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Projective system of topological quasi modules

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Abstract. Quasi module is a new algebraic structure, based on module, which is composed of a semigroup structure and a partial order accompanied with an external ring multiplication. We proposed this structure in our paper [1] while we were studying the hyperspace $\mathscr{C}(M)$ consisting of all nonempty compact subsets of a topological module M over some topological ring R. Quasi module can be considered as a generalisation of module in some sense. In the present paper we have defined topological quasi module and given some examples of it. We have shown that the Cartesian product of arbitrary family of topological quasi modules is again a topological quasi module over some topological unitary ring. Finally we have defined projective system of topological quasi modules and projective limit of this system. We have proved various topological properties of the projective limit of a projective system.

Keywords: Quasi module, topological quasi module, projective system, projective limit.

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1. Introduction

We have introduced the concept of quasi module in our paper [1]. This algebraic structure is a conglomeration of a semigroup structure, an external ring multiplication and a partial order. It can be considered as a generalisation of module in the sense that every quasi module contains a module and conversely every module can be embedded into a quasi module. We have found a number of examples of quasi modules in our papers [1–4]. Let us start with the definition of quasi module.

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Definition 1.1 [1] Let $(X, \leqslant)$ be a partially ordered set, '+' be a binary operation on X [called *addition*] and '$\cdot$': $R \times X \longrightarrow X$ be another composition [called *ring multiplication*, R being a unitary ring]. If the operations and partial order satisfy the following axioms then $(X, +, \cdot, \leqslant)$ is called a *quasi module* (in short *qmod*) over R.

$A_1 : (X, +)$ is a commutative semigroup with identity θ.

$A_2 : x \leqslant y \; (x, y \in X) \Rightarrow x + z \leqslant y + z, \; r \cdot x \leqslant r \cdot y, \; \forall\, z \in X, \forall\, r \in R.$

A_3 : (i) $r \cdot (x + y) = r \cdot x + r \cdot y$,

(ii) $r \cdot (s \cdot x) = (rs) \cdot x$,

(iii) $(r + s) \cdot x \leqslant r \cdot x + s \cdot x$,

(iv) $1 \cdot x = x$, '1' being the multiplicative identity of R,

(v) $0 \cdot x = \theta$ and $r \cdot \theta = \theta \; (r \in R)$

$\forall\, x, y \in X, \; \forall\, r, s \in R.$

$A_4 : x + (-1) \cdot x = \theta$ iff $x \in X_0 := \big\{ z \in X : y \nleqslant z, \forall\, y \in X \smallsetminus \{z\} \big\}$.

A_5 : For each $x \in X, \exists\, y \in X_0$ such that $y \leqslant x$.

The elements of the set X_0 (which are evidently the minimal elements of X with respect to the partial order '$\leqslant$') are called '*one order*' elements of X and, by axiom A_4, these are the *only* invertible elements of X. In [1] we have proved the following proposition.

Proposition 1.2 [1] For any quasi module X over an unitary ring R, the set X_0 of all one order elements of X is a module over R.

Above proposition shows that every quasi module contains a module. The following example shows that every topological module over a topological unitary ring can be embedded into a quasi module over the same ring.

***Example* 1.3** [1] Let M be a topological module over a topological unitary ring R and $\mathscr{C}(M)$ be the collection of all nonempty compact subsets of M. Then $\mathscr{C}(M)$ forms a quasi module over R with usual set-inclusion as partial order and the relevant operations defined as follows : for $A, B \in \mathscr{C}(M)$ and $r \in R$, $A + B := \{a + b : a \in A, b \in B\}$ and $r \cdot A := \{ra : a \in A\}$. The identity element of $\mathscr{C}(M)$ is $\{\theta\}$, where θ is the identity element of M; the set of all one order elements of $\mathscr{C}(M)$ is given by $[\mathscr{C}(M)]_0 = \big\{\{m\} : m \in M\big\}$. If we identify $\big\{\{m\} : m \in M\big\}$ with M we can say that, the topological module M is embedded into the quasi module $\mathscr{C}(M)$.

The embedding of any module (not necessarily topological) into a quasi module can also be done as is explained in the following example.

***Example* 1.4** [1] Let M be a module over a unitary ring R. Let $\widetilde{M} := M \bigcup \{\omega\} \; (\omega \notin M)$. Define '+', '$\cdot$' and the partial order '$\leqslant_p$' as follows:
(i) The operation '+' between any two elements of M is same as in the module M and $x + \omega := \omega$ and $\omega + x := \omega$, $\forall\, x \in \widetilde{M}$.
(ii) The operation '$\cdot$' when applied on M is same as the ring multiplication in the module M and $r \cdot \omega := \omega$, if $r (\neq 0) \in R$ and $0 \cdot \omega := \theta$, θ being the identity element in M.
(iii) $x \leqslant_p \omega$, $\forall\, x \in M$ and $x \leqslant_p x$, $\forall\, x \in \widetilde{M}$.

Then $\big(\widetilde{M}, +, \cdot, \leqslant_p\big)$ is a quasi module over R. Here the set of all one order elements is

M. In other words, M can be embedded into $\widetilde{M}$, $x \mapsto x$ being the embedding.

In this example if we consider $M = \mathbb{C}$, the vector space of all complex numbers as a module over itself then the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ becomes a quasi module over $\mathbb{C}$, provided we define $0 \cdot \infty = 0$ and $z < \infty$, $\forall\, z \in \mathbb{C}$; here the set of all one order elements is $\mathbb{C}$.

We need the following concepts for the development of our theory.

Definition 1.5 [1] A subset Y of a qmod X is said to be a *sub quasi module* (*subqmod* in short) if Y itself is a quasi module with all the compositions and partial order of X being restricted to Y.

