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Atomic systems in n-Hilbert spaces and their tensor products

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Abstract. Concept of a family of local atoms in n-Hilbert space is being studied. K-frame in tensor product of n-Hilbert spaces is described and a characterization is given. Atomic system in tensor product of n-Hilbert spaces is presented and established a relationship between atomic systems in n-Hilbert spaces and their tensor products.

Keywords: Atomic system, K-frame, tensor product of Hilbert spaces, linear n-normed space, n-Hilbert space.

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1. Introduction and preliminaries

In recent times, many generalizations of frames have been appeared. Some of them are K-frame, g-frame, fusion frame and g-fusion frame etc. K-frames for a separable Hilbert spaces were introduced by Lara Gavruta [5] to study atomic decomposition systems for a bounded linear operator. In fact, generalized atomic subspaces for operators in Hilbert spaces were studied by Ghosh and Samanta [7]. K-frame is also presented to reconstruct elements from the range of a bounded linear operator K in a separable Hilbert space and it is a generalization of the ordinary frames. In fact, many properties of ordinary frames may not holds for such generalization of frames. Like K-frame, another generalization of frame is g-fusion frame and it has been studied by several authors [10, 17, 18]. Rabinson [15] presented the basic concepts of tensor product of Hilbert spaces. The tensor product

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of Hilbert spaces X and Y is a certain linear space of operators which was represented by Folland [4] and Kadison and Ringrose [13]. Generalized fusion frame in tensor product of Hilbert spaces was studied by Ghosh and Samanta [8].

In 1970, Diminnie et al. [3] introduced the concept of 2-inner product space. Atomic system in 2-inner product space is studied by Dastourian and Janafada [2]. A generalization of a 2-inner product space for $n \geqslant 2$ was developed by Misiak [14] in 1989.

In this paper, we give a notion of a family of local atoms in n-Hilbert space. Since tensor product of n-Hilbert spaces becomes a n-Hilbert space, we like to study K-frame in this n-Hilbert space. We give a necessary and sufficient condition for being K-frames in n-Hilbert spaces is that of being in their tensor products. Atomic system in tensor product of n-Hilbert spaces is discussed. Finally, we are going to establish a relationship between atomic systems in n-Hilbert spaces and their tensor products.

Throughout this paper, X will denote separable Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle_1$ and $\mathbb{K}$ denotes the field of real or complex numbers. $l^2(\mathbb{N})$ and $l^2(\mathbb{N} \times \mathbb{N})$ denotes the spaces of square summable scalar-valued sequences with index sets $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, respectively. $\mathcal{B}(X)$ denotes the space of all bounded linear operators on X.

Now, we recall some basic definitions and theorems.

Definition 1.1 [5] Let $K \in \mathcal{B}(X)$. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ is called a K-frame for X if there exist positive constants A, B such that

$$A \| K^* f \|_1^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i \rangle_1|^2 \leqslant B \| f \|_1^2 \;\; \forall f \in X. \tag{1}$$

The constants A, B are called frame bounds. If $\{f_i\}_{i=1}^{\infty}$ satisfies only the right inequality of (1), it is called a Bessel sequence with bound B.

Definition 1.2 [5] Let $K \in \mathcal{B}(X)$ and $\{f_i\}_{i=1}^{\infty}$ be a sequence in X. Then $\{f_i\}_{i=1}^{\infty}$ is said to be an atomic system for K if the following statements hold:

(i) the series $\sum_{i=1}^{\infty} c_i f_i$ converges for all $\{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$;

(ii) for every $x \in X$, there exists $a_x = \{a_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$ such that $\| a_x \|_{l^2} \leqslant C \| x \|_1$ and $K(x) = \sum_{i=1}^{\infty} a_i f_i$ for some $C > 0$.

Definition 1.3 [16] Let $(Y, \langle \cdot, \cdot \rangle_2)$ be a Hilbert space. Then the tensor product of X and Y is denoted by $X \otimes Y$ and it is defined to be an inner product space associated with the inner product

$$\langle f \otimes g, f' \otimes g' \rangle = \langle f, f' \rangle_1 \langle g, g' \rangle_2 \;\; \forall f, f' \in X \;\&\; g, g' \in Y.$$

The norm on $X \otimes Y$ is given by

$$\| f \otimes g \| = \| f \|_1 \| g \|_2 \;\; \forall f \in X \;\&\; g \in Y,$$

where $\| \cdot \|_1$ and $\| \cdot \|_2$ are norms induced from $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. The space $X \otimes Y$ is clearly completion with respect to the above inner product. Thus, the space $X \otimes Y$ is a Hilbert space.

For the operators $Q \in \mathcal{B}(X)$ and $T \in \mathcal{B}(Y)$, their tensor product is denoted by $Q \otimes T$ and defined as $(Q \otimes T) A = Q A T^* \;\; \forall \; A \in X \otimes Y$. It can be easily verified that $Q \otimes T \in \mathcal{B}(X \otimes Y)$ [4].

Theorem 1.4 [4] Suppose $Q, Q' \in \mathcal{B}(X)$ and $T, T' \in \mathcal{B}(Y)$. Then

(i) $Q \otimes T \in \mathcal{B}(X \otimes Y)$ and $\|Q \otimes T\| = \|Q\| \|T\|$.
(ii) $(Q \otimes T)(f \otimes g) = Qf \otimes Tg$ for all $f \in X, g \in Y$.
(iii) $(Q \otimes T)(Q' \otimes T') = (QQ') \otimes (TT')$.
(iv) $Q \otimes T$ is invertible if and only if Q and T are invertible, in which case $(Q \otimes T)^{-1} = \left(Q^{-1} \otimes T^{-1}\right)$.
(v) $(Q \otimes T)^* = (Q^* \otimes T^*)$.
(vi) Let $f, f' \in H \setminus \{0\}$ and $g, g' \in K \setminus \{0\}$. If $f \otimes g = f' \otimes g'$, then there exist constants a and b with $ab = 1$ such that $f = af'$ and $g = bg'$.

Definition 1.5 [12] A real valued function $\|\cdot, \cdots, \cdot\| : H^n \to \mathbb{R}$ satisfying the following properties:

(i) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,
(ii) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutations of $x_1, \cdots, x_n$,
(iii) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \|x_1, x_2, \cdots, x_n\|, \alpha \in \mathbb{K}$,
(iv) $\|x + y, x_2, \cdots, x_n\| \leqslant \|x, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$,

for all $x_1, x_2, \cdots, x_n, x, y \in H$, is called n-norm on H. A linear space H together with a n-norm $\|\cdot, \cdots, \cdot\|$ is called a linear n-normed space.

