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The n^{th} commutativity degree of semigroups

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Abstract. For a given positive integer n, the n^{th} commutativity degree of a finite non-commutative semigroup S is defined to be the probability of choosing a pair (x, y) for $x, y \in S$ such that x^n and y commute in S. If for every elements x and y of an associative algebraic structure $(S, .)$ there exists a positive integer r such that $xy = y^r x$, then S is called quasi-commutative. Evidently, every abelian group or commutative semigroup is quasi-commutative. In this paper, we study the n^{th} commutativity degree of certain classes of quasi-commutative semigroups. We show that the n^{th} commutativity degree of such structures is greater than $\frac{1}{2}$. Finally, we compute the n^{th} commutativity degree of a finite class of non-quasi-commutative semigroups and we conclude that it is less than $\frac{1}{2}$.

Keywords: Quasi-commutative semigroups, commutativity degree, probability.

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1. Introduction and preliminaries

For a given finite algebraic structure A, the commutativity degree of A (denoted by $P(A)$) is defined to be the probability of choosing a pair (x, y) of the elements of A such that x commutes with y. Indeed,

$$P(A) = \frac{|\{(x, y) \in A \times A : xy = yx\}|}{|A|^2} = \frac{\sum_{x \in A} |C_A(x)|}{|A|^2},$$

where $C_A(x)$ is the centralizer of x in A. The commutativity degree of groups has been studied extensively by certain authors during the years and recently it is studied and

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considered for finite semigroups. One may find some results around them in [11, 12, 19]. In general, for $n \geqslant 2$, the n^{th} commutativity degree of a finite algebraic structure S (denoted by $P_n(S)$) is defined to be the probability of choosing a pair $(x, y) \in S \times S$ such that x^n and y commute. Then, we have

$$P_n(S) = \frac{|\{(x, y) \in S \times S : x^n y = y x^n\}|}{|S|^2}.$$

In [13], one can see some good results on the n^{th} commutativity degree of certain non-commutative finite groups and semigroups. The quasi-commutativity property in algebraic structures is one of the interesting ideas which has been studied by many authors since 1971. The classification or identification of certain major classes of semigroups has been studied as well. For more and detailed descriptions on the quasi-commutative, quasi-commutative Hamiltonian, quasi-commutative super Hamiltonian and periodic Hamiltonian semigroups, one may consult the prolific articles [16, 17, 20]. A non-commutative semigroup S is called quasi-commutative if for every two elements $x, y \in S$, $xy = y^r x$ holds for some positive integers r. These semigroups introduced and studied by Mukherjee in [16]. For more decomposition property and also the certain infinite classes of such semigroups, see the results studied by Sorouhesh in [21]. According to [21], consider the following presentations:

$$\pi_1 = \langle a, b | a^5 = a, b^2 = a^2, ba = ab^3 \rangle$$

and for a given positive integer k,

$$\pi_2 = \langle a, b, c_1, c_2, \ldots, c_k | a^5 = a, b^2 = a^2, ba = ab^3, c_i^3 = c_i, ac_i = c_i a, bc_i = c_i b, \\ c_i c_j = c_j c_i, 1 \leqslant i, j \leqslant k \rangle,$$

$$\pi_3 = \langle a, b, c_1, c_2, \ldots, c_k, d | a^5 = a, b^2 = a^2, c_i^3 = c_i, d^{p+1} = d, ac_i = c_i a, da = ad, \\ db = bd, ba = ab^3, dc_i = c_i d, bc_i = c_i b, c_i c_j = c_j c_i, 1 \leqslant i, j \leqslant k \rangle,$$

where p is an odd prime.

Our notation is fairly standard and following [1, 2, 7, 19], We recall the notion of a presentation $\langle A|R \rangle$ of a semigroup A. For an alphabet A, let A^+ be the free semigroup over A. For a subset R of $A^+ \times A^+$, let ρ be a congruence relation generated by R. Then the semigroup $S = A^+/\rho$ will be denoted by $\langle A|R \rangle$ that is called a semigroup with the presentation for S. To lessen the likelihood of confusion, for $w_1, w_2 \in A^+$ we write $w_1 \equiv w_2$ if w_1 and w_2 are identical words, and $w_1 = w_2$ if they represent the same element of S (i.e. if $(w_1, w_2) \in \rho$). For example, let $A = \{a, b\}$ and $R = \{ab = ba\}$, then $aba = a^2 b$ and $aba \not\equiv a^2 b$. For more information on the presentation of semigroups, one may consult [2–10] and for a detailed study, one can see [15, 18, 19]. In whole of this paper, we use the well known notation $Sg(\pi)$ to denote the semigroup presented by the presentation π.

