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Some categorical aspects of coarse proximity spaces

Gh. Mirhosseinkhani[a,*]

[a] *Department of Mathematics and Computer Sciences, Sirjan University of Technology, Sirjan, Iran.*

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Abstract. In this paper, we study some categorical structures of the category **CoarsePro**, whose objects are coarse proximity spaces and whose morphisms are coarse proximity maps. We investigate the structure of initial, final, embedding and quotient morphisms in the construct **CoarsePro**. A special attention is paid to investigate quotients by introducing some conditions that they exist. Also, it is shown that bimorphisms are exactly bijective coarse proximity maps, but not isomorphisms. Consequently, **CoarsePro** is not balanced.

Keywords: Coarse proximity space, coarse proximity category, embedding morphism, quotient morphism, bimorphism.

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1. Introduction and Preliminaries

In classical topology, there are various ways for studying small-scale structures on a set. Uniformity and proximity are two different ways for this concept. In 1937, Weil [15] defined the concept of uniformity and in 1950, Efremovič [3, 4] used proximity relations for studying small-scale notions. In contrast to classical topology, coarse topology investigates the large-scale aspects of spaces, including the large-scale analog of uniform spaces, called coarse spaces [13]. Because coarse spaces generalize coarse properties of metric spaces, some topologists have attempted to define a concept of large-scale proximity. In [6], Hartmann defined a binary relation on the power set of a metric space as the negation of asymptotic disjointness. In [9], the authors presented asymptotic resemblance relations as a generalization of the Hausdorff distance relations of metric spaces. Recently, Grzegrzolka and Siegert [5] have defined coarse proximity structures, which are

*Corresponding author.
E-mail address: gh.mirhosseini@yahoo.com, gh.mirhosseini@sirjantech.ac.ir (Gh. Mirhosseinkhani).

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an analog of small-scale proximity spaces in the large-scale context. The relationships between proximity spaces, ideals and grills investigated by many researchers, for example see [8, 10, 11, 14].

In this paper, we study some categorical structures of the category **CoarsePro** of coarse proximity spaces with coarse proximity maps. We show that monomorphisms and epimorphisms are exactly injective and surjective coarse proximity maps, respectively. Thus bimorphisms are bijective coarse proximity maps, but not isomorphisms. Consequently, **CoarsePro** is not balanced. Also, we introduce the structure of initial, final, embedding and quotient morphisms in the construct **CoarsePro**. In particular, we show that the embedding morphisms are precisely the coarse proximity embedding maps and the quotient maps are precisely the coarse proximity quotient maps.

In the following, readers are suggested to refer to [1] for some categorical notions. We first recall some basic results and concepts of proximity and coarse proximity structures given in [5, 12].

Definition 1.1 Let X be a set and $P(X)$ be the power set of X. A (Efremovič) proximity on a set X is a relation δ on $P(X)$ satisfying the following axioms for all $A, B, C \in P(X)$:

(1) $A\delta B$ implies $B\delta A$,
(2) $A\delta B$ implies $A \neq \emptyset$ and $B \neq \emptyset$,
(3) $A \cap B \neq \emptyset$ implies $A\delta B$,
(4) $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$,
(5) $A\bar{\delta}B$ implies that there exists a subset E such that $A\bar{\delta}E$ and $(X - E)\bar{\delta}B$,

where $A\bar{\delta}B$ means $A\delta B$ is not true. If $A\delta B$, then we say that A is close to (or near) B. Axiom 4 is called the union axiom and axiom 5 is called the strong axiom. A pair (X, δ), where X is a set and δ is a proximity on X, is called a proximity space.

Definition 1.2 A function $f : (X, \delta_1) \to (Y, \delta_2)$ is called a proximity map if $A\delta_1 B$ implies $f(A)\delta_2 f(B)$ for all $A, B \subseteq X$.

Definition 1.3 A bornology $\mathcal{B}$ on a set X is a family of subsets of X satisfying:

(1) $\{x\} \in \mathcal{B}$ for all $x \in X$,
(2) $A \subseteq B$ and $B \in \mathcal{B}$ implies $A \in \mathcal{B}$ (i.e., it is closed under taking subsets),
(3) if $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$ (i.e., it is closed under taking finite unions).

The elements of $\mathcal{B}$ are called bounded and subsets of X not in $\mathcal{B}$ are called unbounded.

Bornologies play an important role in the theory of locally convex spaces [7], boundedness in metric spaces [2] and coarse geometry [5].

Example **1.4** The following families are bornologies on a set X:

(1) the finite subsets of X,
(2) the countable subsets of X,
(3) the power set $P(X)$,
(4) the bounded subsets of a metric space X,
(5) the totally bounded subsets of a metric space X,
(6) the subsets of a metric space X with compact closure.