Definition 1.6 [14] Let $n \in \mathbb{N}$ and H be a linear space of dimension greater than or equal to n over the field $\mathbb{K}$. An n-inner product on H is a map

$$(x, y, x_2, \cdots, x_n) \longmapsto \langle x, y | x_2, \cdots, x_n \rangle, \; x, y, x_2, \cdots, x_n \in H$$

from H^{n+1} to the set $\mathbb{K}$ such that for every $x, y, x_1, x_2, \cdots, x_n \in H$,

(i) $\langle x_1, x_1 | x_2, \cdots, x_n \rangle \geqslant 0$ and $\langle x_1, x_1 | x_2, \cdots, x_n \rangle = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent,
(ii) $\langle x, y | x_2, \cdots, x_n \rangle = \langle x, y | x_{i_2}, \cdots, x_{i_n} \rangle$ for every permutations $(i_2, \cdots, i_n)$ of $(2, \cdots, n)$,
(iii) $\langle x, y | x_2, \cdots, x_n \rangle = \overline{\langle y, x | x_2, \cdots, x_n \rangle}$,
(iv) $\langle \alpha x, y | x_2, \cdots, x_n \rangle = \alpha \langle x, y | x_2, \cdots, x_n \rangle$ for $\alpha \in \mathbb{K}$,
(v) $\langle x + y, z | x_2, \cdots, x_n \rangle = \langle x, z | x_2, \cdots, x_n \rangle + \langle y, z | x_2, \cdots, x_n \rangle$.

A linear space H together with an n-inner product $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle$ is called an n-inner product space.

Theorem 1.7 [14] Let H be an n-inner product space. Then

$$|\langle x, y | x_2, \cdots, x_n \rangle| \leqslant \|x, x_2, \cdots, x_n\| \|y, x_2, \cdots, x_n\|,$$

for all $x, y, x_2, \cdots, x_n \in H$, where $\|x_1, x_2, \cdots, x_n\| = \sqrt{\langle x_1, x_1 | x_2, \cdots, x_n \rangle}$ is called Cauchy-Schwarz inequality.

Definition 1.8 [12] A sequence $\{x_k\}$ in linear n-normed space H is said to be convergent to $x \in H$ if $\lim\limits_{k \to \infty} \|x_k - x, e_2, \cdots, e_n\| = 0$ for every $e_2, \cdots, e_n \in H$ and it is called a Cauchy sequence if $\lim\limits_{l, k \to \infty} \|x_l - x_k, e_2, \cdots, e_n\| = 0$ for every $e_2, \cdots, e_n \in H$. The space H is said to be complete if every Cauchy sequence in this space is convergent in H. An n-inner product space is called n-Hilbert space if it is complete with respect to its induce norm.

Definition 1.9 [1] Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-Hilbert space and $\xi \in X$. A sequence

$\{f_i\}_{i=1}^{\infty} \subseteq X$ is said to be a 2-frame associated to ξ if there exist positive constants A, B such that

$$A \, \| f, \xi \|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i \,|\, \xi \rangle|^2 \leqslant B \, \| f, \xi \|^2 \quad \forall f \in X.$$

Theorem 1.10 [1] Let L_ξ denote the 1-dimensional linear subspace of X generated by a fixed $\xi \in X$. Let M_ξ be the algebraic complement of L_ξ. Define $\langle x, y \rangle_\xi = \langle x, y \,|\, \xi \rangle$ on X. This semi-inner product induces an inner product on the quotient space X / L_ξ which is given by

$$\langle x + L_\xi, y + L_\xi \rangle_\xi = \langle x, y \rangle_\xi = \langle x, y \,|\, \xi \rangle \;\; \forall \; x, y \in X.$$

By identifying X / L_ξ with M_ξ in an obvious way, we obtain an inner product on M_ξ. Define $\| x \|_\xi = \sqrt{\langle x, x \rangle_\xi} \;\; (x \in M_\xi)$. Then $(M_\xi, \| \cdot \|_\xi)$ is a norm space. Let X_ξ be the completion of the inner product space M_ξ.

Definition 1.11 [6] Let H be a n-Hilbert space. A sequence $\{f_i\}_{i=1}^{\infty}$ in H is said to be a frame associated to $(a_2, \cdots, a_n)$ if there exists constant $0 < A \leqslant B < \infty$ such that

$$A \, \| f, a_2, \cdots, a_n \|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle|^2 \leqslant B \, \| f, a_2, \cdots, a_n \|^2 \qquad (2)$$

for all $f \in H$. The constants A, B are called frame bounds. If $\{f_i\}_{i=1}^{\infty}$ satisfies only the right inequality of (2), is called a Bessel sequence associated to $(a_2, \cdots, a_n)$ in H with bound B.

Let $a_2, a_3, \cdots, a_n$ be the fixed elements in H and L_F denote the linear subspace of H spanned by the non-empty finite set $F = \{a_2, a_3, \cdots, a_n\}$. Then the quotient space H / L_F is a normed linear space with respect to the norm, $\| x + L_F \|_F = \| x, a_2, \cdots, a_n \|$ for every $x \in H$. Let M_F be the algebraic complement of L_F, then $H = L_F \oplus M_F$. Define $\langle x, y \rangle_F = \langle x, y \,|\, a_2, \cdots, a_n \rangle$ on H. Then $\langle \cdot, \cdot \rangle_F$ is a semi-inner product on H and this semi-inner product induces an inner product on the quotient space H / L_F which is given by

$$\langle x + L_F, y + L_F \rangle_F = \langle x, y \rangle_F = \langle x, y \,|\, a_2, \cdots, a_n \rangle \quad \forall \; x, y \in H.$$

By identifying H / L_F with M_F in an obvious way, we obtain an inner product on M_F. Then M_F is a normed space with respect to the norm $\| \cdot \|_F$ defined by $\| x \|_F = \sqrt{\langle x, x \rangle_F} \; \forall x \in M_F$. Let H_F be the completion of the inner product space M_F [6].

Theorem 1.12 [6] Let H be a n-Hilbert space. Then $\{f_i\}_{i=1}^{\infty} \subseteq H$ is a frame associated to $(a_2, \cdots, a_n)$ with bounds A and B if and only if it is a frame for the Hilbert space H_F with bounds A and B.

For more details on frames in n-Hilbert spaces and their tensor products one can go through the papers [6, 9].

2. Atomic system in n-Hilbert space

In this section, concept of a family of local atoms associated to $(a_2, \cdots, a_n)$ is discussed. Next, we are going to generalize this concept and then define K-frame associated to $(a_2, \cdots, a_n)$ for H, for a given bounded linear operator K.