2. The semigroup $Sg(S_1)$

Let $S_1 = Sg(\pi_1)$, $S_2 = Sg(\pi_2)$ and $S_3 = Sg(\pi_3)$. In this section, we study the behaviour of n^{th} commutativity degree $P_n(S_1)$, $n \geqslant 2$ and in sections 3 and 4, we will consider the n^{th} commutativity degrees $P_n(S_2)$ and $P_n(S_3)$ and will give our main results around.

Our main result on S_1 is considered in the following proposition.

Proposition 2.1 For every positive integer n, $P_n(S_1) > \dfrac{1}{2}$. In fact,

$$P_n(S_1) = \begin{cases} \frac{49}{81} & \text{if } \ n = 1, \\ \\ \frac{56}{81} & \text{if } \ n \geqslant 2 \text{ and n is even}, \\ \\ \frac{46}{81} & \text{if } \ n \geqslant 3 \text{ and n is odd}. \end{cases}$$

Proof. Let $C(x^n) = \{y \mid x^n y = y x^n\}$ be the centralizer of x^n for every element $x \in S_1$. If $n = 1$, then by using the relation of S_1 we obtain

$$\begin{array}{l} C(a) = C(a^3) = \{a^i|\ i = 1, \dots, 4\}, \\ C(b) = C(a^2b) = C(a^4b) = \{a^{2i}, b, a^{2i}b|\ i = 1, 2\}, \\ C(a^2) = C(a^4) = \{a^i, b, a^ib|\ i = 1, \dots, 4\}, \\ C(ab) = C(a^3b) = \{a^{2i}, a^{2i-1}b|\ i = 1, 2\}. \end{array}$$

Consequently, $P_1(S_1) = \frac{49}{81}$.

Now, let $n > 1$. Then we consider two following cases: If n is even, then

$$C(x^n) = \begin{cases} C((a^2)^{\frac{n}{2}}) = C(a^2) = \{a^i, b, a^ib|\ i = 1, \dots, 4\} & \text{if } x = a \\ C((a^2)^n) = C(a^4) = \{a^i, b, a^ib|\ i = 1, \dots, 4\} & \text{if } x = a^2 \\ C((a^3)^n) = C(a^2) & \text{if } x = a^3 \\ C((a^4)^n) = C(a^4) & \text{if } x = a^4 \\ C(b^n) = C(a) = \{a^i|\ i = 1, \dots, 4\} & \text{if } x = b \\ C((ab)^n) = C(a) & \text{if } x = ab \\ C((a^2b)^n) = C(a^{n-1}) & \text{if } x = a^2b \\ C((a^3b)^n) = C(a^{n-1}) & \text{if } x = a^3b \\ C((a^4b)^n) = C((b^5)^n) = C(b^{5n}) = C(a^{5n}) = C(a) & \text{if } x = a^4b \end{cases}.$$

Thus, we have

$$P_n(S_1) = \frac{9+9+9+9+4+4+4+4+4}{9 \times 9} = \frac{56}{81}.$$

A similar proof may be used when n is odd. In this case, we obtain that

$$P_n(S_1) = \frac{4+9+4+9+4+4+4+4+4}{9 \times 9} = \frac{46}{81}.$$

Consequently, for every positive integer $n > 1$, $P_n(S_1) > \frac{1}{2}$. ■

3. The semigroup $Sg(S_2)$

In this section, we try to obtain the n^{th} commutativity degree $P_n(S_2)$ for a positive integer n. Our main result of this section is presented as the following proposition.