Definition 1.5 Let $\mathcal{B}$ be a bornology on a set X. A binary relation $\mathbf{b}$ on $P(X)$ is called a coarse proximity on X if it satisfies the following axioms for all $A, B, C \in P(X)$:

($\mathcal{A}_1$) $A\mathbf{b}B$ implies $B\mathbf{b}A$,
($\mathcal{A}_2$) $A\mathbf{b}B$ implies $A \notin \mathcal{B}$ and $B \notin \mathcal{B}$,

($\mathcal{A}_3$) $A \cap B \notin \mathcal{B}$ implies $A\mathbf{b}B$,
($\mathcal{A}_4$) $(A \cup B)\mathbf{b}C$ if and only if $A\mathbf{b}B$ or $B\mathbf{b}C$,
($\mathcal{A}_5$) $A\bar{\mathbf{b}}B$ implies that there exists a subset E such that $A\bar{\mathbf{b}}E$ and $(X - E)\bar{\mathbf{b}}B$,

where $A\bar{\mathbf{b}}B$ means $A\mathbf{b}B$ is not true. A triple $(X, \mathcal{B}, \mathbf{b})$, where X is a set, $\mathcal{B}$ is a bornology on X and $\mathbf{b}$ a coarse proximity on X, is called a coarse proximity space and the pair $(\mathcal{B}, \mathbf{b})$ a coarse proximity structure on X. Axiom ($\mathcal{A}_4$) is called the union axiom and ($\mathcal{A}_5$) is called the strong axiom.

***Example* 1.6** Let (X, d) be a metric space, $\mathcal{B}_d$ be the collection of all bounded sets of X with respect to the metric d and $\mathbf{b}_d$ be the relation defined by $A\mathbf{b}_dB$. If there exists $\epsilon < \infty$ such that $d(A - D, B - D) < \epsilon$ for all bounded sets D, then this relation is a coarse proximity on X. We call this relation the metric coarse proximity and the associated space $(X, \mathcal{B}_d, \mathbf{b}_d)$ the metric coarse proximity space.

***Example* 1.7** Let $\mathcal{B}$ be a bornology on a set X. For any subsets A and B of X, define

(1) $A\mathbf{b}_1B \Longleftrightarrow A \cap B \notin \mathcal{B}$,
(2) $A\mathbf{b}_2B \Longleftrightarrow A, B \notin \mathcal{B}$.

Then $\mathbf{b}_1$ and $\mathbf{b}_2$ are coarse proximities on X, called the discrete and indiscrete coarse proximity, respectively.

Definition 1.8 Let $(X, \mathcal{B}_1, \mathbf{b}_1)$ and $(Y, \mathcal{B}_2, \mathbf{b}_2)$ be coarse proximity spaces. Let $f : X \to Y$ be a function and A and B subsets of X. Then f is a coarse proximity map provided that the following are satisfied:

(1) $B \in \mathcal{B}_1$ implies $f(B) \in \mathcal{B}_2$,
(2) $A\mathbf{b}_1B$ implies $f(A)\mathbf{b}_2f(B)$.

***Remark* 1** *Notice that a coarse proximity map sends unbounded sets to unbounded sets. For if $B \notin \mathcal{B}_1$, then $B\mathbf{b}_1B$. Thus, $f(B)\mathbf{b}_2f(B)$, implying that $f(B) \notin \mathcal{B}_2$. Consequently, the preimages of bounded sets are bounded.*

Definition 1.9 Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Given subsets $A, B \subseteq X$, we say that B is a $\mathbf{b}$-coarse neighborhood of A, denoted $A \ll B$, if $A\bar{\mathbf{b}}(X - B)$.

Theorem 1.10 Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Let A, B, C and D be subsets of X. Then the relation $\ll$ satisfies the following properties:

(P_1) $X \ll (X - D)$ for all $D \in \mathcal{B}$,
(P_2) $A \ll B$ implies that there exists $D \in \mathcal{B}$ such that $(A - D) \subseteq B$, i.e., $A \subseteq B$ up to some bounded set D,
(P_3) $A \subseteq B \ll C \subseteq D$ implies $A \ll D$,
(P_4) $A \ll (B \cap C)$ if and only if $A \ll B$ and $A \ll C$,
(P_5) $A \ll B$ if and only if $(X - B) \ll (X - A)$,
(P_6) $A \ll B$ implies that there exists $C \subseteq X$ such that $A \ll C \ll B$.

Theorem 1.11 Let X be a set with a bornology $\mathcal{B}$. If $\ll$ is a binary relation on $P(X)$ satisfying (P_1) through (P_6) of Theorem 1.10 and $\mathbf{b}$ is a relation on $P(X)$ defined by

$$A\bar{\mathbf{b}}B \quad \text{if and only if} \quad A \ll (X - B).$$

Then $\mathbf{b}$ is a coarse proximity on X, called the coarse proximity induced by the relation $\ll$. Also, B is a $\mathbf{b}$-neighborhood of A if and only if $A \ll B$.

Proposition 1.12 Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Let A, B and C be subsets of X. Then the following statements hold.

(1) If $A \subseteq C$, $B \subseteq D$, and $A\mathbf{b}B$, then $C\mathbf{b}D$.
(2) $A\mathbf{b}B$ if and only if for all $D_1, D_2 \in \mathcal{B}$, $(A - D_1)\mathbf{b}(B - D_2)$.

