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# The moving frame method and invariant subspace under parametric group actions

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**Abstract.** For a given subspace as a solution space of a linear ODE, we define a special linear parametric group action and prolong it to the jet bundle. We determine these group parameters by moving frame method and prove that these group parameters are the first integrals of the given ODE. These first integrals are used to construct the general form of operators which preserve given subspace invariant.



**Keywords:** First integral, moving frame, group action, differential equation, invariant subspace, invariant operator.

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## 1. Introduction and preliminaries

The invariant subspace method recently proposed by Galaktionov is one of the powerful approaches for constructing exact solutions to nonlinear evolution equations. Indeed, the invariant subspace method generates many interesting exact solutions to nonlinear evolution equations in mechanics and physics and systematical solution procedure was given by Galaktionov and Svirshchevskii [2].

The problem of finding all operators that keep a given linear subspace invariant was discussed in [4, 6]. The most important part of constructing these operators, where preserve given subspace as a solution space of an linear ODE, is determined by the first integrals of the given ODE. In 2007, Galaktionov and Svirshchevskii [3] considered this problem via linear algebra methods.

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In this work, we consider the same problem of invariant subspaces by a new geometric approach via group action and moving frame method. Accordingly, this paper is organized as follows:
After a brief review of basic and common notations, definitions of invariant subspace problem and the Wronskian, we discuss about geometric tools such as jet bundle and group action. Then we define a special linear group action and prolongation of it and by moving frame method, we find the group parameters. We prove in Theorem 2.1 that these group parameters are the first integrals of the given ODE that used to construct the general form of operators which preserve given subspace invariant. Finally, we solve two examples by our method.

Let $V_r = \mathcal{L}\{f_1, \cdots, f_r\}$ be the subspace spanned by $r$ linearly independent real valued functions $f_i$'s on $\mathbb{R}$. There exist two main problems in the context of invariant subspaces. The first problem is the finding all invariant subspaces $V_r$ for a given operator $F$, i.e. $F \mapsto \{V_r\}$. The second one, which we consider it here, is the finding all operators like $F$ such that preserve the given linear subspace $V_r$ invariant, i.e. $V_r \mapsto \{F\}$.

**Definition 1.1** [3] Let $V_r = \mathcal{L}\{f_1, \cdots, f_r\} = \{\sum_{i=1}^{r} c_i f_i(x) | c_i \in \mathbb{R}\}$ be a finite dimensional linear subspace. We say $V_r$ is invariant with respect to the operator $F$ if $F[V_r] \subseteq V_r$.

This means that for arbitrary constant $c_1, \cdots, c_r$, there exist functions $\bar{F}_1, \cdots, \bar{F}_r$ such that $F(\sum_{i=1}^{r} c_i f_i(x)) = \sum_{i=1}^{r} \bar{F}_i(c_1, \cdots, c_r) f_i(x)$.

**Definition 1.2** [4] The Wronskian matrix of order $n$ of the functions $f_1, \cdots, f_r$ is the matrix whose entries consist of partial derivatives of the $f_k$'s up to order $n$ with respect to the $x_i$'s. In the scalar case $x \in \mathbb{R}$, the standard Wronskian matrix $W_{r-1} = W_{r-1}[f_1, \cdots, f_r]$ have a square matrix form as follow:

$$\begin{pmatrix} f_1(x) & \cdots & f_1^{(r-1)}(x) \\ \vdots & \ddots & \vdots \\ f_r(x) & \cdots & f_r^{(r-1)}(x) \end{pmatrix}.$$

**Definition 1.3** [4] We define the Wronskian matrix rank function

$$\rho_n(x) = rank\ W_n[f_1, \cdots, f_r](x).$$

A subspace $V_r = \mathcal{L}\{f_1, \cdots, f_r\}$ is called regular if it has uniformly bounded Wronskian order at each point, that is, there exists a finite $n$ such that the minimal such $n \leqslant r$ called the order of $V_r$.

**Definition 1.4** A group action is called semi-regular if all of its orbits have the same dimension. A group action is called regular if in addition, each point $z \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset thereof.

**Definition 1.5** A transformation group acts freely on $M$ if the isotropy subgroups are all trivial, $stab_G(z) = \{e\}$ for all $z \in M$.

**Definition 1.6** Given a transformation group $G$ acting on a manifold $M$, a moving frame is a smooth $G$-equivalent map $\rho : M \to G$.

Fels and Olver [1] formulated a new, powerful and constructive approach to the equivalent moving frame theory that can be systematically applied to general transformation groups. All classical moving frames can be reinterpreted in this manner, but the equivalent approach applies in far broader generality. Cartan's construction of the moving frame through the normalization process is interpreted with the choice of a cross-section to the group orbits. The existence of a moving frame requires freeness of the underlying group action. Classically, non-free actions are made free by prolonging to jet spaces.

