(z_n, z_m) &\leqslant sd(z_n, z_{n+1}) + sd(z_{n+1}, z_m)\\ &\leqslant sd(z_n, z_{n+1}) + s^2 d(z_{n+1}, z_{n+2}) + s^2 d(z_{n+2}, z_m)\\ &\leqslant sd(z_n, z_{n+1}) + s^2 d(z_{n+1}, z_{n+2}) + ... + s^{m-n} d(z_{m-1}, z_m)\\ &\ \ \vdots\\ &\leqslant sv^n(1 + sv + ... + s^{m-n-1}v^{m-n-1}) d(z_0, z_1) \qquad (vs < 1)\\ &< \frac{sv^n}{1 - sv} d(z_0, z_1), \end{aligned}$$

which implies that $\lim_{m,n\to\infty} d(z_n, z_m) = 0$. Hence, $\{z_n\}$ is a Cauchy sequence. Due to the completeness of the b-metric space, there exists $z \in \mathcal{X}$ so that $z_n \to z$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Ax_{2n+2} = z.$$

Now we demonstrate that z is a common fixed point of f, g, A and B. Since A is continuous, we have $\lim_{n\to\infty} A^2x_{2n+2} = Az$ and $\lim_{n\to\infty} Afx_{2n} = Az$. Since f and A are compatible,

$$\lim_{n\to\infty} d(fAx_{2n}, Afx_{2n}) = 0.$$

Thus, we have $\lim_{n\to\infty} fAx_{2n} = Az$. Consider $x = Ax_{2n}$ and $y = x_{2n+1}$ in (1). Then, we get

$$\begin{aligned} d(fAx_{2n}, gx_{2n+1}) &\leqslant v\max\{\phi(d(fAx_{2n}, A^2x_{2n})), \phi(d(gx_{2n+1}, Bx_{2n+1})), \phi(d(A^2x_{2n}, Bx_{2n+1}))\} \\ &\quad + u\min\{d(fx_{2n+1}, gx_{2n+1}), d(fAx_{2n}, gAx_{2n})\} \\ &< v\max\{d(fAx_{2n}, A^2x_{2n}), d(gx_{2n+1}, Bx_{2n+1}), d(A^2x_{2n}, Bx_{2n+1})\} \\ &\quad + u\min\{d(fx_{2n+1}, gx_{2n+1}), d(fAx_{2n}, gAx_{2n})\}. \end{aligned}$$

Now, we have

$$\lim_{n\to\infty} d(Afx_{2n}, gx_{2n+1}) = d(Az, z) \leqslant v\max\{\phi((Az, z)), 0, 0\}.$$

Consequently, $d(Az, z) \leqslant vd(Az, z)$ with $0 < v < \frac{1}{s}$. Hence, $Az = z$. Similarly, since B is continuous and B and g are compatible, we get $Bz = z$. Also, by (1), we obtain

$$\begin{aligned} d(fz, gx_{2n+1}) &\leqslant v\max\{\phi(d(fz, Az)), \phi(d(gx_{2n+1}, Bx_{2n+1})), \phi(d(Az, Bx_{2n+1}))\} \\ &\quad + u\min\{d(fx_{2n+1}, gx_{2n+1}), d(fz, gz)\}. \end{aligned}$$

By taking $n \to \infty$ and since $Az = Bz = z$, we have

$$d(fz, z) \leqslant v\max\{\phi(d(fz, z)), \phi(d(z, z))\},$$

which induces that $fz = z$ (by $0 < v < \frac{1}{s}$). Similarly $gz = z$. Thus, $Az = Bz = fz = gz = z$ and the proof ends. ■

***Example* 2.5** Consider a b-metric by $d(x, y) = |x - y|^2$ for all $x, y, \in \mathcal{X} = [0, 1]$ with the parameter $s = 2$. Define the mappings f, g, A and B on $\mathcal{X}$ by $f(x) = x$, $g(x) = 2x$, $A(x) = 4x$ and $B(x) = 8x$. Clearly, $f(\mathcal{X}) \subset B(\mathcal{X})$ and $g(\mathcal{X}) \subset A(\mathcal{X})$. Also, two pairs $\{f, A\}$, and $\{g, B\}$ are compatible. Further, for $\phi(t) = \frac{t}{2}$ and for all $x, y \in \mathcal{X}$, we get

$$\begin{aligned} \phi(d(fx, gy)) = |x - 2y|^2 &= \frac{1}{16}(|4x - 8y|^2) \\ &= \frac{1}{8}\phi(d(Ax, By)) \\ &\leqslant \frac{1}{8}\max\{\phi(d(fx, Ax)), \phi(d(gz, Bz)), \phi(d(Ax, By))\} \\ &\quad + u\min\{d(fy, gy), d(fx, gx)\}. \end{aligned}$$