Definition 2.1 Let $(H, \|\cdot, \cdots, \cdot\|)$ be a linear n-normed space and $a_2, \cdots, a_n$ be fixed elements in H. Let W be a subspace of H and $\langle a_i \rangle$ denote the subspaces of H generated by a_i, for $i = 2, 3, \cdots, n$. Then a map $T : W \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle \to \mathbb{K}$ is called a b-linear functional on $W \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle$, if for every $x, y \in W$ and $k \in \mathbb{K}$, the following conditions hold:

(i) $T(x + y, a_2, \cdots, a_n) = T(x, a_2, \cdots, a_n) + T(y, a_2, \cdots, a_n)$
(ii) $T(kx, a_2, \cdots, a_n) = k\,T(x, a_2, \cdots, a_n)$.

A b-linear functional is said to be bounded if there exists a real number $M > 0$ such that

$$|T(x, a_2, \cdots, a_n)| \leqslant M\,\|x, a_2, \cdots, a_n\| \;\; \forall\, x \in W.$$

Some properties of bounded b-linear functional defined on $H \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle$ have been discussed in [11].

Definition 2.2 Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence associated to $(a_2, \cdots, a_n)$ in H and Y be a closed subspace of H. Then $\{f_i\}_{i=1}^{\infty}$ is said to be a family of local atoms associated to $(a_2, \cdots, a_n)$ for Y if there exists a sequence of bounded b-linear functionals $\{T_i\}_{i=1}^{\infty}$ defined on $H \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle$ such that

(i) $\sum\limits_{i=1}^{\infty} |T_i(f, a_2, \cdots, a_n)|^2 \leqslant C\,\|f, a_2, \cdots, a_n\|^2$ for some $C > 0$.

(ii) $f = \sum\limits_{i=1}^{\infty} T_i(f, a_2, \cdots, a_n)\, f_i$ for all $f \in Y$.

Theorem 2.3 Let $\{f_i\}_{i=1}^{\infty}$ be a family of local atoms associated to $(a_2, \cdots, a_n)$ for Y, where Y be a closed subspace of H. Then the family $\{f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for Y.

Proof. Since $\{f_i\}_{i=1}^{\infty}$ is a family of local atoms associated to $(a_2, \cdots, a_n)$ for Y, there exists a sequence of bounded b-linear functionals $\{T_i\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} |T_i(f, a_2, \cdots, a_n)|^2 \leqslant C\,\|f, a_2, \cdots, a_n\|^2, f \in Y,$$

for some $C > 0$. Now, for each $f \in Y$,

$$\begin{aligned}
\|f, a_2, \cdots, a_n\|^4 &= \langle f, f \,|\, a_2, \cdots, a_n \rangle^2 \\
&= \left\langle f, \sum_{i=1}^{\infty} T_i(f, a_2, \cdots, a_n)\, f_i \,\middle|\, a_2, \cdots, a_n \right\rangle^2 \\
&= \left(\sum_{i=1}^{\infty} \overline{T_i(f, a_2, \cdots, a_n)}\, \langle f, f_i \,|\, a_2, \cdots, a_n \rangle \right)^2 \\
&\leqslant \sum_{i=1}^{\infty} |T_i(f, a_2, \cdots, a_n)|^2 \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle|^2 \\
&\leqslant C\,\|f, a_2, \cdots, a_n\|^2 \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle|^2
\end{aligned}$$

$$\Rightarrow \frac{1}{C} \| f , a_2 , \cdots , a_n \|^2 \leqslant \sum_{i=1}^{\infty} | \langle f , f_i | a_2 , \cdots , a_n \rangle |^2 .$$

Also, $\{ f_i \}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots , a_n)$ in Y. Hence, $\{ f_i \}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots , a_n)$ for Y. ■

Theorem 2.4 Let $\{ f_i \}_{i=1}^{\infty}$ be a Bessel sequence associated to $(a_2, \cdots , a_n)$ in H and Y be a closed subspace of H. If there exists a Bessel sequence associated to $(a_2, \cdots , a_n)$ in H, say $\{ g_i \}_{i=1}^{\infty}$ such that

$$P_Y (f) = \sum_{i=1}^{\infty} \langle f , g_i | a_2 , \cdots , a_n \rangle f_i, \tag{3}$$

for all $f \in H_F$, where P_Y is the orthogonal projection onto Y, then $\{ f_i \}_{i=1}^{\infty}$ is a family of local atoms associated to $(a_2, \cdots , a_n)$ for Y.

Proof. Let us take $f \in Y$ then by (3), we can write

$$f = P_Y (f) = \sum_{i=1}^{\infty} \langle f , g_i | a_2 , \cdots , a_n \rangle f_i.$$

Now, for each $f \in Y$, we define

$$T_i (f, a_2, \cdots , a_n) = \langle f , g_i | a_2 , \cdots , a_n \rangle .$$

Then, for each $f \in Y$, we have

$$f = \sum_{i=1}^{\infty} T_i (f, a_2, \cdots , a_n) f_i.$$

Also, for any i, we have

$$\begin{aligned} | T_i (f, a_2, \cdots , a_n) | &= | \langle f , g_i | a_2 , \cdots , a_n \rangle | \\ &\leqslant \| f, a_2, \cdots , a_n \| \, \| g_i, a_2, \cdots , a_n \| \\ &\leqslant M \, \| f, a_2, \cdots , a_n \| \end{aligned}$$

where $M = \sup_i \| g_i, a_2, \cdots , a_n \|$. This verifies that each T_i are bounded b-linear functionals defined on $Y \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle$. On the other hand, since $\{ g_i \}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots , a_n)$, we get

$$\begin{aligned} \sum_{i=1}^{\infty} | T_i (f, a_2, \cdots , a_n) |^2 &= \sum_{i=1}^{\infty} | \langle f, g_i | a_2, \cdots , a_n \rangle |^2 \\ &\leqslant B \| f, a_2, \cdots , a_n \|^2 \end{aligned}$$

This completes the proof. ■

Now, we are going to generalize the concept of a family of local atoms associated to $(a_2, \cdots, a_n)$.

Definition 2.5 Let K be a bounded linear operator on H_F and $\{f_i\}_{i=1}^{\infty}$ be a sequence of vectors in H. Then $\{f_i\}_{i=1}^{\infty}$ is said to be an atomic system associated to $(a_2, \cdots, a_n)$ for K in H if

(i) $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in H.

(ii) For any $f \in H_F$, there exists $\{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$ such that $K(f) = \sum\limits_{i=1}^{\infty} c_i f_i$, where $\|\{c_i\}_{i=1}^{\infty}\|_{l^2} \leqslant C \|f, a_2, \cdots, a_n\|$ and $C > 0$.