Theorem 1.6 [1] A nonempty subset Y of a qmod X (over a unitary ring R) is a sub quasi module iff Y satisfies the following conditions:
(i) $rx + sy \in Y$, $\forall\, r, s \in R$, $\forall\, x, y \in Y$;
(ii) $Y_0 \subseteq X_0 \cap Y$, where $Y_0 := \big\{z \in Y : y \nleqslant z, \forall\, y \in Y \smallsetminus \{z\}\big\}$;
(iii) $\forall\, y \in Y$, $\exists\, y_0 \in Y_0$ such that $y_0 \leqslant y$.

If Y is a subqmod of a qmod X then actually $Y_0 = X_0 \cap Y$, since for any $Y \subset X$ we always have $X_0 \cap Y \subseteq Y_0$.

Definition 1.7 [1] A mapping $f : X \longrightarrow Y$ (X, Y being two quasi modules over a unitary ring R) is called an *order-morphism* if the following conditions hold:
(i) $f(x + y) = f(x) + f(y)$, $\forall\, x, y \in X$;
(ii) $f(rx) = rf(x)$, $\forall\, r \in R$, $\forall\, x \in X$;
(iii) $x \leqslant y$ $(x, y \in X) \Rightarrow f(x) \leqslant f(y)$;
(iv) $p \leqslant q$ $\big(p, q \in f(X)\big) \Rightarrow f^{-1}(p) \subseteq\downarrow f^{-1}(q)$ and $f^{-1}(q) \subseteq\uparrow f^{-1}(p)$, where $\uparrow A := \{x \in X : x \geqslant a$ for some $a \in A\}$ and $\downarrow A := \{x \in X : x \leqslant a$ for some $a \in A\}$ for any $A \subseteq X$.

A surjective (injective, bijective) order-morphism is called an *order-epimorphism* (*order-monomorphism*, *order-isomorphism*).

Using these notions we have introduced the concept of exact sequence of quasi modules and proved some theorems related to these sequences in our paper [4]. In our papers [1] and [2] we have proved first and second order-isomorphism theorems respectively. In our paper [3] we have introduced the concept of chain conditions on quasi modules and defined Noetherian and Artinian quasi modules. Also we have discussed various ways of construction of Noetherian and Artinian quasi modules. We have also discussed in the same paper how exact sequences influence on Noetherian and Artinian qmods.

In the present paper in the next section we have introduced the concept of topological quasi module. We have given examples of topological quasi modules. We have shown that the Cartesian product of arbitrary family of topological quasi modules is again a topological quasi module with respect to the product topology.

In the last section we have introduced the concept of projective system of quasi modules and defined the projective limit of this system. Finally we have found various topological properties of the projective limit of projective system of topological quasi modules over a topological unitary ring.

2. Topological quasi module

In this section we shall topologize a quasi module. For this we need the following concept.

Definition 2.1 [6] Let Z be a topological space and '$\leqslant$' be a partial order on Z. Then '$\leqslant$' is called a *closed order* if the graph of the partial order $G_{\leqslant} := \{(x,y) \in Z \times Z : x \leqslant y\}$ is a closed set in the product space $Z \times Z$.

Theorem 2.2 [6] A partial order '$\leqslant$' in a topological space Z will be a closed order iff for any $x, y \in Z$ with $x \nleqslant y$, $\exists$ open neighbourhoods U, V of x, y respectively in Z such that $(\uparrow U) \cap (\downarrow V) = \emptyset$, where $\uparrow U := \{x \in Z : x \geqslant u \text{ for some } u \in U\}$ and $\downarrow V := \{x \in Z : x \leqslant v \text{ for some } v \in V\}$.

Corollary 2.3 If Z is a discrete topological space then any partial order '$\leqslant$' on Z is closed.

Proof. Let $x, y \in Z$ with $x \nleqslant y$. Then $\{x\}, \{y\}$ are open neighbourhoods of x, y respectively in Z. We claim that $\uparrow \{x\} \cap \downarrow \{y\} = \emptyset$. If not, $\exists\, z \in \uparrow \{x\} \cap \downarrow \{y\} \Rightarrow x \leqslant z \leqslant y$ —— a contradiction. Then by Theorem 2.2, the partial order of Z is closed. ■

Definition 2.4 A quasi module X over a topological unitary ring R is said to be a *topological quasi module* if X is equipped with a topology such that the addition and ring multiplication are continuous and the partial order is closed.

Definition 2.5 A toplogical space (Z, τ) is called Hausdorff if for any two distinct points $x, y \in Z$, there exists open neighbourhoods . U, V of x, y respectively in τ such that $U \cap V = \emptyset$.

Remark 1 *From the definition of topological quasi module, using the Theorem 2.2, we can say that every topological quasi module is Hausdorff. Again restriction of a continuous function being continuous we can say that X_0 becomes a Hausdorff topological module whenever X is a topological quasi module.*

We now give examples of topological quasi modules.

***Example* 2.6** Let us recall the Example 2.4 of [1] described as follows:

Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}^+ := \{n \in \mathbb{Z} : n \geqslant 0\}$. Then under the usual addition, $\mathbb{Z}^+$ is a commutative semigroup with the identity 0. Also it is a partially ordered set with respect to the usual order ($\leqslant$) of integers. If we define the ring multiplication '$\cdot$' : $\mathbb{Z} \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ by $(m, n) \longmapsto |m|n$, then it is a routine work to verify that $(\mathbb{Z}^+, +, \cdot, \leqslant)$ is a quasi module over $\mathbb{Z}$. Here the set of all one order elements is given by $[\mathbb{Z}^+]_0 = \{0\}$.