Hence, all conditions of Theorem 2.4 are held with $v = \frac{1}{8}$ and $u = 0$. Obviously, f, g, A and B have a common fixed point at $x = 0$.

Now, we define a new notion of contractions which is named a (q, p, ϕ)-contraction.

Definition 2.6 Consider a b-metric space $(\mathcal{X}, d)$ with a parameter $s \geqslant 1$ and two mappings $f, g : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and two self-mappings T and R on $\mathcal{X}$. If there exist a ϕ-function

ϕ and two constants $q \in (0, \frac{1}{s})$ and $p \geqslant 0$ so that

$$\begin{aligned} d(f(x,y), g(w,z)) \leqslant q \max\{ & \frac{1}{2}(\phi(d(Rx,Tw)) - \phi(d(Ry,Tz))), \\ & \frac{1}{2}(\phi(d(g(w,z),Tw)) - \phi(d(g(z,w),Tz))), \\ & \frac{1}{2}(\phi(d(f(x,y),Rx)) - \phi(d(f(y,x),Ry)))\} \qquad (6) \\ & + p \min\{\frac{1}{2}(d(f(w,z),g(w,z)) + d(f(z,w),g(z,w))), \\ & \frac{1}{2}(d(f(x,y),g(x,y)) + d(f(y,x),g(y,x)))\} \end{aligned}$$

for each $x, y, z, w \in \mathcal{X}$, then (f, g, R, T) is named a (q, p, ϕ)-contraction.

In 2006, Bhaskar and Lakshmikantham [6] defined the concept of a coupled fixed point and proved some fixed point results for a mixed monotone mapping. For more details on coupled, tripled and n-tuple fixed point theorems, we refer to [8, 9, 13, 16, 17] and references therein. The second result of this article is related to the existence of common coupled fixed point for four mappings.

Definition 2.7 [12] Consider a nonempty set $\mathcal{X}$ and mappings $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$. F and g is said to be commutative if $F(gx, gy) = g(F(x, y))$ for each $x, y \in \mathcal{X}$.

In the sequel, denote $\mathcal{X} \times \cdots \times \mathcal{X}$ by $\mathcal{X}^n$, where $\mathcal{X}$ is a non-empty set and $n \in \mathbb{N}$.

Lemma 2.8 [8] Let $(\mathcal{X}, d)$ be a b-metric space with a parameter $s \geqslant 1$. Then the following assertions hold:

1. $(\mathcal{X}^n, D)$ is a b-metric space with
$$D((x_1, \cdots, x_n), (y_1, \cdots, y_n)) = \max[d(x_1, y_1), d(x_2, y_2), \cdots, d(x_n, y_n)].$$
2. The mappings $f : \mathcal{X}^n \to \mathcal{X}$, $g : \mathcal{X}^n \to \mathcal{X}$, $T : \mathcal{X} \to \mathcal{X}$ and $R : \mathcal{X} \to \mathcal{X}$ have a n-tuple common fixed point if and only if the mappings $F : \mathcal{X}^n \to \mathcal{X}^n$, $G : \mathcal{X}^n \to \mathcal{X}^n$, $\mathcal{T} : \mathcal{X}^n \to \mathcal{X}^n$ and $\mathcal{R} : \mathcal{X}^n \to \mathcal{X}^n$ defined by
$$\begin{aligned} &F(x_1, x_2, \cdots, x_n) = (f(x_1, x_2, \cdots, x_n), f(x_2, \cdots, x_n, x_1), \cdots, f(x_n, x_1, \cdots, x_{n-1})), \\ &G(x_1, x_2, \cdots, x_n) = (g(x_1, x_2, \cdots, x_n), g(x_2, \cdots, x_n, x_1), \cdots, g(x_n, x_1, \cdots, x_{n-1})), \\ &\mathcal{T}(x_1, x_2, \cdots, x_n) = (Tx_1, Tx_2, \cdots, Tx_n), \mathcal{R}(x_1, x_2, \cdots, x_n) = (Rx_1, Rx_2, \cdots, Rx_n) \end{aligned}$$
have a common fixed point in $\mathcal{X}^n$.
3. $(\mathcal{X}, d)$ is complete if and only if $(\mathcal{X}^n, D)$ is complete.