2. Monomorphisms and epimorphisms

In this section, we show that monomorphisms and epimorphisms in **CoarsePro** are exactly injective and surjective coarse proximity maps, respectively. Consequently, **CoarsePro** is not balanced.

A coarse proximity space $(X, \mathcal{B}, \mathbf{b})$ is said to be bounded if X is bounded, i.e., $X \in \mathcal{B}$, otherwise X is said to be unbounded. The following lemma is an immediate consequence of the definition of boundedness.

Lemma 2.1 A coarse proximity space $(X, \mathcal{B}, \mathbf{b})$ is bounded if and only if $\mathcal{B}$ is the power set of X if and only if $\mathbf{b}$ is the empty relation.

By the previous lemma we have the following two results.

Corollary 2.2 The full subcategory of **CoarsePro** consisting of all bounded coarse proximity space is isomorphic to the full subcategory of topological spaces consisting of all discrete topological spaces and hence is isomorphic to the category **Set**.

Corollary 2.3 Let $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ be any function whose domain is bounded. Then f is a coarse proximity map if and only if $B \in \mathcal{B}_1$ implies $f(B) \in \mathcal{B}_2$. In particular, since every finite coarse proximity space is bounded and every function preserves finite subsets, it follows that if X is finite, then f is a coarse proximity map.

Theorem 2.4 A morphism f in **CoarsePro** is a monomorphism if and only if f is injective.

Proof. It is clear that every injective coarse proximity map is a monomorphism. Conversely, let $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ be a monomorphism and $f(x) = f(y)$. Assume that $(\{1\}, \mathcal{B}_0, \emptyset)$ is the one point coarse proximity space, where $\mathcal{B}_0 = \{\emptyset, \{1\}\}$. Then by Corollary 2.3, the functions $\alpha, \beta : (\{1\}, \mathcal{B}_0, \emptyset) \to (X, \mathcal{B}_1, \mathbf{b}_1)$ defined by $\alpha(1) = x$ and $\beta(1) = y$ are coarse proximity maps such that $f \circ \alpha = f \circ \beta$. Thus $\alpha = \beta$ and hence $x = y$. ■

Lemma 2.5 Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space and a be a point such that $a \notin X$. Let $X_a = X \cup \{a\}$, $\mathcal{B}_a$ be the set $\{C \mid C \subseteq B \cup \{a\} \quad \text{for some} \quad B \in \mathcal{B}\}$ and $\mathbf{b}_a$ a relation on X_a defined as follows:

$$A\mathbf{b}_a B \Longleftrightarrow (A - \{a\})\mathbf{b}(B - \{a\}).$$

Then $(\mathcal{B}_a, \mathbf{b}_a)$ is a coarse proximity structure on X_a such that is coarser than of $(\mathcal{B}, \mathbf{b})$.

Proof. It is easy to verify that $\mathcal{B}_a$ is a bornology on X_a and $\mathcal{B} \subseteq \mathcal{B}_a$. Now, we show that $\mathbf{b}_a$ is a coarse proximity. Since $\mathbf{b}$ is symmetric, so is $\mathbf{b}_a$. To show $(\mathcal{A}_2)$, let $A\mathbf{b}_a B$. Then $(A - \{a\})\mathbf{b}(B - \{a\})$, so $(A - \{a\}) \notin \mathcal{B}$ and $(B - \{a\}) \notin \mathcal{B}$. Hence $A \notin \mathcal{B}_a$ and $B \notin \mathcal{B}_a$. To show $(\mathcal{A}_3)$, let $A\bar{\mathbf{b}}_a B$. Then $(A - \{a\})\bar{\mathbf{b}}(B - \{a\})$, so $((A \cap B) - \{a\}) \in \mathcal{B}$ and hence $(A \cap B) \in \mathcal{B}_a$. The union axiom of $\mathbf{b}_0$ follows immediately from the union axiom of $\mathbf{b}_a$. To show the strong axiom, assume $A\bar{\mathbf{b}}_a B$, i.e, $(A - \{a\})\bar{\mathbf{b}}(B - \{a\})$. By the strong axiom of $\mathbf{b}$, there exists a subset $E \subseteq X$ such that $(A - \{a\})\bar{\mathbf{b}}E$ and $(X - E)\bar{\mathbf{b}}(B - \{a\})$. Since

$a \notin X$, we have $(A - \{a\})\bar{\mathbf{b}}(E - \{a\})$ and $((X_a - E) - \{a\})\bar{\mathbf{b}}(B - \{a\})$. Thus $A\bar{\mathbf{b}}_a E$ and $(X_a - E)\bar{\mathbf{b}}_a B$. Finally, it is easy to check that $(\mathcal{B}, \mathbf{b})$ is finer than of $(\mathcal{B}_a, \mathbf{b}_a)$. ■

Theorem 2.6 A morphism f in **CoarsePro** is an epimorphism if and only if f is surjective.