Topologization of $\mathbb{Z}^+$: To topologize this qmod consider $\mathbb{Z}^+$ with the discrete topology. Since with respect to the discrete topology any function is continuous it follows that the addition and ring multiplication on $\mathbb{Z}^+$ are continuous. Also, from Corollary 2.3, it follows that the partial order of $\mathbb{Z}^+$ is closed. Thus $\mathbb{Z}^+$ is a topological quasi module.

***Example* 2.7** Let M be a module over the unitary ring $\mathbb{Z}$ of integers. Let us consider the Example 2.1 of quasi module in [3] given as follows:

Let $X := \mathbb{Z}^+ \times M$. We define addition on X component-wise and ring multiplication on X by $(n, (m, a)) \longmapsto (|n|m, na)$, $\forall\, n \in \mathbb{Z}$ and $\forall\, (m, a) \in X$. We define a partial order '$\preceq$' on X as follows: $(n, a) \preceq (m, b)$ if and only if $n \leqslant m$ and $a = b$. Then X with aforesaid operations and partial order is a quasi module over $\mathbb{Z}$. Here the set of all one order elements $X_0 = \{(0, r) : r \in M\}$ can be identified with M through the map $(0, x) \mapsto x$ $(x \in M)$.

This example also justifies that every module over $\mathbb{Z}$ can be embedded into a quasi module over $\mathbb{Z}$. We now make this $\mathbb{Z}^+ \times M$ a topological quasi module.

Topologization of $\mathbb{Z}^+ \times M$: To topologize $\mathbb{Z}^+ \times M$ we have to first consider M to be a Hausdorff topological module over the ring $\mathbb{Z}$, where $\mathbb{Z}$ is considered with the discrete topology. We now consider $\mathbb{Z}^+$ with the discrete topology. Then we give the *product topology* on $\mathbb{Z}^+ \times M$. In this product topology a net $\{(a_n, x_n)\}_{n\in D}$ in $\mathbb{Z}^+ \times M$ (D being some directed set) is convergent iff the nets $\{a_n\}_{n\in D}$ and $\{x_n\}_{n\in D}$ are convergent in $\mathbb{Z}^+$ and M respectively. We show below that $\mathbb{Z}^+ \times M$ is a topological quasi module.

First of all let $\{(a_n, x_n)\}_{n\in D}$ and $\{(a'_n, x'_n)\}_{n\in D}$ be two nets in $\mathbb{Z}^+ \times M$ (D being some directed set) converging to (a, x) and (a', x') respectively. Then $a_n \to a$, $a'_n \to a'$ in $\mathbb{Z}^+$ and $x_n \to x$, $x'_n \to x'$ in M. So M being a topological module we have $a_n + a'_n \to a + a'$ in $\mathbb{Z}^+$ and $x_n + x'_n \to x + x'$ in M. Therefore $(a_n + a'_n, x_n + x'_n) \to (a + a', x + x')$ in $\mathbb{Z}^+ \times M$ justifying that the addition in $\mathbb{Z}^+ \times M$ is continuous. Now if $\{r_n\}_{n\in D}$ is a net in $\mathbb{Z}$ converging to r then $|r_n| \to |r|$ and $r_n x_n \to rx$ [$\because M$ is a topological module] and hence $(|r_n|a_n, r_n x_n) \to (|r|a, rx)$. This ensures that the ring multiplication of $\mathbb{Z}^+ \times M$ is continuous.

It now only remains to show that the partial order of $\mathbb{Z}^+ \times M$ is closed. For this let $\{(a_n, x_n)\}_{n\in D}$ and $\{(a'_n, x'_n)\}_{n\in D}$ be two nets in $\mathbb{Z}^+ \times M$ (D being some directed set) with $(a_n, x_n) \preceq (a'_n, x'_n)$, $\forall\, n \in D$. Also let these nets converge to (a, x) and (a', x') respectively. Then $a_n \leqslant a'_n$ and $x_n = x'_n$, $\forall\, n \in D$. So $\lim\limits_n a_n \leqslant \lim\limits_n a'_n$ and $\lim\limits_n x_n = \lim\limits_n x'_n$ $\Longrightarrow a \leqslant a'$ and $x = x' \Rightarrow (a, x) \preceq (a', x')$. This justifies that the partial order of $\mathbb{Z}^+ \times M$ is closed. Thus $\mathbb{Z}^+ \times M$ is a topological quasi module whenever M is a Hausdorff topological module.

Remark 2 *In Example 2.7 if we consider $M = \{\theta\}$, the trivial module over $\mathbb{Z}$, then the resulting quasi module $\mathbb{Z}^+ \times \{\theta\}$ can clearly be identified with the quasi module $\mathbb{Z}^+$ explained in Example 2.6.*

***Example* 2.8** Let us consider the ring of integers $\mathbb{Z}$ which can be thought of as a topological module over the ring $\mathbb{Z}$ with respect to the discrete topology on $\mathbb{Z}$. Then as described in Example 1.3, the set $\mathscr{C}(\mathbb{Z})$ of all nonempty compact subsets of $\mathbb{Z}$ form a quasi module over $\mathbb{Z}$. We now consider an interesting sub collection of $\mathscr{C}(\mathbb{Z})$. Let

$$\mathscr{C}_s(\mathbb{Z}) := \Big\{A \in \mathscr{C}(\mathbb{Z}) : 0 \in A, A \text{ is symmetric about } 0\Big\}$$

We show below that $\mathscr{C}_s(\mathbb{Z})$ is a quasi module over $\mathbb{Z}$ with respect to the addition, ring multiplication and partial order as defined on $\mathscr{C}(\mathbb{Z})$ [see Example 1.3].