Proposition 3.1 For every positive integer n, $P_n(S_2) \geqslant \frac{1}{2}$. In fact,

$$P_n(S_2) = \begin{cases} \frac{17}{25} & \text{if } n = 1 \\ \frac{3}{4} & \text{if } n \geqslant 2 \, and \ n \ is \ even \\ \frac{1}{2} & \text{if } n \geqslant 3 \, and \ n \ is \ odd. \end{cases}$$

Proof. For $n = 1$, we may use a similar proof to that of the semigroup S_1 and then we get the following results, where $1 \leqslant i, j \leqslant k$ and $\ell = 1, 2$.

$$\begin{aligned}
&C(a) = C(a^3) = \{a^t,\ c_i^\ell,\ a^t c_i^\ell, c_i^\ell c_j^\ell, a^t c_i^\ell c_j^\ell | t = 1, \ldots, 4\},\\
&C(a^2) = C(a^4) = \{a^t, b, a^t b, c_i^\ell, a^t c_i^\ell, bc_i^\ell, a^t bc_i^\ell, c_i^\ell c_j^\ell, a^t c_i^\ell c_j^\ell, bc_i^\ell c_j^\ell, a^t bc_i^\ell c_j^\ell | t = 1, \ldots, 4\},\\
&C(b) = \{a^t, b, a^t b, c_i^\ell, a^t c_i^\ell, bc_i^\ell, a^t bc_i^\ell, c_i^\ell c_j^\ell, a^t c_i^\ell c_j^\ell, bc_i^\ell c_j^\ell, a^t bc_i^\ell c_j^\ell | t = 1, 2\},\\
&C(ab) = C(a^3 b) = \{a^t, a^{2t-1} b, c_i^\ell, a^t c_i^\ell, a^{2t-1} bc_i^\ell, c_i^\ell c_j^\ell, a^t c_i^\ell c_j^\ell, a^{2t-1} bc_i^\ell c_j^\ell | t = 1, 2\},\\
&C(a^2 b) = C(a^4 b) = C(b),\\
&C(c_i^\ell) = C(c_i^\ell c_j^\ell) = C(a^2),\\
&C(a^t c_i^\ell) = C(a^t c_i^\ell c_j^\ell) = C(a^t), \qquad (t = 1, \ldots, 4),\\
&C(bc_i^\ell) = C(bc_i^\ell c_j^\ell) = C(b),\\
&C(a^t bc_i^\ell) = C(a^t bc_i^\ell c_j^\ell) = C(a^t b), \qquad (t = 1, \ldots, 4).
\end{aligned}$$

Since $|S_2| = 10 \times 3^k - 1$, then

$$P_1(S_2) = \frac{4 \times 3^k(5(3^k) - 1) + 3 \times 3^k(6(3^k) - 1) + (3(3^k) - 1)(10(3^k) - 1)}{(10(3^k) - 1)^2}$$

$$= \frac{68(3^{2k}) - 20(3^k) + 1}{100(3^{2k}) - 20(3^k) + 1},$$

$$\lim_{k \to \infty} \frac{68(3^{2k}) - 20(3^k) + 1}{100(3^{2k}) - 20(3^k) + 1} = \lim_{k \to \infty} \frac{3^{2k}(68 - \frac{20}{3^k} + \frac{1}{3^{2k}})}{3^{2k}(100 - \frac{20}{3^k} + \frac{1}{3^{2k}})} = \frac{17}{25}.$$

For $n > 1$, we consider two cases. If n is even, then $C(x^n)$ is equal to

$$\begin{cases}
C(a^2) & if\ x = a^t, t = 1, 3\\
C((a^t)^n) = C(a^4) & \text{if } x = a^t, t = 2, 4\\
C(b^n) = C(a) & \text{if } x = b\\
C((a^t b)^n) = C(a) & \text{if } x = a^t b, t = 1, 4\\
C((a^t b)^n) = C(a^{n-1}) & \text{if } x = a^t b, t = 2, 3\\
C((c_i^\ell)^n) = C((c_i^\ell c_j^\ell)^n) = C(c_i^2) & \text{if } x = c_i^\ell, c_i^\ell c_j^\ell, \ell = 1, 2\\
C((a^t c_i^\ell)^n) = C((a^t c_i^\ell c_j^\ell)^n) = C(a^2 c_i^2) & \text{if } x = a^t c_i^\ell, a^t c_i^\ell c_j^\ell, \ell = 1, 2,\ t = 1, 3\\
C((a^t c_i^\ell)^n) = C((a^t c_i^\ell c_j^\ell)^n) = C(a^4 c_i^2) & \text{if } x = a^t c_i^\ell, a^t c_i^\ell c_j^\ell, \ell = 1, 2, t = 2, 4\\
C((bc_i^\ell)^n) = C((bc_i^\ell c_j^\ell)^n) = C(ac_i^2) & \text{if } x = bc_i^\ell, bc_i^\ell c_j^\ell, \ell = 1, 2\\
C((a^t bc_i^\ell)^n) = C((a^t bc_i^\ell c_j^\ell)^n) = C(ac_i^2) & \text{if } x = a^t bc_i^\ell, a^t bc_i^\ell c_j^\ell, \ell = 1, 2, t = 1, 4\\
C((a^t bc_i^\ell)^n) = C((a^t bc_i^\ell c_j^\ell)^n) = C(a^{n-1} c_i^2) & if\ x = a^t bc_i^\ell, a^t bc_i^\ell c_j^\ell, \ell = 1, 2, t = 2, 3,
\end{cases}$$