Proof. It is clear that every surjective coarse proximity map is an epimorphism. Conversely, assume that $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}, \mathbf{b})$ is an epimorphism such that is not surjective. Then there is $y_0 \in Y$ such that $y_0 \notin f(X)$. Let $(Y_a, \mathcal{B}_a, \mathbf{b}_a)$ be the coarse proximity space defined as Lemma 2.5. Since $(\mathcal{B}, \mathbf{b})$ is finer than of $(\mathcal{B}_a, \mathbf{b}_a)$, the inclusion map $e : (Y, \mathcal{B}, \mathbf{b}) \to (Y, \mathcal{B}_a, \mathbf{b}_a)$ is a coarse proximity map. Let $\beta : (Y, \mathcal{B}, \mathbf{b}) \to (Y, \mathcal{B}_a, \mathbf{b}_a)$ be a function defined by $\beta(y) = y$ for all $y \neq y_0$ and $\beta(y_0) = a$. Then for every $B \subseteq Y$ we have

$$\beta(B) = \begin{cases} B, & \text{if } y_0 \notin B, \\ (B \cup \{a\}) - \{y_o\}, & \text{if } y_0 \in B. \end{cases}$$

We show that β is a coarse proximity map. It is clear that $B \in \mathcal{B}$ implies $\beta(B) \in \mathcal{B}_a$. If $A\mathbf{b}B$, then by Proposition 1.12, $(\beta(A) - \{a\})\mathbf{b}(\beta(B) - \{a\})$ and hence $\beta(A)\mathbf{b}_a\beta(B)$. Now, take $\alpha = e$, then α and β are coarse proximity maps such that $\alpha \circ f = \beta \circ f$. Thus $\alpha = \beta$ and hence $a = \beta(y_0) = \alpha(y_0) = y_0 \in Y$, which is a contradiction. ■

Recall that a morphism in a category **C** is called a bimorphism if it is both epimorphism and monomorphism; and the category **C** is called balanced if bimmorphisms are exactly the isomorphisms. According to Theorems 2.4 and 2.6, a bimorphism in **CoarsePro** is a bijective coarse proximity map. Therefore, a bimorphism in **CoarsePro** need not be an isomorphism, for example, assume that X is a infinite set with the bornology $\mathcal{B}$ of all finite subsets of X. If $\mathbf{b}_1$ and $\mathbf{b}_2$ are the discrete and indiscrete coarse proximities on X, then the identity map $id : (X, \mathcal{B}, \mathbf{b}_1) \to (X, \mathcal{B}, \mathbf{b}_2)$ is a bimorphism but not an isomorphism. Thus, we have the following result:

Corollary 2.7 The category **CoarsePro** is not balanced.

3. Initial and final Structures

In this section, we introduce the structure of final, initial, embedding and quotient morphisms in the construct **CoarsePro**.
Let $(\mathbf{A}, |-|)$ be a concrete category over a category **X**. An **A**-morphism $f : A \to B$ is called initial provided that for any **A**-object C an **X**-morphism $g : |C| \to |A|$ is an **A**-morphism whenever $f \circ g : |C| \to |B|$ is an **A**-morphism. An initial morphism $f : A \to B$ that has a monomorphic underlying **X**-morphism $f : |A| \to |B|$ is called an embedding. The concepts of final morphism and quotient morphism are dual to the concepts of initial morphism and embedding, respectively. A concrete category over the category **Set** of sets is called a construct [1].

Definition 3.1 Given a set X and two coarse proximity structures $(\mathcal{B}_1, \mathbf{b}_1)$, $(\mathcal{B}_2, \mathbf{b}_2)$ on X, we say that $(\mathcal{B}_1, \mathbf{b}_1)$ is finer than $(\mathcal{B}_2, \mathbf{b}_2)$ or $(\mathcal{B}_2, \mathbf{b}_2)$ is coarser than $(\mathcal{B}_1, \mathbf{b}_1)$, if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathbf{b}_1 \subseteq \mathbf{b}_2$, i.e., $A\mathbf{b}_1B$ implies $A\mathbf{b}_2B$.

Theorem 3.2 Let $f : X \to (Y, \mathcal{B}_1, \mathbf{b}_1)$ be a function such that codomain be a coarse proximity spaces. Then the coarsest coarse proximity structure $(\mathcal{B}_0, \mathbf{b}_0)$ on X for which f is a coarse proximity map, is defined by

(1) $\mathcal{B}_0 = \{B \subseteq X \mid f(B) \in \mathcal{B}_1\}$,
(2) $A\mathbf{b}_0B$ if and only if $f(A)\mathbf{b}_1f(B)$.