Note that the Lemma 2.8 is a two-way relationship. Thus, we can obtain n-tuple fixed point results from fixed point theorems and conversely.

The second result of this work is the following theorem.

Theorem 2.9 Assume that T and R are two mappings on a complete b-metric space $\mathcal{X}$ with a parameter $s \geqslant 1$ and f and g are two mappings on $\mathcal{X} \times \mathcal{X}$ and provided that the pairs $\{f, R\}$ and $\{g, T\}$ are commutative and $f(\mathcal{X} \times \mathcal{X}) \subset T(\mathcal{X})$ and $g(\mathcal{X} \times \mathcal{X}) \subset R(\mathcal{X})$. If (f, g, R, T) is a (q, p, ϕ)-contraction, then f, g, R and T have a common coupled fixed point so that R and T are continuous.

Proof. Let us define $D : \mathcal{X}^2 \times \mathcal{X}^2 \to [0,\infty)$ by $D((x_1,x_2),(y_1,y_2)) = \max[d(x_1,y_1), d(x_2,y_2)]$, $F,G : \mathcal{X}^2 \to \mathcal{X}^2$ by $F(x,y) = (f(x,y), f(y,x))$ and $G(x,y) = (g(x,y), g(y,x))$, and $\mathcal{T},\mathcal{R} : \mathcal{X}^2 \to \mathcal{X}^2$ by $\mathcal{T}(x,y) = (Tx,Ty)$ and $\mathcal{R}(x,y) = (Rx,Ry)$. Using Lemma 2.8, $(\mathcal{X}^2, D)$ is a complete b-metric space. Also, $(x,y) \in \mathcal{X}^2$ is a common coupled fixed point of f,g and T,R if and only if it is a common fixed point of F,G and $\mathcal{T},\mathcal{R}$. On the other hands, from (6), we have either

$$
\begin{aligned}
D(F(x,y),G(w,z)) &= D((f(x,y),f(y,x)),(g(w,z),g(z,w)))\\
&= \max[d(f(x,y),g(w,z)), d(f(y,x),g(z,w))]\\
&= d(f(x,y),g(w,z))\\
&\leqslant q\max\{\frac{1}{2}(\phi(d(Rx,Tw)) - \phi(d(Ry,Tz))),\\
&\quad \frac{1}{2}(\phi(d(g(w,z),Tw)) - \phi(d(g(z,w),Tz))),\\
&\quad \frac{1}{2}(\phi(d(f(x,y),Rx)) - \phi(d(f(y,x),Ry)))\}\\
&\quad + p\min\{\frac{1}{2}(d(f(w,z),g(w,z)) + d(f(z,w),g(z,w))),\\
&\quad \frac{1}{2}(d(f(x,y),g(x,y)) + d(f(y,x),g(y,x)))\}\\
&\leqslant q\max\{\phi(D(\mathcal{R}(x,y),\mathcal{T}(w,z))), \phi(D(G(x,y),\mathcal{T}(w,z))),\\
&\quad \phi(D(F(x,y),\mathcal{R}(w,z)))\}\\
&\quad + p\min\{D(F(w,z),G(w,z)), D(F(x,y),G(x,y))\}
\end{aligned}
$$