Justification:

$\mathbf{A_1}$: Since in the discrete topology on $\mathbb{Z}$ a set $A \subseteq \mathbb{Z}$ is compact iff it is finite it follows that every element of $\mathscr{C}(\mathbb{Z})$ is finite. So an arbitrary element of $\mathscr{C}_s(\mathbb{Z})$ is of the form $\{-n_p, -n_{p-1}, \ldots, -n_1, 0, n_1, \ldots, n_{p-1}, n_p\}$, where $n_i \in \mathbb{Z}^+ := \{m \in \mathbb{Z} : m \geqslant 0\}$ for $i = 1, 2, \ldots, p$ with $0 \leqslant n_1 < n_2 < \cdots < n_p$, since a set $A \subseteq \mathbb{Z}$ is symmetric about 0 iff $A = -A$. Then for any $A, B \in \mathscr{C}_s(\mathbb{Z})$, $-(A + B) = -A - B = A + B$. This shows that $A + B$ is also compact, symmetric and contains 0; in other words, $\mathscr{C}_s(\mathbb{Z})$ is closed under the aforesaid addition. Also for any $A \in \mathscr{C}_s(\mathbb{Z})$ we have $A + \{0\} = A$. Thus under addition $\mathscr{C}_s(\mathbb{Z})$ is a commutative semigroup with the identity $\{0\}$.

$\mathbf{A_2}$: $\mathscr{C}_s(\mathbb{Z})$ is also closed under the above ring multiplication, since multiplication of a finite subset of $\mathbb{Z}$ by an integer is again finite and this ring multiplication preserves the symmetry. Now it is clear that for any $A, B, C \in \mathscr{C}_s(\mathbb{Z})$ and $r \in \mathbb{Z}$, $A \subseteq B \Rightarrow A + C \subseteq B + C$ and $r \cdot A \subseteq r \cdot B$.

$\mathbf{A_3}$: For any $A, B \in \mathscr{C}_s(\mathbb{Z})$ and $r, s \in \mathbb{Z}$, we have

(i) $r \cdot (A + B) = \big\{r(a + b) : a \in A, b \in B\big\} = \big\{ra + rb : a \in A, b \in B\big\} = \big\{ra : a \in A\big\} + \big\{rb : b \in B\big\} = r \cdot A + r \cdot B$.

(ii) $(rs) \cdot A = r \cdot (s \cdot A)$.

(iii) $(r + s) \cdot A = \big\{(r + s)a : a \in A\big\} \subseteq \big\{ra + sb : a, b \in A\big\} = r \cdot A + s \cdot A$.

(iv) $1 \cdot A = A$.

(v) $0 \cdot A = \{0\}$ for any $A \in \mathscr{C}_s(\mathbb{Z})$ and $r \cdot \{0\} = \{0\}$ for any $r \in \mathbb{Z}$.

$\mathbf{A_4}$: $\{0\}$ is the only minimal element of $\mathscr{C}_s(\mathbb{Z})$, since for any finite subset A of $\mathbb{Z}$ symmetric about 0 and containing 0 we must have $A \supseteq \{0\}$. So $[\mathscr{C}_s(\mathbb{Z})]_0 = \big\{\{0\}\big\}$. Now, for $A \in \mathscr{C}_s(\mathbb{Z})$, $A - A = \{0\} \iff A = \{0\}$. Consequently, $A - A = \{0\} \iff A \in [\mathscr{C}_s(\mathbb{Z})]_0$.

$\mathbf{A_5}$: For any $A \in \mathscr{C}_s(\mathbb{Z})$ we always have $A \supseteq \{0\}$.

Thus it follows that $\mathscr{C}_s(\mathbb{Z})$ is a quasi module over $\mathbb{Z}$.

We now topologize $\mathscr{C}_s(\mathbb{Z})$ with the Vietoris topology which will make it a topological quasi module over $\mathbb{Z}$ which is equipped with the discrete topology. For this we have to describe first the Vietoris topology [5].

Let W be an open set in $\mathbb{Z}$. We define

$$W^+ := \Big\{A \in \mathscr{C}_s(\mathbb{Z}) : A \subseteq W\Big\},\ W^- := \Big\{A \in \mathscr{C}_s(\mathbb{Z}) : A \cap W \neq \emptyset\Big\}.$$

Then $\mathscr{S} := \Big\{W^+ : W \text{ is open in } \mathbb{Z}\Big\} \bigcup \Big\{W^- : W \text{ is open in } \mathbb{Z}\Big\}$ forms a subbase for some topology on $\mathscr{C}_s(\mathbb{Z})$. This topology is known as the *Vietoris topology* or *finite topology*. Since for any open sets $W_1, W_2, \ldots, W_m$ in $\mathbb{Z}$, $W_1^+ \cap W_2^+ \cap \cdots \cap W_m^+ = (W_1 \cap W_2 \cap \cdots \cap W_m)^+$ we can say that an arbitrary basic open set in the Vietoris topology takes the form

$$\mathcal{V} := V_1^- \cap V_2^- \cap \cdots \cap V_n^- \cap V_0^+$$

for some open sets V_i $(0 \leqslant i \leqslant n)$ in $\mathbb{Z}$. Also we may assume that $V_i \subseteq V_0$, for all $i = 1, 2, \ldots, n$. Now we claim that the Vietoris topology on $\mathscr{C}_s(\mathbb{Z})$ is the discrete topology.

To justify our assertion it is enough to prove that each singleton in $\mathscr{C}_s(\mathbb{Z})$ is an open set in the Vietoris topology. For this let $A \in \mathscr{C}_s(\mathbb{Z})$. Without any loss of generality we may assume that

$$A := \{-n_p, -n_{p-1}, \ldots, -n_1, 0, n_1, \ldots, n_{p-1}, n_p\},$$

where $n_i \in \mathbb{N}$ for $i = 1, 2, \ldots, p$ with $n_1 < n_2 < \cdots < n_p$. Then

$$\{A\} = \{n_1\}^- \cap \{n_2\}^- \cap \cdots \cap \{n_p\}^- \cap A^+.$$

Now, each singleton $\{n_i\}$ and A itself being open in the discrete space $\mathbb{Z}$. It follows that $\{A\}$ is open in the Vietoris topology in $\mathscr{C}_s(\mathbb{Z})$. If $A = \{0\}$ then we can write $\{A\} = \{0\}^+$, where $\{0\}$ is clearly open in $\mathbb{Z}$. Thus $\{A\}$ is open in the Vietoris topology in this case also.