Definition 2.6 Let K be a bounded linear operator on H_F. Then a sequence $\{f_i\}_{i=1}^{\infty} \subseteq H$ is said to be a K-frame associated to $(a_2, \cdots, a_n)$ for H if there exist constants $A, B > 0$ such that for each $f \in H_F$,

$$A \|K^* f, a_2, \cdots, a_n\|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \leqslant B \|f, a_2, \cdots, a_n\|^2.$$

Theorem 2.7 Let $\{f_i\}_{i=1}^{\infty}$ be a K-frame associated to $(a_2, \cdots, a_n)$ for H. Then there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ associated to $(a_2, \cdots, a_n)$ such that

$$K^* f = \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle g_i \quad \forall f \in H_F.$$

Proof. According to the Theorem 3 of [5], there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ associated to $(a_2, \cdots, a_n)$ such that

$$K f = \sum_{i=1}^{\infty} \langle f, g_i | a_2, \cdots, a_n \rangle f_i \quad \forall f \in H_F.$$

Now, for each $f, g \in H_F$, we have

$$\begin{aligned} \langle K f, g | a_2, \cdots, a_n \rangle &= \left\langle \sum_{i=1}^{\infty} \langle f, g_i | a_2, \cdots, a_n \rangle f_i, g | a_2, \cdots, a_n \right\rangle \\ &= \sum_{i=1}^{\infty} \langle f, g_i | a_2, \cdots, a_n \rangle \langle f_i, g | a_2, \cdots, a_n \rangle \\ &= \left\langle f, \sum_{i=1}^{\infty} \langle g, f_i | a_2, \cdots, a_n \rangle g_i | a_2, \cdots, a_n \right\rangle. \end{aligned}$$

This shows that

$$K^* f = \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle g_i \quad \forall f \in H_F.$$

This completes the proof. ■

3. Atomic system in Tensor product of n-Hilbert spaces

Let H_1 and H_2 be two n-Hilbert spaces associated with the n-inner products $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_1$ and $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_2$, respectively. The tensor product of H_1 and H_2 is denoted by $H_1 \otimes H_2$ and it is defined to be an n-inner product space associated with the n-inner product given by

$$\begin{aligned} &\langle f \otimes g, f_1 \otimes g_1 | f_2 \otimes g_2, \cdots, f_n \otimes g_n \rangle \\ &= \langle f, f_1 | f_2, \cdots, f_n \rangle_1 \langle g, g_1 | g_2, \cdots, g_n \rangle_2 \end{aligned} \tag{4}$$

for all $f, f_1, f_2, \cdots, f_n \in H_1$ and $g, g_1, g_2, \cdots, g_n \in H_2$.
The n-norm on $H_1 \otimes H_2$ is defined by

$$\begin{aligned} &\| f_1 \otimes g_1, f_2 \otimes g_2, \cdots, f_n \otimes g_n \| \\ &= \| f_1, f_2, \cdots, f_n \|_1 \, \| g_1, g_2, \cdots, g_n \|_2 \end{aligned} \tag{5}$$

for all $f_1, f_2, \cdots, f_n \in H_1$ and $g_1, g_2, \cdots, g_n \in H_2$, where $\| \cdot, \cdots, \cdot \|_1$ and $\| \cdot, \cdots, \cdot \|_2$ are n-norm generated by $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_1$ and $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle_2$, respectively. Clearly, the space $H_1 \otimes H_2$ is completion with respect to the above n-inner product. Therefore, the space $H_1 \otimes H_2$ is an n-Hilbert space.

Remark 1 *Let $G = \{ b_2, b_3, \cdots, b_n \}$ be a non-empty finite set, where $b_2, b_3, \cdots, b_n$ be the fixed elements in H_2. Then we define the Hilbert space K_G with respect to the inner product is given by*

$$\langle x + L_G, y + L_G \rangle_G = \langle x, y \rangle_G = \langle x, y | b_2, \cdots, b_n \rangle_2 \quad \forall \; x, y \in H_2,$$

where L_G denote the linear subspace of H_2 spanned by the set G. According to the definition 1.3, $H_F \otimes K_G$ is the Hilbert space with respect to the inner product:

$$\left\langle x \otimes y, x' \otimes y' \right\rangle = \left\langle x, x' \right\rangle_F \left\langle y, y' \right\rangle_G \quad \forall \, x, x' \in H_F \;\&\; y, y' \in K_G.$$

Definition 3.1 Let $K_1 \in \mathcal{B}(H_F)$ and $K_2 \in \mathcal{B}(K_G)$. Then the sequence of vectors $\{ f_i \otimes g_j \}_{i,j=1}^{\infty} \subseteq H_1 \otimes H_2$ is said to be a $K_1 \otimes K_2$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$ if there exist $A, B > 0$ such that

$$\begin{aligned} &A \| (K_1 \otimes K_2)^* (f \otimes g), a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 \\ &\leqslant \sum_{i,j=1}^{\infty} | \langle f \otimes g, f_i \otimes g_j | a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle |^2 \\ &\leqslant B \| f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 \quad \forall \, f \otimes g \in H_F \otimes K_G. \end{aligned} \tag{6}$$

If $A = B$, then the sequence is called a tight $K_1 \otimes K_2$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$. If $K_1 = I_F$ and $K_2 = I_G$, then by the Theorem 1.12, it is a frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$, where I_F and I_G are identity operators on H_F and K_G, respectively. If only the last inequality of (6) is true then the sequence $\{ f_i \otimes g_j \}_{i,j=1}^{\infty}$ is called a Bessel sequence associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ in $H_1 \otimes H_2$. Thus, every $K_1 \otimes K_2$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ is a Bessel sequence associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$.

Theorem 3.2 Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$ be two sequences in H_1 and H_2. Then $\{f_i\}_{i=1}^{\infty}$ is a K_1-frame associated to $(a_2, \cdots, a_n)$ for H_1 and $\{g_j\}_{j=1}^{\infty}$ is a K_2-frame associated to $(b_2, \cdots, b_n)$ for H_2 if and only if the sequence $\{f_i \otimes g_j\}_{i,j=1}^{\infty}$ is a $K_1 \otimes K_2$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$.