where $1 \leqslant i, j \leqslant k$. Therefore,

$$P_n(S_2) = \frac{5 \times 3^k(5(3^k) - 1) + (5(3^k) - 1)(10(3^k) - 1)}{(10(3^k) - 1)^2} = \frac{75(3^{2k}) - 20(3^k) + 1}{100(3^{2k}) - 20(3^k) + 1}.$$

This gives us $\lim_{k \to \infty} P_n(S_2) = \frac{3}{4}$.
For the odd values of n, we may use a similar method and conclude that

$$P_n(S_2) = \frac{10(10(3^k) - 1) + (10(3^k) - 1)(5(3^k) - 1)}{(10(3^k) - 1)^2} = \frac{50(3^{2k}) + 85(3^k) - 9}{100(3^{2k}) - 20(3^k) + 1}$$

and $\lim_{k \to \infty} P_n(S_2) = \frac{1}{2}$. Finally, for every positive integer n, we have $P_n(S_2) \geqslant \frac{1}{2}$. ∎

4. The semigroup $Sg(S_3)$

In this section, we compute the n^{th} commutative degree of the semigroup S_3. The following proposition show our main result of this section.

Proposition 4.1 For every positive integer n, $P_n(S_3) \geqslant \frac{1}{2}$. In fact,

$$P_n(S_3) = \begin{cases} \frac{17}{25} \text{ if } n = 1 \\ \frac{3}{4} \text{ if } n \geqslant 2, \text{ and n is even} \\ \frac{1}{2} \text{ if } n \geqslant 3, \text{ and n is odd.} \end{cases}$$

Proof. Since S_3 is a finite quasi-commutative semigroup of order $10(3^k)(1+p) - 1$ (see [5]), where p is an odd prime and k is a positive integer then to estimate $P_1(S_3)$ we need to compute $|C(x)|$'s for all $x \in S_3$, for instance using $|C(a)|$ we get

$$|C(a)| = 5 \times 3^k - 1 + p + (5 \times 3^k - 1)p = 5 \times 3^k(1 + p) - 1.$$

Consequently,

$$\begin{aligned} P_1(S_3) = & \frac{(4(3^k)(1+p) - 1)(5(3^k)(1+p) - 1)}{(10(3^k)(1+p) - 1)^2} \\ & + \frac{(3(3^k)(1+p) - 1)(6(3^k)(1+p) - 1)}{(10(3^k)(1+p) - 1)^2} \\ & + \frac{(3(3^k)(1+p) - 1)(10(3^k)(1+p) - 1)}{(10(3^k)(1+p) - 1)^2}. \end{aligned}$$

As an immediate result and using an almost tedious hand calculation, we see that

$$\lim_{k \to \infty} P_1(S_3) = \frac{17}{25}.$$

For every values of $n \geqslant 2$, we may consider two cases. Let n be even. Then