It is now our endeavour to prove that $\mathscr{C}_s(\mathbb{Z})$ with the Vietoris topology is a topological quasi module. First of all, since the Vietoris topology on $\mathscr{C}_s(\mathbb{Z})$ reduces to simply the discrete topology it is evident that the addition and ring multiplication which are respectively the mappings

$$\left.\begin{array}{r} \mathscr{C}_s(\mathbb{Z}) \times \mathscr{C}_s(\mathbb{Z}) \longrightarrow \mathscr{C}_s(\mathbb{Z}) \\ (A, B) \longmapsto A + B \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{r} \mathbb{Z} \times \mathscr{C}_s(\mathbb{Z}) \longrightarrow \mathscr{C}_s(\mathbb{Z}) \\ (r, A) \longmapsto r \cdot A \end{array}\right\}$$

are continuous, $\mathbb{Z}$ being endowed with the discrete topology.

Again by Corollary 2.3 we can say that the partial order of $\mathscr{C}_s(\mathbb{Z})$ is closed, the Vietoris topology on $\mathscr{C}_s(\mathbb{Z})$ being the discrete topology.

Thus $\mathscr{C}_s(\mathbb{Z})$ is a topological quasi module (over $\mathbb{Z}$) with the Vietoris topology.

Definition 2.9 [1] **(Arbitrary product of qmods)** Let $\{X_\mu : \mu \in \Lambda\}$ be an arbitrary family of quasi modules over a unitary ring R. Let $X := \prod_{\mu \in \Lambda} X_\mu$ be the Cartesian product of these quasi modules defined as : $x \in X$ if and only if $x : \Lambda \longrightarrow \bigcup_{\mu \in \Lambda} X_\mu$ is a map such that $x(\mu) \in X_\mu$, $\forall\, \mu \in \Lambda$. Then clearly X is nonempty, since Λ is nonempty and each X_μ contains at least the additive identity θ_μ (say).

Let us denote $x_\mu := x(\mu)$, $\forall\, \mu \in \Lambda$. Also we write each $x \in X$ as $x = (x_\mu)$, where $x_\mu = p_\mu(x)$, $p_\mu : X \longrightarrow X_\mu$ being the projection map, $\forall\, \mu \in \Lambda$.

The following theorem shows that this Cartesian product becomes a quasi module with respect to suitably defined operations and partial order.

Theorem 2.10 [1] If $\{X_\mu : \mu \in \Lambda\}$ is an arbitrary family of quasi modules over a unitary ring R then the Cartesian product $X := \prod\limits_{\mu\in\Lambda} X_\mu$ is a quasi module over R with respect to the following operations and partial order: for $x = (x_\mu),\ y = (y_\mu) \in X$ and $r \in R$
(i) $x + y := (x_\mu + y_\mu)$;
(ii) $r.x := (rx_\mu)$;
(iii) $x \leqslant y$ if $x_\mu \leqslant y_\mu$, $\forall\, \mu \in \Lambda$.

Corollary 2.11 [1] For any family of qmods $\{X_\mu : \mu \in \Lambda\}$ over some common unitary ring R, $\left[\prod\limits_{\mu\in\Lambda} X_\mu\right]_0 = \prod\limits_{\mu\in\Lambda}[X_\mu]_0$.

Proposition 2.12 [1] Let $\{X_i : i \in \Lambda\}$ be an arbitrary family of quasi modules over a unitary ring R and $X := \prod\limits_{i\in\Lambda} X_i$ be the product qmod of these qmods. Then each projection map $p_j : X \longrightarrow X_j$ is an order-epimorphism.

Proposition 2.13 If $f : X \to Y$ is an order-morphism between two qmods X, Y over a unitary ring R then for any set $B \subseteq f(X)$,
(i) $\uparrow f^{-1}(B) = f^{-1}(\uparrow B)$,
(ii) $\downarrow f^{-1}(B) = f^{-1}(\downarrow B)$.

Proof. (i) Let $x \in\uparrow f^{-1}(B)$. Then $\exists\, w \in f^{-1}(B)$ such that $x \geqslant w \Rightarrow f(x) \geqslant f(w)$ and $f(w) \in B \Rightarrow f(x) \in\uparrow B \Rightarrow x \in f^{-1}(\uparrow B)$. Conversely, let $y \in f^{-1}(\uparrow B)$. Then $f(y) \in\uparrow B$ $\Rightarrow \exists\, b \in B$ such that $f(y) \geqslant b$. Since f is an order-morphism and $b \in B \subseteq f(X)$ we have $f^{-1}(f(y)) \subseteq\uparrow f^{-1}(b) \subseteq\uparrow f^{-1}(B) \Rightarrow y \in\uparrow f^{-1}(B)$.

(ii) Same as above. ■

Theorem 2.14 If $\{X_i : i \in \Lambda\}$ is a family of topological quasi modules over a topological unitary ring R then $\prod\limits_{i\in\Lambda} X_i$ is a topological quasi module with respect to the product topology.