Proof. It is easy to verify that $\mathcal{B}_0$ is a bornology on X. Now, we show that $\mathbf{b}_0$ is a coarse proximity. Since $\mathbf{b}_1$ is symmetric, so is $\mathbf{b}_0$. To show $(\mathcal{A}_2)$, let $A\mathbf{b}_0B$. Then $f(A)\mathbf{b}_1f(B)$, so $f(A) \notin \mathcal{B}_1$ and $f(A) \notin \mathcal{B}_1$. Hence $A \notin \mathcal{B}_0$ and $B \notin \mathcal{B}_0$. To show $(\mathcal{A}_3)$, let $A\bar{\mathbf{b}}_0B$. Then $f(A)\bar{\mathbf{b}}_1f(B)$, so $f(A) \cap f(B) \in \mathcal{B}_1$ and hence $f(A \cap B) \in \mathcal{B}_1$. Thus $A \cap B \in \mathcal{B}_0$. The union axiom of $\mathbf{b}_0$ follows immediately from the union axiom of $\mathbf{b}_1$. To show the strong axiom, assume $A\bar{\mathbf{b}}_0B$, i.e., $f(A)\bar{\mathbf{b}}_1f(B)$. By the strong axiom of $\mathbf{b}_1$, there exists a subset $F \subseteq Y$ such that $f(A)\bar{\mathbf{b}}_1F$ and $(Y - F)\bar{\mathbf{b}}_1B$. Take $E = f^{-1}(F)$, then $f(E) \subseteq F$ and $f(X - E) \subseteq (Y - F)$. By Proposition 1.12, $f(A)\bar{\mathbf{b}}_1f(E)$ and $f(X - E)\bar{\mathbf{b}}_1B$. Thus $A\bar{\mathbf{b}}_0E$ and $(X - E)\bar{\mathbf{b}}_0B$. Finally, it is clear that $f : (X, \mathcal{B}_0, \mathbf{b}_0) \to (Y, \mathcal{B}_1, \mathbf{b}_1)$ is a coarse proximity map. Assume that $(\mathcal{B}_2, \mathbf{b}_2)$ is another coarse proximity structure on X such that $f : (X, \mathcal{B}_2, \mathbf{b}_2) \to (Y, \mathcal{B}_1, \mathbf{b}_1)$ is a coarse proximity map. If $B \in \mathcal{B}_2$, then $f(B) \in \mathcal{B}_1$ and hence $B \in \mathcal{B}_0$, which shows that $\mathcal{B}_2 \subseteq \mathcal{B}_0$. If $A\mathbf{b}_2B$, then $f(A)\mathbf{b}_1f(B)$, so $A\mathbf{b}_0B$. Hence $\mathbf{b}_2 \subseteq \mathbf{b}_0$. ■

Definition 3.3 The coarse proximity structure $(\mathcal{B}_0, \mathbf{b}_0)$ defined in Theorem 3.2, is called the induced coarse proximity structure by $(f, \mathcal{B}_1, \mathbf{b}_1)$.

Theorem 3.4 Let $f : (X, \mathcal{B}, \mathbf{b}) \to (Y, \mathcal{B}_1, \mathbf{b}_1)$ be a morphism in the construct **CoarsePro**. Then f is initial if and only if $(\mathcal{B}, \mathbf{b})$ is the induced coarse proximity structure by $(f, \mathcal{B}_1, \mathbf{b}_1)$.

Proof. Let f be an initial morphism and $(\mathcal{B}_0, \mathbf{b}_0)$ be the induced coarse proximity structure by $(f, \mathcal{B}_1, \mathbf{b}_1)$. Then by Theorem 3.2, we have $\mathcal{B} \subseteq \mathcal{B}_0$ and $\mathbf{b} \subseteq \mathbf{b}_0$. Now, assume that $id : (X, \mathcal{B}_0, \mathbf{b}_0) \to (Y, \mathcal{B}, \mathbf{b})$ is the identity function. Then $f \circ id : (X, \mathcal{B}_0, \mathbf{b}_0) \to (Y, \mathcal{B}_1, \mathbf{b}_1)$ is a coarse proximity map, so is id. Thus we have $\mathcal{B}_0 \subseteq \mathcal{B}$ and $\mathbf{b}_0 \subseteq \mathbf{b}$, which shows that $(\mathcal{B}, \mathbf{b}) = (\mathcal{B}_0, \mathbf{b}_0)$. Conversely, let $(\mathcal{B}, \mathbf{b})$ be the induced coarse proximity structure by $(f, \mathcal{B}_1, \mathbf{b}_1)$ and $g : (Z, \mathcal{B}_2, \mathbf{b}_2) \to (X, \mathcal{B}, \mathbf{b})$ a function such that $f \circ g$ be a coarse proximity map. Then $B \in \mathcal{B}_2$ implies $f(g(B)) \in \mathcal{B}_1$. Hence $g(B) \in \mathcal{B}$. Also, $A\mathbf{b}_2B$ implies $f(g(A))\mathbf{b}_1f(g(B))$, so $g(A)\mathbf{b}g(B)$, which shows that g is a coarse proximity map. ■

Definition 3.5 An injective coarse proximity map $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ is called a coarse proximity embedding map if $(\mathcal{B}_1, \mathbf{b}_1)$ is the induced coarse proximity structure by $(f, \mathcal{B}_2, \mathbf{b}_2)$.