Proof. Suppose that the sequence $\{f_i \otimes g_j\}_{i,j=1}^{\infty}$ is a $K_1 \otimes K_2$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$. Then for each $f \otimes g \in H_F \otimes K_G - \{\theta \otimes \theta\}$, there exist constants $A, B > 0$ such that

$$\begin{aligned}
&A \,\|(K_1 \otimes K_2)^*(f \otimes g), a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2 \\
&\qquad \leqslant \sum_{i,j=1}^{\infty} |\langle f \otimes g, f_i \otimes g_j \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n\rangle|^2 \\
&\qquad \leqslant B\,\|f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2 \\
&\Rightarrow A\,\|K_1^* f \otimes K_2^* g, a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2 \\
&\qquad \leqslant \sum_{i,j=1}^{\infty} |\langle f \otimes g, f_i \otimes g_j \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n\rangle|^2 \\
&\qquad \leqslant B\,\|f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2 \\
&\Rightarrow A\,\|K_1^* f, a_2, \cdots, a_n\|_1^2\,\|K_2^* g, b_2, \cdots, b_n\|_2^2 \\
&\quad \leqslant \left(\sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n\rangle_1|^2\right)\left(\sum_{j=1}^{\infty} |\langle g, g_j \,|\, b_2, \cdots, b_n\rangle_2|^2\right) \\
&\quad \leqslant B\,\|f, a_2, \cdots, a_n\|_1^2\,\|g, b_2, \cdots, b_n\|_2^2.
\end{aligned}$$

Since $f \otimes g \in H \otimes K$ is non-zero element i.e., $f \in H$ and $g \in K$ are non-zero elements. Here, we may assume that every f_i and $a_2, \cdots, a_n$ are linearly independent and every g_j and $b_2, \cdots, b_n$ are linearly independent. Hence

$$\sum_{j=1}^{\infty} |\langle g, g_j \,|\, b_2, \cdots, b_n\rangle_2|^2,\ \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n\rangle_1|^2$$

are non-zero. Therefore, by the above inequality, we get

$$\begin{aligned}
&\frac{A\,\|K_2^* g, b_2, \cdots, b_n\|_2^2}{\sum_{j=1}^{\infty} |\langle g, g_j \,|\, b_2, \cdots, b_n\rangle_2|^2}\,\|K_1^* f, a_2, \cdots, a_n\|_1^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n\rangle_1|^2 \\
&\qquad \leqslant \frac{B\,\|K_2^* g, b_2, \cdots, b_n\|_2^2}{\sum_{j=1}^{\infty} |\langle g, g_j \,|\, b_2, \cdots, b_n\rangle_2|^2}\,\|f, a_2, \cdots, a_n\|_1^2
\end{aligned}$$

$$\Rightarrow A_1\,\|K_1^* f, a_2, \cdots, a_n\|_1^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n\rangle_1|^2 \leqslant B_1\,\|f, a_2, \cdots, a_n\|_1^2,$$

where

$$A_1 = \inf_{g \in K_G} \frac{A \, \| K_2^* g, b_2, \cdots, b_n \|_2^2}{\sum\limits_{j=1}^{\infty} \left| \langle g, g_j \,|\, b_2, \cdots, b_n \rangle_2 \right|^2} \text{ and } B_1 = \sup_{g \in K_G} \frac{B \, \| K_2^* g, b_2, \cdots, b_n \|_2^2}{\sum\limits_{j=1}^{\infty} \left| \langle g, g_j \,|\, b_2, \cdots, b_n \rangle_2 \right|^2}.$$

This shows that $\{ f_i \}_{i=1}^{\infty}$ is a K_1-frame associated to $(a_2, \cdots, a_n)$ for H_1. Similarly, it can be shown that $\{ g_j \}_{j=1}^{\infty}$ is a K_2-frame associated to $(b_2, \cdots, b_n)$ for H_2.

Conversely, suppose that $\{ f_i \}_{i=1}^{\infty}$ is a K_1-frame associated to $(a_2, \cdots, a_n)$ for H_1 with bounds A, B and $\{ g_j \}_{j=1}^{\infty}$ is a K_2-frame associated to $(b_2, \cdots, b_n)$ for H_2 with bounds C, D. Then, for all $f \in H_F$ and $g \in K_G$, we have

$$A \, \| K_1^* f, a_2, \cdots, a_n \|_1^2 \leqslant \sum_{i=1}^{\infty} | \langle f, f_i \,|\, a_2, \cdots, a_n \rangle_1 |^2 \leqslant B \, \| f, a_2, \cdots, a_n \|_1^2,$$

$$C \, \| K_2^* g, b_2, \cdots, b_n \|_2^2 \leqslant \sum_{j=1}^{\infty} \left| \langle g, g_j \,|\, b_2, \cdots, b_n \rangle_2 \right|^2 \leqslant D \, \| g, b_2, \cdots, b_n \|_2^2.$$

Multiplying the above two inequalities and using (4) and (5), we get

$$\begin{aligned} &AC \, \| (K_1 \otimes K_2)^* \, (f \otimes g), a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 \\ &\leqslant \sum_{i,j=1}^{\infty} | \langle f \otimes g, f_i \otimes g_j \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle |^2 \\ &\leqslant BD \, \| f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 \;\; \forall \, f \otimes g \in H_F \otimes K_G. \end{aligned}$$

Hence, $\{ f_i \otimes g_j \}_{i,j=1}^{\infty}$ is a $K_1 \otimes K_2$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$. This completes the proof. ■

Theorem 3.3 Let $\{ f_i \}_{i=1}^{\infty}$ and $\{ g_j \}_{j=1}^{\infty}$ be the sequences of vectors in n-Hilbert spaces H_1 and H_2. Then the sequence $\{ f_i \otimes g_j \}_{i,j=1}^{\infty} \subseteq H_1 \otimes H_2$ is a Bessel sequence associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ in $H_1 \otimes H_2$ if and only if $\{ f_i \}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in H_1 and $\{ g_j \}_{j=1}^{\infty}$ is a Bessel sequence associated to $(b_2, \cdots, b_n)$ in H_2.

Proof. Since every $K_1 \otimes K_2$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ is a Bessel sequence associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$, the proof of this theorem directly follows from the Theorem 3.2. ■

Theorem 3.4 Let $\{ f_i \}_{i=1}^{\infty}$ be a K_1-frame associated to $(a_2, \cdots, a_n)$ for H_1 with bounds A, B and $\{ g_j \}_{j=1}^{\infty}$ be a K_2-frame associated to $(b_2, \cdots, b_n)$ for H_2 with bounds C, D, respectively.

(*i*) If $T_1 \otimes T_2 \in \mathcal{B} (H_F \otimes K_G)$ is an isometry such that $(K_1 \otimes K_2)^* (T_1 \otimes T_2) = (T_1 \otimes T_2) \, (K_1 \otimes K_2)^*$, then the sequence $\Delta = \{ (T_1 \otimes T_2)^* \, (f_i \otimes g_j) \}_{i,j=1}^{\infty}$ is a $K_1 \otimes K_2$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$.