Proof. Let $X := \prod\limits_{i\in\Lambda} X_i$ and $p_i : X \to X_i$ be the i-th projection map. Then the *product topology* on X is the smallest topology on X such that each p_i is continuous. We now prove this theorem through the following steps:
<u>**Step-I:**</u> Let A_X and A_i denote respectively the addition on X and X_i, for all $i \in \Lambda$. Then for any $x = (x_i), y = (y_i) \in X$ we have $p_i \circ A_X(x, y) = p_i(x + y) = x_i + y_i = A_i(x_i, y_i) = A_i \circ (p_i \times p_i)(x, y) \Rightarrow p_i \circ A_X = A_i \circ (p_i \times p_i)$. Since each X_i is a topological quasi module so A_i is continuous, for each $i \in \Lambda$. This justifies that $A_i \circ (p_i \times p_i)$ and hence $p_i \circ A_X$ is continuous for each i. Then by the theory of product topology it follows that $A_X : X \times X \to X$ is continuous.
<u>**Step-II:**</u> If M_X and M_i denote respectively the ring multiplication on X and X_i, for all $i \in \Lambda$ then $p_i \circ M_X = M_i \circ (id_R \times p_i)$, $id_R : R \to R$ being the identity map. In fact, for any $x = (x_i) \in X$ and $r \in R$ we have $p_i \circ M_X(r, x) = p_i(rx) = rx_i = M_i(r, x_i) = M_i \circ (id_R \times p_i)(r, x)$. Then each M_i being continuous [as each X_i is a topological qmod]

we can say that $M_i \circ (id_R \times p_i)$ is continuous for each i [$\because id_R$ is always continuous] and hence each $p_i \circ M_X$ is continuous. The continuity of M_X then follows from the standard result of product topology.

Step-III : Let $x = (x_i), y = (y_i) \in X$ be such that $x \not\leqslant y$. Then $\exists j \in \Lambda$ such that $x_j \not\leqslant y_j$. So the partial order of X_j being closed we can find two open neighbourhoods . U_j, V_j of x_j, y_j respectively in X_j such that $\uparrow U_j \cap \downarrow V_j = \emptyset$, by Theorem 2.2. Then $p_j^{-1}(U_j)$ and $p_j^{-1}(V_j)$ are open neighbourhoods . of x and y respectively in X. Now by the Propositions 2.12 and 2.13 we have $\uparrow p_j^{-1}(U_j) \bigcap \downarrow p_j^{-1}(V_j) = p_j^{-1}(\uparrow U_j) \bigcap p_j^{-1}(\downarrow V_j) = p_j^{-1}(\uparrow U_j \bigcap \downarrow V_j) = p_j^{-1}(\emptyset) = \emptyset$. This justifies that the partial order of X is closed, by Theorem 2.2. ∎

3. Projective System

In this section, we shall study projective system of topological quasi modules. For this let us first define a projective system of quasi modules.

Definition 3.1 A *projective system* (or *inverse system*) of quasi modules is defined as a triplet

$$\Big((D, \leqslant), \{X_i\}_{i\in D}, \{f_i^j\}_{i\leqslant j}\Big)$$

where, $(D, \leqslant)$ is a directed set, $\{X_i : i \in D\}$ is a family of quasi modules indexed by D over a common unitary ring R and $\big\{f_i^j : X_j \longrightarrow X_i \,\big|\, i \leqslant j,\ i, j \in D\big\}$ is a family of order-epimorphisms, called *bonding maps*, such that

(i) f_i^i is the identity map on X_i, for each $i \in D$

(ii) $f_i^j \circ f_j^k = f_i^k$, for all $i \leqslant j \leqslant k$ in D.

We shall denote this projective system as $\big\{D, X_i, f_i^j\big\}_{i\leqslant j}$.

The *projective limit* (or *inverse limit*) of the projective system $\big\{D, X_i, f_i^j\big\}_{i\leqslant j}$, denoted as $\varprojlim_{i\in D} X_i$, is defined by

$$\varprojlim_{i\in D} X_i := \left\{ x = (x_i)_i \in \prod_{i\in D} X_i : f_j^k(x_k) = x_j, \text{ for all } j \leqslant k,\ j, k \in D \right\}$$

where $\prod_{i\in D} X_i$ is the Cartesian product of quasi modules described in Definition 2.9 and Theorem 2.10.

We first show that the projective limit of a projective system is a quasi module.

Theorem 3.2 Let $\big\{D, X_i, f_i^j\big\}_{i\leqslant j}$ be a projective system of quasi modules over a unitary ring R. Then its projective limit is a sub quasi module of $\prod_{i\in D} X_i$.

Proof. For convenience let us denote $X \equiv \varprojlim_{i\in D} X_i$. To show that X is a subqmod of $\prod_{i\in D} X_i$ we shall justify for X three necessary and sufficient conditions stated in Theorem 1.6.

(i) First of all note that $\theta = (\theta_i)_i \in X$, since each f_i^j is an order-morphism. Let $x = (x_i)_i,\ y = (y_i)_i \in X$ and $r, s \in R$. Then $f_j^k(rx_k + sy_k) = rf_j^k(x_k) + sf_j^k(y_k) = rx_j + sy_j$, for all $j \leqslant k$ in $D \Rightarrow rx + sy \in X$.