By Theorem 3.4, we have the following corollary.

Corollary 3.6 The embedding morphisms are precisely the coarse proximity embedding maps.

Remark 2 *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space, Y a subset of X and $e : Y \to X$ be the inclusion map. If $(\mathcal{B}_0, \mathbf{b}_0)$ is the induced coarse proximity structure on Y by $(e, \mathcal{B}, \mathbf{b})$, then e is a coarse proximity map. In particular, we have $\mathcal{B}_0 = \{B \cap Y \mid B \in \mathcal{B}\}$; and $A\mathbf{b}_0B$ if and only if $A\mathbf{b}B$ for all $A, B \subseteq Y$. The coarse proximity space $(Y, \mathcal{B}_0, \mathbf{b}_0)$ is called the coarse proximity subspace of X.*

Recall that a morphism $e : E \to A$ is called an equalizer of the morphisms $f, g : A \to B$ provided that the following conditions hold:

(1) $f \circ e = g \circ e$,
(2) for any morphism $e' : E' \to A$ with $f \circ e' = g \circ e'$, there exists a unique morphism $\bar{e} : E' \to E$ such that $e' = e \circ \bar{e}$.

Theorem 3.7 The category **CoarsePro** has equalizers.

Proof. Let $f, g : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ be a pair of morphisms. Let E be the subset $\{x \in X \mid f(x) = g(x)\}$ of X and $(E, \mathcal{B}_0, \mathbf{b}_0)$ be the coarse proximity subspace of X. Then by Remark *2*, the inclusion map $e : (E, \mathcal{B}_0, \mathbf{b}_0) \to (X, \mathcal{B}_1, \mathbf{b}_1)$ is a coarse proximity map such that $f \circ e = g \circ e$. It is easy to verify that e is an equalizer of $f$and g. ■

Lemma 3.8 Let $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ be a function between coarse proximity spaces. Then the following statements are equivalent:

(1) $A\mathbf{b}_1 B$ implies $f(A)\mathbf{b}_2 f(B)$,
(2) $C\bar{\mathbf{b}}_2 D$ implies $f^{-1}(C)\bar{\mathbf{b}}_1 f^{-1}(D)$,
(3) $C \ll_2 D$ implies $f^{-1}(C) \ll_1 f^{-1}(D)$.

Proof. (1)⇒(2): Let C and D be subsets of Y such that $f^{-1}(C)\mathbf{b}_1 f^{-1}(D)$. By assumption we have $ff^{-1}(C)\mathbf{b}_1 ff^{-1}(D)$. Hence by Remark *1*, $C\mathbf{b}_2 D$.
(2)⇒(3): Let C and D be subsets of Y such that $C \ll_2 D$. This means that $C\bar{\mathbf{b}}_2(Y - D)$. By assumption we have $f^{-1}(C)\mathbf{b}_1(X - f^{-1}(D))$, which means that $f^{-1}(C) \ll_1 f^{-1}(D)$.
(3)⇒(1): Let A and B be subsets of X such that $f(A)\bar{\mathbf{b}}_2 f(B)$. This means that $f(A) \ll_2 (Y - f(B))$. By assumption we have $f^{-1}f(A) \ll_1 (X - f^{-1}f(B))$, which means that $f^{-1}f(A)\bar{\mathbf{b}}_1 f^{-1}f(B)$. Hence by Remark *1*, $A\bar{\mathbf{b}}_1 B$. ■

The following lemma is an immediate consequence of Remark *1*.

Lemma 3.9 Let $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ be a coarse proximity map. Then the following statements hold.

(1) X is bounded if and only if $f(X)$ is bounded.
(2) $B \in \mathcal{B}_1$ if and only if $f^{-1}f(B) \in \mathcal{B}_1$.
(3) If $\mathcal{B}'$ is the set $\{C \subseteq Y \mid f^{-1}(C) \in \mathcal{B}_1\}$, then $\mathcal{B}'$ is a bornology on Y, such that $\mathcal{B}_2 \subseteq \mathcal{B}'$. Moreover, if f is surjective, then $\mathcal{B}_2 = \mathcal{B}'$.

Remark 3 *By Lemma 3.9, if* $(X, \mathcal{B}_1, \mathbf{b}_1)$ *and* $(Y, \mathcal{B}_2, \mathbf{b}_2)$ *are coarse proximity spaces such that* X *is unbounded and* Y *is bounded, then there is no coarse proximity map from* $(X, \mathcal{B}_1, \mathbf{b}_1)$ *to* $(Y, \mathcal{B}_2, \mathbf{b}_2)$*. Thus, the quotient of coarse proximity spaces need not exist, in general.*

In the following, we investigate the structure of quotient morphisms, and introduce the conditions that the quotient of a coarse proximity space exists.

Theorem 3.10 Suppose that $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to Y$ is a surjective function such that domain is a coarse proximity space and f satisfies the following conditions:

(1) $f^{-1}(\{y\}) \in \mathcal{B}_1$ for all $y \in Y$,
(2) $f^{-1}f(B) \in \mathcal{B}_1$ for all $B \in \mathcal{B}_1$.