(*ii*) The sequence $\Gamma = \{ (L_1 \otimes L_2) \, (f_i \otimes g_j) \}_{i,j=1}^{\infty}$ is a $(L_1 \otimes L_2) \, (K_1 \otimes K_2)$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$, for some operator $L_1 \otimes L_2 \in \mathcal{B} (H_F \otimes K_G)$.

Proof. (i) For each $f \otimes g \in H_F \otimes K_G$, we have

$$\sum_{i,j=1}^{\infty} \left|\left\langle f \otimes g, (T_1 \otimes T_2)^* (f_i \otimes g_j) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \right\rangle\right|^2$$

$$= \sum_{i,j=1}^{\infty} \left|\left\langle f \otimes g, T_1^* f_i \otimes T_2^* g_j \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \right\rangle\right|^2$$

$$= \left(\sum_{i=1}^{\infty} \left|\left\langle f, T_1^* f_i \,|\, a_2, \cdots, a_n \right\rangle_1\right|^2\right) \left(\sum_{j=1}^{\infty} \left|\left\langle g, T_2^* g_j \,|\, b_2, \cdots, b_n \right\rangle_2\right|^2\right)$$

$$= \left(\sum_{i=1}^{\infty} \left|\left\langle T_1 f, f_i \,|\, a_2, \cdots, a_n \right\rangle_1\right|^2\right) \left(\sum_{j=1}^{\infty} \left|\left\langle T_2 g, g_j \,|\, b_2, \cdots, b_n \right\rangle_2\right|^2\right) \tag{7}$$

$$\leqslant B \, \|T_1 f, a_2, \cdots, a_n\|_1^2 \; D \, \|T_2 g, a_2, \cdots, a_n\|_2^2$$

[since $\{f_i\}_{i=1}^{\infty}$ is a K_1-frame associated to $(a_2, \cdots, a_n)$ and $\{g_j\}_{j=1}^{\infty}$ is a K_2-frame associated to $(b_2, \cdots, b_n)$]

$$\leqslant B D \, \|T_1\|^2 \, \|T_2\|^2 \, \|f, a_2, \cdots, a_n\|_1^2 \, \|g, b_2, \cdots, b_n\|_2^2$$

$$= B D \, \|T_1 \otimes T_2\|^2 \, \|f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2.$$

On the other hand, since $\{f_i\}_{i=1}^{\infty}$ is a K_1-frame associated to $(a_2, \cdots, a_n)$ for H_1 and $\{g_j\}_{j=1}^{\infty}$ is a K_2-frame associated to $(b_2, \cdots, b_n)$ for H_2, from (7), we have

$$\sum_{i,j=1}^{\infty} \left|\left\langle f \otimes g, (T_1 \otimes T_2)^* (f_i \otimes g_j) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \right\rangle\right|^2$$

$$\geqslant A \, \|K_1^* T_1 f, a_2, \cdots, a_n\|_1^2 \; C \, \|K_2^* T_2 g, b_2, \cdots, b_n\|_2^2$$

$$= A C \, \langle K_1^* T_1 f, K_1^* T_1 f \,|\, a_2, \cdots, a_n \rangle_1 \, \langle K_2^* T_2 g, K_2^* T_2 g \,|\, b_2, \cdots, b_n \rangle_2$$

$$= A C \, \langle K_1^* T_1 f \otimes K_2^* T_2 g, \; K_1^* T_1 f \otimes K_2^* T_2 g \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle$$

$$= A C \, \langle (K_1 \otimes K_2)^* (T_1 \otimes T_2)(f \otimes g), (K_1 \otimes K_2)^* (T_1 \otimes T_2)(f \otimes g) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle$$

$$= A C \, \langle (T_1 \otimes T_2)(K_1 \otimes K_2)^* (f \otimes g), (T_1 \otimes T_2)(K_1 \otimes K_2)^* (f \otimes g) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle$$

[since $(K_1 \otimes K_2)^* (T_1 \otimes T_2) = (T_1 \otimes T_2)(K_1 \otimes K_2)^*$]

$$= A C \, \langle (K_1 \otimes K_2)^* (f \otimes g), (K_1 \otimes K_2)^* (f \otimes g) \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle$$

[since $(T_1 \otimes T_2)$ is an isometry]

$$= A C \, \langle K_1^* f \otimes K_2^* g, \; K_1^* f \otimes K_2^* g \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle$$

$$= A C \, \langle K_1^* f, K_1^* f \,|\, a_2, \cdots, a_n \rangle_1 \, \langle K_2^* g, K_2^* g \,|\, b_2, \cdots, b_n \rangle_2$$

$$= A C \, \|K_1^* f, a_2, \cdots, a_n\|_1^2 \, \|K_2^* g, b_2, \cdots, b_n\|_2^2$$

$$= A C \, \|K_1^* f \otimes K_2^* g, a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2$$

$$= A C \, \|(K_1 \otimes K_2)^* (f \otimes g), a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2.$$

Hence, Δ is a $K_1 \otimes K_2$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$.

(ii) According to the proof of (i), it is easy to verify that for each $f \otimes g \in H_F \otimes K_G$, we have

$$\sum_{i,j=1}^{\infty} |\langle f \otimes g, (L_1 \otimes L_2)(f_i \otimes g_j) | a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle|^2$$
$$\leqslant B D \, \| L_1 \otimes L_2 \|^2 \, \| f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2$$

On the other hand,

$$\sum_{i,j=1}^{\infty} |\langle f \otimes g, (L_1 \otimes L_2)(f_i \otimes g_j) | a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle|^2$$
$$\geqslant A \, \| K_1^* L_1^* f, a_2, \cdots, a_n \|_1^2 \, C \, \| K_2^* L_2^* g, b_2, \cdots, b_n \|_2^2$$
$$= A C \, \| K_1^* L_1^* f \otimes K_2^* L_2^* g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2$$
$$= A C \, \| (K_1^* L_1^* \otimes K_2^* L_2^*)(f \otimes g), a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2$$
$$= A C \, \| [(L_1 \otimes L_2)(K_1 \otimes K_2)]^* (f \otimes g), a_2 \otimes b_2, \cdots, a_n \otimes b_n \|^2 .$$

Hence, Γ is a $(L_1 \otimes L_2)(K_1 \otimes K_2)$-frame associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $H_1 \otimes H_2$. ■

Definition 3.5 Let K_1 and K_2 be bounded linear operators on the Hilbert spaces H_F and K_G. Then the sequence of vectors $\{f_i \otimes g_j\}_{i,j=1}^{\infty} \subseteq H_1 \otimes H_2$ is said to be an atomic system associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $K_1 \otimes K_2 \in \mathcal{B}(H_F \otimes K_G)$ in $H_1 \otimes H_2$ if

(i) $\{f_i \otimes g_j\}_{i,j=1}^{\infty}$ is a Bessel sequence associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ in $H_1 \otimes H_2$.