(ii) We now show that

$$X_0 = X \bigcap \left[\prod_{i\in D} X_i\right]_0 \tag{1}$$

$$= X \bigcap \prod_{i\in D} [X_i]_0 \ \cdots\cdots\cdots \text{[by Corollary 2.11]} \tag{2}$$

$$= \left\{ u = (u_i)_i \in \prod_{i\in D} [X_i]_0 : f_j^k(u_k) = u_j, \text{ for all } j \leqslant k, \ j,k \in D \right\} \tag{3}$$

For this it is enough to show that $X_0 \subseteq X \bigcap \left[\prod_{i\in D} X_i\right]_0$, since the other inclusion is always true, as explained in the Theorem 1.6. We shall prove this contrapositively. For this let $x = (x_i)_i \in X$ but $x \notin \prod_{i\in D}[X_i]_0$. We claim that $x \notin X_0$ (such an assumption is reasonable, since $X_0 \subseteq X$ always). To justify this claim we have to find some element $y \in X$ such that $y \leqslant x$ but $y \neq x$. In fact, we shall find something more i.e $y \in X_0$. We shall determine such an y through the following steps:

Step-I: $x \notin \prod_{i\in D}[X_i]_0 \Rightarrow \exists\, j \in D$ such that $x_j \notin [X_j]_0 \Rightarrow \exists\, y_j \in [X_j]_0$ such that $y_j \leqslant x_j$ but $y_j \neq x_j$. Now $x \in X \Rightarrow f_i^k(x_k) = x_i$, whenever $i \leqslant k$ in $D \Rightarrow f_i^k \circ p_k = p_i$, whenever $i \leqslant k$, where $p_i : \prod_{d\in D} X_d \longrightarrow X_i$ is the i-th projection map which is an order-epimorphism (by Proposition 2.12). This justifies that each f_i^k ($i \leqslant k$ in D) is an order-epimorphism. So $y_j \leqslant x_j \Rightarrow$ for any $i \geqslant j$ in D, $x_i \in (f_j^i)^{-1} f_j^i(x_i) = (f_j^i)^{-1}(x_j) \subseteq\uparrow (f_j^i)^{-1}(y_j) \Rightarrow \exists\, z_i \in X_i$ such that $f_j^i(z_i) = y_j$ and $z_i \leqslant x_i \Rightarrow \exists\, y_i \in [X_i]_0$ such that $y_i \leqslant z_i \leqslant x_i$. Then $f_j^i(y_i) \leqslant f_j^i(z_i) = y_j \Rightarrow f_j^i(y_i) = y_j$ [$\because y_j$ is of order one]. Thus for any $i \geqslant j$ in D, we have found some $y_i \in [X_i]_0$ such that $f_j^i(y_i) = y_j$ and $y_i \leqslant x_i$.

Step-II: For any $i \leqslant j$, $f_i^j(y_j) \leqslant f_i^j(x_j) = x_i$ [$\because y_j \leqslant x_j$]. We define $y_i := f_i^j(y_j)$. Then $y_i \leqslant x_i$ and $y_i \in [X_i]_0$, since f_i^j being an order-morphism sends each one order element of X_j to an one order element of X_i.

Step-III: Now let $D_j := \{i \in D :$ either $i \leqslant j$ or $i \geqslant j\}$ and $i \notin D_j$. Then $\exists\, k \in D$ such that $k \geqslant i, j$ [$\because D$ is a directed set]. So by Step-I, we can find $y_k \in [X_k]_0$ such that $y_k \leqslant x_k$ and $f_j^k(y_k) = y_j \Rightarrow f_i^k(y_k) \leqslant f_i^k(x_k) = x_i$. We define $y_i := f_i^k(y_k)$. Then $y_i \leqslant x_i$ and $y_i \in [X_i]_0$, since y_k is of order one. Thus for any $i \notin D_j$ we have found $y_i \in [X_i]_0$ such that $y_i \leqslant x_i$, $y_i = f_i^k(y_k)$ and $f_j^k(y_k) = y_j$, where $k \geqslant i, j$.

Step-IV : Let us define $y : D \longrightarrow \bigcup_{i\in D} X_i$ by $y(i) := y_i, \forall\, i \in D$, where y_i's are as defined in the steps I, II, III. Then $y \in \prod_{i\in D} X_i$. Also $y \leqslant x$, since $y_i \leqslant x_i$, $\forall\, i \in D$ [shown in steps I, II, III]. Moreover, $y \neq x$, since $y_j \neq x_j$. Again, $y \in \prod_{i\in D}[X_i]_0 = \left[\prod_{i\in D} X_i\right]_0$, since $y_i \in [X_i]_0$, $\forall\, i \in D$ [by construction of y_i's].

Step-V: In this step we shall prove that $y \in X$. For this let $m, n \in D$ such that $m \leqslant n$.

Then $\exists\, q \in D$ such $q \geqslant m, n, j$. So by similar arguments as used in step-III, we have $f_m^q(y_q) = y_m$ and $f_n^q(y_q) = y_n$, where $y_q \in [X_q]_0$ is such that $f_j^q(y_q) = y_j$. Therefore $y_m = f_m^q(y_q) = f_m^n \circ f_n^q(y_q) = f_m^n(y_n)$. This, together with step-IV, justifies that $y \in X$.

Thus in view of steps (I) through (V) it follows that $x \notin X_0$. So contrapositively we have $X_0 \subseteq X \bigcap \left[\prod_{i \in D} X_i\right]_0$ and hence our assertion as in equations 1 to 3 is justified i.e

$$X_0 = \left\{(u_i)_i \in \prod_{i \in D}[X_i]_0 : f_j^k(u_k) = u_j, \text{ for all } j \leqslant k,\ j,k \in D\right\}$$

(iii) Let $x = (x_i)_i \in X$. Then by similar arguments as in the above steps (I) through (V), we can find some

$$y \in \left\{(u_i)_i \in \prod_{i \in D}[X_i]_0 : f_j^k(u_k) = u_j, \text{ for all } j \leqslant k,\ j,k \in D\right\} = X_0$$

such that $y \leqslant x$.

Therefore by Theorem 1.6 it follows that $X \equiv \varprojlim_{i \in D} X_i$ is a sub quasi module of $\prod_{i \in D} X_i$.

■

Theorem 3.3 Let $\{D, X_i, f_i^j\}_{i \leqslant j}$ be a projective system of topological quasi modules (over a topological unitary ring R) and continuous bonding maps. Then its projective limit is closed in $\prod_{i \in D} X_i$, endowed with the product topology.