Then the finest coarse proximity structure $(\mathcal{B}_0, \mathbf{b}_0)$ on Y for which f is a coarse proximity map, is defined by

(1) $\mathcal{B}_0 = \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{B}_1\}$,
(2) $\mathbf{b}_0$ is induced by the relation $\ll_0$ defined as follows: $A \ll_0 B$ if and only if for each binary rational $s \in [0, 1]$ there is a subset A_s of Y such that $A_0 = A$, $A_1 = B$ and $s < t$ implies $f^{-1}(A_s) \ll_1 f^{-1}(A_t)$.

Proof. It is easy to show that $\mathcal{B}_0$ is a bornology on Y. Now, we show that $\ll_0$ satisfies in the conditions (P_1) through (P_6) of Theorem 1.10. To show (P_1), let $D \in \mathcal{B}_0$. Take $A_0 = Y$ and $A_s = Y - D$ for all $s \neq 0$, then we have $f^{-1}(A_0) = X \ll_1 (X - f^{-1}(D)) = f^{-1}(A_s)$ for

all $s \neq 0$. Since $(X - f^{-1}(D)) \ll_1 (X - f^{-1}(D))$, it follows that $f^{-1}(A_s) \ll_1 f^{-1}(A_t)$ for $s < t$. Thus $Y \ll_0 (Y - D)$. To show (P_2), let $A \ll_0 B$. By the definition of $\ll_0$, we have $f^{-1}(A) \ll_1 f^{-1}(B)$. Thus by Theorem 1.10, $f^{-1}(A - B) \in \mathcal{B}_1$ and hence $(A - B) \in \mathcal{B}_0$. To show (P_3), let $A \subseteq B \ll_0 C$. Then for each binary rational $s \in [0, 1]$ there is a subset B_s of Y such that $B_0 = B$, $B_1 = C$ and $s < t$ implies $f^{-1}(B_s) \ll_1 f^{-1}(A_t)$. Take $A_0 = A$ and $A_s = B_s$ for all $s \neq 0$, then we have $f^{-1}(A_0) \subseteq f^{-1}(B_0) \ll_1 f^{-1}(B_s) = f^{-1}(B_s)$ for all $s \neq 0$. Since $f^{-1}(A_s) \ll_1 f^{-1}(A_t)$ for $0 < s < t$, it follows that $A \ll_0 C$. Similarly, it is easy to show that $B \ll_0 C \subseteq D$ implies $B \ll_0 D$. To show (P_4), By condition (P_3), $A \ll_0 (B_1 \cap B_2)$ implies $A \ll_0 B_1$ and $A \ll_0 B_2$. Conversely, assume that $A \ll_0 B_1$ and $A \ll_0 B_2$. Then for each binary rational $s \in [0, 1]$ there are subsets A_s and A'_s of Y such that $A_0 = A$, $A_1 = B_1$, $A'_0 = A$, $A'_1 = B_2$ and $s < t$ implies $f^{-1}(A_s) \ll_1 f^{-1}(A_t)$ and $f^{-1}(A'_s) \ll_1 f^{-1}(A'_t)$. Let $A''_s = A_s \cap A'_s$. Then $A''_0 = A$, $A''_1 = B_1 \cap B_2$ and $s < t$ implies $f^{-1}(A''_s) = f^{-1}(A_s \cap A'_s) \ll_1 f^{-1}(A_t \cap A'_t) = f^{-1}(A''_t)$. Thus we have $A \ll_0 (B_1 \cap B_2)$. To show (P_5), assume that $A \ll_0 B$. Then for each binary rational $s \in [0, 1]$ there is a subset A_s of Y such that $A_0 = A$, $A_1 = B$ and $s < t$ implies $f^{-1}(A_s) \ll_1 f^{-1}(A_t)$. Let $(Y - B)_s = Y - A_{(1-s)}$. Then we have $(Y - B)_0 = Y - B$, $(Y - B)_1 = Y - A$ and $s < t$ implies $f^{-1}(Y - A_{(1-s)}) \ll_1 f^{-1}(Y - A_{(1-t)})$. Hence $f^{-1}((Y - B)_s) \ll_1 f^{-1}((Y - B)_t)$, which shows that $(Y - B) \ll_0 (Y - A)$. To show (P_6), assume that $A \ll_0 B$. Then for each binary rational $s \in [0, 1]$ there is a subset A_s of Y such that $A_0 = A$, $A_1 = B$ and $s < t$ implies $f^{-1}(A_s) \ll_1 f^{-1}(A_t)$. Take $C = A_{\frac{1}{2}}$ and $A'_s = A_{\frac{s}{2}}$, $A''_s = A_{\frac{1+s}{2}}$ for all s. Then we have $A'_0 = A$, $A'_1 = A''_0 = C$, $A''_1 = B$ and $s < t$ implies $f^{-1}(A'_s) \ll_1 f^{-1}(A'_t)$ and $f^{-1}(A''_s) \ll_1 f^{-1}(A''_t)$. Thus we have $A \ll_0 C \ll_0 B$.
Finally, we show that $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_0, \mathbf{b}_0)$ is a coarse proximity map. If $B \in \mathcal{B}_1$, then by assumption $f^{-1}f(B) \in \mathcal{B}_1$ and hence $f(B) \in \mathcal{B}_0$. On the other hand, by the definition of $\ll_0$, $A \ll_0 B$ implies $f^{-1}(A) \ll_1 f^{-1}(B)$. Thus by Lemma 3.8, f is a coarse proximity map. Now, assume that $(\mathcal{B}_2, \mathbf{b}_2)$ is another coarse proximity structure on Y such that $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ is a coarse proximity map. Then by Lemma 3.9, we have $\mathcal{B}_2 = \mathcal{B}_0$. Let $A\bar{\mathbf{b}}_2 B$, i.e., $A \ll_2 (Y - B)$. Since $\ll_2$ satisfies the condition (P_6) of Theorem 1.10, for each binary rational $0 < s < 1$ there is a subset A_s of Y such that $s < t$ implies $A \ll_2 A_s \ll_2 A_t \ll_2 (Y - B)$. Take $A_0 = A$ and $A_1 = Y - B$. Then by Lemma 3.8, we have that $f^{-1}(A_s) \ll_1 f^{-1}(A_t)$ for each binary rational $s \in [0, 1]$. Hence $A \ll_0 (Y - B)$, i.e., $A\bar{\mathbf{b}}_0 B$. Thus $(\mathcal{B}_0, \mathbf{b}_0)$ is finer than $(\mathcal{B}_2, \mathbf{b}_2)$. ■