(ii) For any $f \otimes g \in H_F \otimes K_G$, there exists $c \otimes d = \{c_i d_j\}_{i,j=1}^{\infty} \in l^2(\mathbb{N} \times \mathbb{N})$ such that

$$(K_1 \otimes K_2)(f \otimes g) = \sum_{i,j=1}^{\infty} c_i d_j (f_i \otimes g_j),$$

and for some $C > 0$,

$$\| c \otimes d \|_{l^2} \leqslant C \, \| f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n \|,$$

where $c = \{c_i\}_{i=1}^{\infty}$ and $d = \{d_j\}_{j=1}^{\infty}$ are in $l^2(\mathbb{N})$.

Theorem 3.6 Let $\{f_i\}_{i=1}^{\infty}$ be an atomic system associated to $(a_2, \cdots, a_n)$ for K_1 in H_1 and $\{g_j\}_{j=1}^{\infty}$ be an atomic system associated to $(b_2, \cdots, b_n)$ for K_2 in H_2. Then the sequence $\{f_i \otimes g_j\}_{i,j=1}^{\infty}$ is an atomic system associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $K_1 \otimes K_2$ in $H_1 \otimes H_2$.

Proof. Since $\{f_i\}_{i=1}^{\infty}$ is an atomic system associated to $(a_2, \cdots, a_n)$ for K_1 in H_1 and $\{g_j\}_{j=1}^{\infty}$ is an atomic system associated to $(b_2, \cdots, b_n)$ for K_2 in H_2, by the definition 2.5, $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in H_1 and $\{g_j\}_{j=1}^{\infty}$ is a

Bessel sequence associated to $(b_2, \cdots, b_n)$ in H_2, respectively. Then by the Theorem 3.3, $\{f_i \otimes g_j\}_{i,j=1}^{\infty}$ is a Bessel sequence associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ in $H_1 \otimes H_2$. Also, for any $f \in H_F$ and $g \in K_G$,

$$K_1 f = \sum_{i=1}^{\infty} c_i f_i \text{ with } \|\{c_i\}_{i=1}^{\infty}\|_{l^2} \leqslant C_1 \|f, a_2, \cdots, a_n\|_1 \text{ for some } C_1 > 0,$$

$$K_2 g = \sum_{j=1}^{\infty} d_j g_j \text{ with } \left\|\{d_j\}_{j=1}^{\infty}\right\|_{l^2} \leqslant C_2 \|g, b_2, \cdots, b_n\|_2 \text{ for some } C_2 > 0.$$

Therefore, for each $f \otimes g \in H_F \otimes K_G$, we have

$$\begin{aligned}(K_1 \otimes K_2)(f \otimes g) &= K_1 f \otimes K_2 g \\ &= \left(\sum_{i=1}^{\infty} c_i f_i\right) \otimes \left(\sum_{j=1}^{\infty} d_j g_j\right) = \sum_{i,j=1}^{\infty} c_i d_j (f_i \otimes g_j)\end{aligned}$$

On the other hand,

$$\begin{aligned}&\|\{c_i\}_{i=1}^{\infty}\|_{l^2} \left\|\{d_j\}_{j=1}^{\infty}\right\|_{l^2} \leqslant C_1 \|f, a_2, \cdots, a_n\|_1 C_2 \|g, b_2, \cdots, b_n\|_2 \\ &\Rightarrow \|c \otimes d\|_{l^2} \leqslant C \|f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n\|,\end{aligned}$$

where $C = C_1 C_2 > 0$. This completes the proof. ∎

Theorem 3.7 If the sequence $\{f_i \otimes g_j\}_{i,j=1}^{\infty}$ is an atomic system associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ for $K_1 \otimes K_2$ in $H_1 \otimes H_2$. Then $\{A f_i\}_{i=1}^{\infty}$ is an atomic system associated to $(a_2, \cdots, a_n)$ for K_1 in H_1 and $\{B g_j\}_{j=1}^{\infty}$ is an atomic system associated to $(b_2, \cdots, b_n)$ for K_2 in H_2, respectively, where A and B are constants with $AB = 1$.

Proof. By definition 3.5, the sequence $\{f_i \otimes g_j\}_{i,j=1}^{\infty}$ is a Bessel sequence associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ in $H_1 \otimes H_2$, and therefore by Theorem 3.3, $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in H_1 and $\{g_j\}_{i=1}^{\infty}$ is a Bessel sequence associated to $(b_2, \cdots, b_n)$ in H_2, respectively. Also, for any $f \otimes g \in H_F \otimes K_G$, there exists $c \otimes d = \{c_i d_j\}_{i,j=1}^{\infty}$ in $l^2(\mathbb{N} \times \mathbb{N})$ such that

$$(K_1 \otimes K_2)(f \otimes g) = \sum_{i,j=1}^{\infty} c_i d_j (f_i \otimes g_j) = \left(\sum_{i=1}^{\infty} c_i f_i\right) \otimes \left(\sum_{j=1}^{\infty} d_j g_j\right).$$

By (vi) of Theorem 1.4, there exist constants A, B with $AB = 1$ such that

$$K_1 f = \sum_{i=1}^{\infty} c_i (A f_i) \text{ and } K_2 g = \sum_{j=1}^{\infty} d_j (B g_j).$$

On the other hand, for some $C > 0$, using (5), we have

$$\begin{aligned}
&\|c \otimes d\|_{l^2} \leqslant C \, \|f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n\| \\
&\Rightarrow \|\{c_i\}_{i=1}^{\infty}\|_{l^2} \left\|\{d_j\}_{j=1}^{\infty}\right\|_{l^2} \leqslant C \, \|f, a_2, \cdots, a_n\|_1 \, \|g, b_2, \cdots, b_n\|_2 \\
&\Rightarrow \|\{c_i\}_{i=1}^{\infty}\|_{l^2} \leqslant \frac{C \, \|g, b_2, \cdots, b_n\|_2}{\left\|\{d_j\}_{j=1}^{\infty}\right\|_{l^2}} \, \|f, a_2, \cdots, a_n\|_1 = C_1 \, \|f, a_2, \cdots, a_n\|_1,
\end{aligned}$$

where $C_1 = \dfrac{C \, \|g, b_2, \cdots, b_n\|_2}{\left\|\{d_j\}_{j=1}^{\infty}\right\|_{l^2}} > 0$. Similarly, it can be shown that

$$\left\|\{d_j\}_{j=1}^{\infty}\right\|_{l^2} \leqslant C_2 \, \|g, b_2, \cdots, b_n\|_2,$$

where $C_2 = \dfrac{C \, \|f, a_2, \cdots, a_n\|_1}{\|\{c_i\}_{i=1}^{\infty}\|_{l^2}}$. This completes the proof. ■