Proof. By Theorem 3.2, the projective limit of the given projective system exists and is a sub quasi module of $\prod_{i \in D} X_i$. Again by Theorem 2.14, $\prod_{i \in D} X_i$ is a topological quasi module over R with respect to the product topology.

Now fix $j, k \in D$ such that $j \leqslant k$ and let

$$T_j^k := \left\{(u_i)_i \in \prod_{i \in D} X_i : f_j^k(u_k) = u_j\right\} \tag{4}$$

We first show that T_j^k is closed in $\prod_{i \in D} X_i$. For this let us consider the map

$$\left.\begin{aligned} \Psi : \prod_{i \in D} X_i &\longrightarrow X_j \times X_j \\ x &\longmapsto \left(x_j, f_j^k(x_k)\right) \end{aligned}\right\}, \text{ whenever } x = (x_i)_i \in \prod_{i \in D} X_i$$

This map is clearly continuous, since the maps $p_j : x \mapsto x_j$ and $f_j^k \circ p_k : x \mapsto f_j^k(x_k)$ are continuous (the projection maps p_i's and bonding maps f_j^k's being continuous). Now the set $\Delta(X_j) := \{(x_j, x_j) : x_j \in X_j\}$, called *diagonal on* X_j, being a closed set in $X_j \times X_j$ it follows that $T_j^k = \Psi^{-1}(\Delta(X_j))$ [To justify this equality it is enough to note that all projection maps p_i's and bonding maps f_j^k's are surjective.] is closed in $\prod_{i \in D} X_i$. Then $\varprojlim_{i \in D} X_i = \bigcap \left\{T_j^k : j \leqslant k, j, k \in D\right\}$ is closed in $\prod_{i \in D} X_i$. ■

Theorem 3.4 Let $\{D, X_i, f_i^j\}_{i\leqslant j}$ be a projective system of topological quasi modules (over a topological unitary ring R) and continuous bonding maps. If moreover, each X_i is compact and connected (i.e a continuum) then its projective limit is also compact and connected in $\prod_{i\in D} X_i$, endowed with the product topology.

Proof. Since each X_i is compact, by Tychonoff product theorem $\prod_{i\in D} X_i$ is compact. So the projective limit $\varprojlim_{i\in D} X_i$ being closed in $\prod_{i\in D} X_i$ (by Theorem 3.3) is compact.

To prove that the projective limit $\varprojlim_{i\in D} X_i$ is connected let us first construct the sets

$$T^k := \left\{(u_i)_i \in \prod_{i\in D} X_i : f_j^k(u_k) = u_j, \forall\, j \leqslant k, j \in D\right\} \text{ for each } k \in D \tag{5}$$

Then $T^k = \bigcap \{T_j^k : j \leqslant k, j \in D\}$, where T_j^k's are as defined in 4 in the proof of the theorem 3.3. So each T_j^k $(j \leqslant k, j, k \in D)$ being closed (proved in theorem 3.3) it follows that T^k is closed for each $k \in D$. Also $\theta = (\theta_i)_i \in T^k$, $\forall\, k \in D$ i.e each T^k is non-empty.

We now show that each T^k is connected. For this let $\Phi : \prod_{i\in D} X_i \longrightarrow \prod_{i\in D} X_i$ be the map defined by

$$p_i \circ \Phi(x) := \begin{cases} p_i(x), & \text{if } i \nleqslant k \\ f_i^k \circ p_k(x), & \text{if } i \leqslant k \end{cases}$$

Then each $p_i \circ \Phi$ is continuous and hence Φ is continuous. So $T^k = \Phi\left(\prod_{i\in D} X_i\right)$ implies that T^k is connected for each $k \in D$, since $\prod_{i\in D} X_i$ is connected as each X_i is so.

Now note that $X := \varprojlim_{i\in D} X_i = \bigcap \left\{T^k : k \in D\right\}$. To prove that X is connected let $X \subseteq U \cup V$, where U, V are disjoint open sets in $\prod_{i\in D} X_i$. We claim that either $X \subseteq U$ or $X \subseteq V$. To justify this claim we first show that $T^k \subseteq U \cup V = W$ (say), for some $k \in D$. If not, $T^k \cap W^c \neq \emptyset$, $\forall\, k \in D$, W^c being the complement of W in $\prod_{i\in D} X_i$. Then $\mathscr{F} := \{T^k \cap W^c : k \in D\}$ is a family of non-empty closed sets in $\prod_{i\in D} X_i$ having finite intersection property. In fact, for any finite subset $D_0 \subseteq D$, $\exists\, n \in D$ such that $n \geqslant k$, $\forall\, k \in D_0$ [by directed property of D]. Now for any $k \leqslant k'$ in D, $T^{k'} \subseteq T^k$, since $f_j^k \circ f_k^{k'} = f_j^{k'}$ for any $j \leqslant k$. So $T^n \subseteq \bigcap_{k\in D_0} T^k$ justifying that $\mathscr{F}$ has finite intersection property. Thus $\prod_{i\in D} X_i$ being compact, we have $\bigcap_{k\in D} (T^k \cap W^c) \neq \emptyset \Rightarrow \left(\bigcap_{k\in D} T^k\right) \bigcap W^c = \left(\varprojlim_{i\in D} X_i\right) \bigcap W^c \neq \emptyset$ $\Rightarrow X \nsubseteq W = U \cup V$ —— a contradiction. So $\exists\, p \in D$ such that $T^p \subseteq U \cup V$. Then

T^p being connected we have either $T^p \subseteq U$ or $T^p \subseteq V \Rightarrow$ either $X \subseteq U$ or $X \subseteq V$. So $X \equiv \varprojlim_{i \in D} X_i$ is connected. ■

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