Definition 3.11 The coarse proximity structure $(\mathcal{B}_0, \mathbf{b}_0)$ defined in Theorem 3.10, is called the coinduced coarse proximity structure by $(f, \mathcal{B}_1, \mathbf{b}_1)$.

Definition 3.12 A surjective coarse proximity map $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ is called a coarse proximity quotient map if $(\mathcal{B}_2, \mathbf{b}_2)$ is the coinduced coarse proximity structure by $(f, \mathcal{B}_1, \mathbf{b}_1)$.

Theorem 3.13 In the construct **CoarsePro** the quotient morphisms are precisely the coarse proximity quotient maps.

Proof. Let $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ be a surjective and final morphism, and $(\mathcal{B}_0, \mathbf{b}_0)$ be the coinduced coarse proximity structure by $(f, \mathcal{B}_1, \mathbf{b}_1)$. Then by Lemma 3.9 and Theorem 3.10, we have $\mathcal{B}_0 \subseteq \mathcal{B}_0$ and $\mathbf{b}_0 \subseteq \mathbf{b}_2$. Now, assume that $id : (Y, \mathcal{B}_2, \mathbf{b}_2) \to (Y, \mathcal{B}_0, \mathbf{b}_0)$ is the identity function. Then $id \circ f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ is a coarse proximity map, so is id. Thus we have $\mathcal{B}_2 \subseteq \mathcal{B}_0$ and $\mathbf{b}_2 \subseteq \mathbf{b}_0$, which shows that $(\mathcal{B}_2, \mathbf{b}_2) = (\mathcal{B}_0, \mathbf{b}_0)$. Conversely, let $f : (X, \mathcal{B}_1, \mathbf{b}_1) \to (Y, \mathcal{B}_2, \mathbf{b}_2)$ be a coarse proximity quotient map and $g : (Y, \mathcal{B}_2, \mathbf{b}_2) \to (Z, \mathcal{B}_3, \mathbf{b}_3)$ a function such that $g \circ f$ be a coarse proximity map. Then $B \in \mathcal{B}_2$ implies $f^{-1}(B) \in \mathcal{B}_1$. Hence $g(B) = g \circ f(f^{-1}(B)) \in \mathcal{B}_3$. Now, let

$C \ll_3 D$. Since $\ll_3$ satisfies the condition (P_6) of Theorem 1.10, for each binary rational $0 < s < 1$ there is a subset C_s of Z such that $s < t$ implies $C \ll_3 C_s \ll_3 C_t \ll_3 D$. Take $A_0 = g^{-1}(C)$, $A_1 = g^{-1}(D)$ and $A_s = g^{-1}(C_s)$ for $0 < s < 1$. Then we have that $f^{-1}(A_s) \ll_1 f^{-1}(A_t)$ for each binary rational $s \in [0, 1]$. Thus $g^{-1}(C) \ll_2 g^{-1}(D)$, which shows that g is a coarse proximity map. ■

4. Conclusion

In this paper, we have studied some categorical structures of the category **CoarsePro**, whose objects are coarse proximity spaces and whose morphisms are coarse proximity maps. We have investigated the structure of initial, final, embedding and quotient morphisms in the construct **CoarsePro**. Also, it is shown that bimorphisms are exactly bijective coarse proximity maps, but not isomorphisms. Consequently, **CoarsePro** is not balanced.

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