Theorem 3.8 Let $\{f_i\}_{i=1}^{\infty}$ be an atomic system associated to $(a_2, \cdots, a_n)$ for K_1 in H_1 and $\{g_j\}_{j=1}^{\infty}$ be an atomic system associated to $(b_2, \cdots, b_n)$ for K_2 in H_2, respectively. Then $\{f_i \otimes g_j\}_{i,j=1}^{\infty}$ is a $K_1 \otimes K_2$-frame associated to $(a_2, \cdots, a_n)$.

Proof. By Theorem 3.3, the sequence $\{f_i \otimes g_j\}_{i,j=1}^{\infty}$ is a Bessel sequence associated to $(a_2 \otimes b_2, \cdots, a_n \otimes b_n)$ in $H_1 \otimes H_2$. Then, for all $f \otimes g \in H_F \otimes K_G$, there exists $B > 0$ such that

$$\begin{aligned}
&\sum_{i,j=1}^{\infty} |\langle f \otimes g, f_i \otimes g_j \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle|^2 \\
&\leqslant B \, \|f \otimes g, a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2.
\end{aligned}$$

Also, for any $f_1 \in H_F$ and $g_1 \in K_G$, we have

$$K_1 f_1 = \sum_{i=1}^{\infty} c_i f_i \text{ with } \|\{c_i\}_{i=1}^{\infty}\|_{l^2} \leqslant C_1 \, \|f_1, a_2, \cdots, a_n\|_1,$$

for some $C_1 > 0$, and

$$K_2 g_1 = \sum_{j=1}^{\infty} d_j g_j \text{ with } \left\|\{d_j\}_{j=1}^{\infty}\right\|_{l^2} \leqslant C_2 \, \|g_1, b_2, \cdots, b_n\|_2$$

for some $C_2 > 0$. Now, for each $f \otimes g \in H_F \otimes K_G$, we have

$$\|(K_1 \otimes K_2)^*(f \otimes g), a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2 = \|K_1^* f \otimes K_2^* g, a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2$$

$$= \|K_1^* f, a_2, \cdots, a_n\|_1^2 \, \|K_2^* g, b_2, \cdots, b_n\|_2^2 \quad [\text{ by } (5)\,]$$

$$= \sup_{\|f_1, a_2, \cdots, a_n\|_1 = 1} |\langle K_1^* f, f_1 \,|\, a_2, \cdots, a_n \rangle_1|^2 \sup_{\|g_1, b_2, \cdots, b_n\|_2 = 1} |\langle K_2^* g, g_1 \,|\, b_2, \cdots, b_n \rangle_2|^2$$

$$= \sup_{\|f_1, a_2, \cdots, a_n\|_1 = 1} |\langle f, K_1 f_1 \,|\, a_2, \cdots, a_n \rangle_1|^2 \sup_{\|g_1, b_2, \cdots, b_n\|_2 = 1} |\langle g, K_2 g_1 \,|\, b_2, \cdots, b_n \rangle_2|^2$$

$$= \sup_{\|f_1, a_2, \cdots, a_n\|_1 = 1} \left| \left\langle f, \sum_{i=1}^{\infty} c_i f_i \,|\, a_2, \cdots, a_n \right\rangle_1 \right|^2 \sup_{\|g_1, b_2, \cdots, b_n\|_2 = 1} \left| \left\langle g, \sum_{j=1}^{\infty} d_j g_j \,|\, b_2, \cdots, b_n \right\rangle_2 \right|^2$$

$$= \sup_{\|f_1, a_2, \cdots, a_n\|_1 = 1} \left| \sum_{i=1}^{\infty} \overline{c_i} \langle f, f_i \,|\, a_2, \cdots, a_n \rangle_1 \right|^2 \sup_{\|g_1, b_2, \cdots, b_n\|_2 = 1} \left| \sum_{j=1}^{\infty} \overline{d_j} \langle g, g_j \,|\, b_2, \cdots, b_n \rangle_2 \right|^2$$

$$\leqslant \sup_{\|f_1, a_2, \cdots, a_n\|_1 = 1} \left\{ \sum_{i=1}^{\infty} |c_i|^2 \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle_1|^2 \right\} \times$$

$$\sup_{\|g_1, b_2, \cdots, b_n\|_2 = 1} \left\{ \sum_{j=1}^{\infty} |d_j|^2 \sum_{j=1}^{\infty} |\langle g, g_j \,|\, b_2, \cdots, b_n \rangle_2|^2 \right\}$$

$$\leqslant \sup_{\|f_1, a_2, \cdots, a_n\|_1 = 1} \left\{ C_1^2 \|f_1, a_2, \cdots, a_n\|_1^2 \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle_1|^2 \right\} \times$$

$$\sup_{\|g_1, b_2, \cdots, b_n\|_2 = 1} \left\{ C_2^2 \|g_1, b_2, \cdots, b_n\|_1^2 \sum_{j=1}^{\infty} |\langle g, g_j \,|\, b_2, \cdots, b_n \rangle_2|^2 \right\}$$

$$= C_1^2 C_2^2 \sum_{i,j=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle_1|^2 \, |\langle g, g_j \,|\, b_2, \cdots, b_n \rangle_2|^2$$

$$= C_1^2 C_2^2 \sum_{i,j=1}^{\infty} |\langle f \otimes g, f_i \otimes g_j \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle|^2 \quad [\text{ by } (4)\,].$$

$$\Rightarrow \frac{1}{C_1^2 C_2^2} \|(K_1 \otimes K_2)^*(f \otimes g), a_2 \otimes b_2, \cdots, a_n \otimes b_n\|^2$$

$$\leqslant \sum_{i,j=1}^{\infty} |\langle f \otimes g, f_i \otimes g_j \,|\, a_2 \otimes b_2, \cdots, a_n \otimes b_n \rangle|^2.$$

This completes the proof. ∎

4. Conclusion

In this paper, in the setting of n-Hilbert space, we give the idea of atomic system and establish some characterizations of them. Yet it remains to establish another few important concepts of frame theory like, perturbation, stability etc. in the setting of n-Hilbert space.

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