or

$$
\begin{aligned}
D(F(x,y),G(w,z)) &= D((f(x,y),f(y,x)),(g(w,z),g(z,w)))\\
&= \max[d(f(x,y),g(w,z)), d(f(y,x),g(z,w))]\\
&= d(f(y,x),g(z,w))\\
&\leqslant q\max\{\frac{1}{2}(\phi(d(Ry,Tz)) - \phi(d(Rx,Tw))),\\
&\quad \frac{1}{2}(\phi(d(g(z,w),Tz)) - \phi(d(g(w,z),Tw))),\\
&\quad \frac{1}{2}(\phi(d(f(y,x),Ry)) - \phi(d(f(x,y),Rx)))\}\\
&\quad + p\min\{\frac{1}{2}(d(f(z,w),g(z,w)) + d(f(w,z),g(w,z))),\\
&\quad \frac{1}{2}(d(f(y,x),g(y,x)) + d(f(x,y),g(x,y)))\}\\
&\leqslant q\max\{\phi(D(\mathcal{R}(y,x),\mathcal{T}(z,w))), \phi(D(G(y,x),\mathcal{T}(z,w))),\\
&\quad \phi(D(F(y,x),\mathcal{R}(z,w)))\}\\
&\quad + p\min\{D(F(z,w),G(z,w)), D(F(y,x),G(y,x))\}
\end{aligned}
$$

Now, by Theorem 2.4, $F, G, \mathcal{R}$ and $\mathcal{T}$ have a common fixed point and by Lemma 2.8, f, g, R and T have a common coupled fixed point. This completes the proof. ■

3. Application

Assume the systems of integral equations:

$$\begin{cases} x(t) = \int_a^b M(t,s)K(s,x(s),y(s))ds, \\ y(t) = \int_a^b M(t,s)K(s,y(s),x(s))ds \end{cases} \tag{7}$$

for all $t \in I = [a,b]$, where $M \in C(I \times I, [0,\infty))$ and $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Also, let $C(I,\mathbb{R})$ be the Banach space of all real continuous functions considered on I with the sup norm. Consider the b-metric $d(x,y) = ||x-y||^2$ for every $x, y \in C(I,\mathbb{R})$. Then the space $(C(I,\mathbb{R}), d)$ is a complete b-metric space with the parameter $s = 2$.

Theorem 3.1 Let $(C(I,\mathbb{R}), d)$ be a complete b-metric space. Suppose $f : C(I,\mathbb{R}) \times C(I,\mathbb{R}) \to C(I,\mathbb{R})$ is an operator such that

$$f(x,y)t = \frac{1}{2}\Big(\int_a^b M(t,s)K(s,x(s),y(s))ds\Big),$$

where $M \in C(I \times I, [0,\infty))$ and $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ be an operator satisfying the following conditions:

(i) $||K||_\infty = \sup_{s\in I,\ x,y\in C(I,\mathbb{R})} |K(s,x(s),y(s))| < \infty$,

(ii) for every $x, y \in C(I,\mathbb{R})$ and all $t \in I$, we have

$$||K(t,x(t),y(t)) - K(t,u(t),v(t))|| \leqslant \max_{t\in I}|x(t)-u(t)|^2 - \max_{t\in I}|y(t)-v(t)|^2,$$

(iii) $\sup_{t\in I} \int_a^b M(t,s)ds < \frac{1}{s}$.

Then the system (7) has a common solution.

Proof. Consider a complete b-metric $d(x,y) = \max_{t\in I}(|x(t)-y(t)|^2)$ for each $x, y \in C(I,\mathbb{R})$. By a simple computation, we get

$$d(f(x,y), g(u,v)) \leqslant \frac{1}{2}[d(Rx,Tu)) - d(Ry,Tv)](\max_{s\in I}\int_a^b M(t,s)ds)$$

for every $x, y, u, v \in C(I,\mathbb{R})$, where $f(x,y) = g(x,y)$ and $Rx = Tx = Ix = x$. Let $q = \max_{s\in I}\int_a^b M(t,s)ds$ and $\phi(t) = t$. Then we conclude that

$$\begin{aligned} d(f(x,y), g(u,v)) &\leqslant q(\frac{1}{2}(\phi(d(Rx,Tu)) - \phi(d(Ry,Tv))), \\ &\leqslant q\max\{\frac{1}{2}(\phi(d(Rx,Tu)) - \phi(d(Ry,Tv))), \\ &\quad \frac{1}{2}(\phi(d(g(u,v),Tu)) - \phi(d(g(v,u),g(v,u),Tv)))\} \end{aligned}$$

for every $x, y, u, v \in C(I, \mathbb{R})$. By applying Theorem 2.9 with $\phi(t) = t$, $p = 0$ and $Rx = Tx = Ix = x$, the operators f and g have a common coupled fixed point, which is the common solution of the system (7). ■

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