$$|C(x^n)| = \begin{cases} |C((a^t)^n)| = |C(a^2)| = 10(3^k)(1+p) - 1 \\ |C(b^n)| = |C((a^tb)^n)| = |C(a)| = 5(3^k)(1+p) - 1 \\ |C((c_i^\ell)^n)| = |C((c_i^\ell c_j^\ell)^n)| = |C(c_i^2)| = 10(3^k)(1+p) - 1 \\ |C((a^tc_i^\ell)^n)| = |C((a^tc_i^\ell c_j^\ell)^n)| = |C(a^2c_i^2)| = 10(3^k)(1+p) - 1 \\ |C((a^tbc_i^\ell)^n)| = |C((a^tbc_i^\ell c_j^\ell)^n)| = |C(a^2bc_i^2)| = 10(3^k)(1+p) - 1 \\ |C(d^n)| = |C(d)| = 10(3^k)(1+p) - 1 \\ |C((d^p)^n)| = |C(d)| = 10(3^k)(1+p) - 1 \\ |C((a^td)^n)| = |C(a^2d)| = 10(3^k)(1+p) - 1 \\ |C((bd^p)^n)| = |C(d)| = 10(3^k)(1+p) - 1 \\ |C((a^tbd^p)^n)| = |C(d)| = 10(3^k)(1+p) - 1 \\ |C((dc_i^\ell)^n)| = |C((dc_i^\ell c_j^\ell)^n)| = |C(dc_i^2)| = 10(3^k)(1+p) - 1 \\ |C((a^tdc_i)^n)| = |C((a^tdc_i^\ell c_j^\ell)^n)| = |C(a^2dc_i^2)| = 10(3^k)(1+p) - 1 \\ |C((a^tbd^pc_i^\ell c_j^\ell)^n)| = 10(3^k)(1+p) - 1, \end{cases}$$

where $1 \leqslant i, j \leqslant k$, $t = 1, \dots 4, \ell = 1, 2$ and p is an odd prime. This computation yields us:

$$\begin{aligned} P_n(S_3) &= \frac{(5(3^k)(1+p)-1)(5(3^k)(1+p)-1) + (5(3^k)(p+1)-1)(10(3^k)(1+p)-1)}{(10(3^k)(1+p)-1)^2} \\ &= \frac{75(3^{2k})p^2 + 150(3^{2k})p + 75(3^{2k}) - 25(3^k)p - 25(3^k) + 2}{100(3^{2k})p^2 + 200(3^{2k})p + 1003^{2k} - 20(3^k)p - 20(3^k) + 1}. \end{aligned}$$

Now, let n be odd. Then, in a similar way as above, we conclude that

$$\begin{aligned} P_n(S_3) =& \frac{(10(1+p))(10(3^k)(1+p)-1) + (5(3^k)(p+1)-1)(10(3^k)(1+p)-1)}{(10(3^k)(1+p)-1)^2} \\ =& \frac{50(3^{2k})p^2 + 100(3^{2k})p + 100(3^k)p^2 + 185(3^k)p + 50(3^{2k}) + 85(3^k) + 10p + 11}{100(3^{2k})p^2 + 200(3^{2k})p + 1003^{2k} - 20(3^k)p - 20(3^k) + 1}, \end{aligned}$$

For every odd prime p, if k tends to infinity, then we get

$$\lim_{k\to\infty} P_n(S_3) = \frac{3}{4} \; or \; \lim_{k\to\infty} P_n(S_3) \geqslant \frac{1}{2},$$

if n is even either odd, respectively. Hence, $P_n(S_3) \geqslant \frac{1}{2}$ for every positive integer n. ■

5. Conclusion

For all of the considered quasi-commutative semigroups in the last sections, the n^{th} commutativity degree was greater than of equal to $\frac{1}{2}$.

Challenging on getting probabilities less than $\frac{1}{2}$ will be of interest and we suppose that the quasi-commutativity property invites the probability to be $\geqslant \frac{1}{2}$, we consider a finite class of non-quasi-commutative semigroups.

$$\pi_4 = \langle a, b|\ a^{m+1} = a, b^3 = b, ba = a^{m-1}b\rangle,\ (m \geqslant 3).$$

This class studied for its finiteness property in [14]. Now, we show that:

Proposition 5.1 Let $S_4 = S_g(\pi_4)$, for $m \geqslant 11$, we have $P_1(S_4) < \frac{1}{2}$.

Proof. We may easily get that $|S_4| = 3m+2$ and by using a similar method to calculate $P_1(S_4)$, as in the last sections we get

$$|C(x)| = \begin{cases} |C(a^i)| = |C(a^ib^2)| = |C(a)| = 2m+1 & \text{if } x = a^i,\ a^ib^2,\ 1 \leqslant i \leqslant m-1, \\ |C(a^m)| = |C(a^mb^2)| = |C(b^2)| = 3m+2 & \text{if } x = a^m, \\ |C(b)| = |\{a^m,\ b,\ b^2,\ ab,\ a^mb^2\}| = 5 & \text{if } x = b, \\ |C(a^ib)| = |\{a^m,\ b^2,\ a^ib,\ a^mb^2\}| = 4 & \text{if } x = a^ib,\ 1 \leqslant i \leqslant m-1, \\ |C(a^mb)| = |\{a^m,\ b,\ b^2,\ a^mb,\ a^mb^2\}| = 5 & \text{if } x = a^mb. \end{cases}$$

Therefore,

$$P_1(S_4) = \frac{(2m-2)(2m+1) + 4(m-1) + 3(3m+2) + 2 \times 5}{(3m+2)^2} = \frac{4m^2 + 11m + 10}{9m^2 + 12m + 14}.$$

Obviously, $P_1(S_4) > \frac{4}{9}$ for $m \geqslant 11$. Hence, $\frac{4}{9} < P_1(S_4) < \frac{1}{2}$. ■

An example of the n-almost commutative semigroup, i.e.; the semigroup when n^{th} commutativity degree is equal to 1, is another result of the study of S_4 as follows:

Proposition 5.2 For every even positive integers m and n such that $m = n$, $m, n \geqslant 4$, we have $P_n(S_4) = 1$, where $S_4 = S_g(\pi_4)$.

Proof.

$$|C(x^n)| = \begin{cases} |C((a^ib^2)^n)| = |C((a^ib)^n)| = |C((a^i)^n)| = |C(a^m)| = 3m+2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x = a^i,\ a^ib,\ a^ib^2, \\ \\ |C((b)^n)| = |C((b^2)^n)| = |C(b^2)| = 3m+2 \quad \text{if } x = b,\ b^2, \end{cases}$$

where $1 \leqslant i \leqslant m$. Hence, $P_1(S_4) = \dfrac{(3m+2)(3m+2)}{(3m+2)^2} = 1$. ■

Consider a finite class of non commutative semigroups $\pi_5 =< a, b|\ a^2 = b^m,\ bab = a >$ of order $|S_5| = 5m-1$.

Proposition 5.3 For $m > 2$, $P_1(S_5) < \frac{1}{2}$, where $S_5 = S_g(\pi_5)$.

Proof. As a similar of proof in the last sections for calculate $P_1(S_5)$, we get

$$|C(x)| = \begin{cases} 4 \text{ if } & x = a,\ a^3,\ ab,\ ba,\ ab^2,\ b^2a,\ a^2ba,\ a^3b, \\ 11 \text{ if } & x = b,\ b^2,\ b^3,\ a^2b,\ aba,\ a^2b^2,\ ab^2a,\ a^4b,\ a^3ba, \\ 19 \text{ if } & x = a^2,\ a^4. \end{cases}$$

Therefore,

$$P_1(S_5) = \frac{8\times 4 + 2\times 19 + 9\times 11}{19\times 19} = \frac{169}{361} < \frac{1}{2}.$$

Now, for when m be even and greater than 2, we have

$$|C(x)| = \begin{cases} |C(a)| = |C(a^3)| = |C(ab^i)| = |C(b^ia)| = 4 & \text{if } x = a,\ a^3, 1 \leqslant i \leqslant m-1, \\ |C(b^i)| = 3m-1 & \text{if } 1 \leqslant i \leqslant m-1, \\ |C(a^2b^i)| = |C(a^4b^i)| = 3m-1 & \text{if } 1 \leqslant i \leqslant m-1, \\ |C(a^i)| = 5m-1 & \text{if } i = 2,\ 4, \end{cases}$$

and we conclude that

$$P_1(S_5) = \frac{2m\times 4 + 2\times(5m-1) + 3(m-1)(3m-1)}{(5m-1)^2} = \frac{9m^2+6m+1}{25m^2-10m+1}$$

and

$$\lim_{m\to\infty} \frac{9m^2+6m+1}{25m^2-10m+1} = \lim_{m\to\infty} \frac{m^2(9+\frac{6}{m}+\frac{1}{m^2})}{m^2(25-\frac{10}{m}+\frac{1}{m^2})} = \frac{9}{25}.$$

Thus, $P_1(S_5) < \frac{1}{2}$. ■

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