Suppose the group $G$ acts semi-regularly on $m$-dimensional manifold $M$ with $r$-dimensional orbits. A cross section is a $(m-r)$-dimensional submanifold $\mathcal{K} \subset M$ such that $\mathcal{K}$ intersects each orbit transversally. Introducing local coordinates $z=(x,u)$ on $M$, the cross section $\mathcal{K}$ will be defined by $r$ equations

$$k_1(z) = c_1 \quad , \quad \cdots \quad , \quad k_r(z) = c_r,$$

where $k_1(z), \cdots, k_r(z)$ are scalar valued functions, while $c_1, \cdots c_r$ are suitably chosen constants. The cross section is regular if $\mathcal{K}$ intersects each orbit at most once. If $G$ acts semi-regularly, then *Implicit Function Theorem* guarantees the existence of local cross sections at any point of $M$. Regular actions admit regular local cross sections. There exists a moving frame in a neighborhood of a point $z \in M$ if and only if $G$ acts freely and regularly near $z$.

Since our interest is the studying of the prolonged group actions, we need to know not only how the group transformations act on the independent and dependent variables, but also how they act on the derivatives of the dependent variables. For the total space $E = X \times U \simeq R^p \times R^q$, the $n$-th jet space $J^n = J^n E = X \times U^{(n)}$ is the Euclidean space of dimension $p + q^{(n)} = p + q\binom{p+n}{n}$ whose coordinates consist of the $p$ independent variables $x^i$, the $q$ dependent variables $u^\alpha$, and the derivative coordinates $u^\alpha_J$ for $\alpha = 1, ..., q$ of orders $1 \leqslant \sharp J \leqslant n$. The points in the vertical space (fiber) $U^{(n)}$ are denoted by $u^{(n)}$, and consist of all the dependent variables and their derivatives up to order $n$. Thus, the coordinates of a typical point $z^{(n)} \in J^n$ are denoted by $(x, u^{(n)})$.

Since the derivative coordinates $u^{(n)}$ form a subset of the derivative coordinates $u^{(n+k)}$, there is a natural projection $\pi^{n+k}_n : J^{n+k} \to J^n$ on the jet spaces with $\pi^{n+k}_n(x, u^{n+k}) = (x, u^{(n)})$. In particular, $\pi^n_0(x, u^{(n)}) = (x, u)$ is the projection from $J^n$ to $E = J^0$. If $M \subset E$ is an open subset, then $J^n M = (\pi^n_0)^{-1} M \subset J^n E$ is the open subset of the $n$-th jet space which projects back down to $M$. At a point $x \in X$, two functions have the same $n$-th order prolongation and so determine the same point of $J^n$ if and only if they have $n$-th order contact, meaning that they and their first $n$ derivatives agree at the point, which is the same as requiring that they have the same $n$-th order Taylor polynomial at the point $x$.

Thus, a more intrinsic way of defining the jet space $J^n$ is to consider it as the set of equivalence classes of smooth functions using the equivalence relation of $n$-th order contact. A smooth function $u = f(x)$ from $X$ to $U$ has $n$-th prolongation $u^{(n)} = f^{(n)}(x)$ (also known as the $n$-jet) and denoted by $j^n f$, which is the function from $X$ to $U^{(n)}$ defined by evaluating all the partial derivatives of $f$ up to order $n$. Thus, the individual coordinate functions of $f^{(n)}$ are $u^\alpha_J = \partial_J f^\alpha(x)$. In particular, $f^{(0)} = f$.

Any transformation group $G$ acting on $M$ preserves the order of contact between submanifolds. Therefore, there is an induced action of $G$ on the $n$-th order jet bundle $J^n = J^n(M, p)$ known as the $n$-th prolongation of $G$, and denoted by $G^{(n)}$. The formulas for the prolonged group action are found by implicit differentiation, and the cross section $\mathcal{K}^n$ is prescribed by setting a collection of $r = \dim G$ independent $n$-th order differential functions to suitably chosen constants

$$k_1(z^{(n)}) = c_1 \quad , \quad \cdots \quad , \quad k_r(z^{(n)}) = c_r.$$

**Definition 1.7** Any $n$-th order moving frame is a map $\rho^{(n)} : J^n \to G$, which is $G$-

equivariant with respect to the prolonged action $G^n$ on $J^n$.

**Lemma 1.8** [1] The $n$-th order moving frame exists in a neighborhood of a point $z^{(n)} \in J^{(n)}$ if and only if $z^{(n)} \in J^{(n)}$ is a regular jet.

For details about higher order moving frame method in general there is a good reference by Fels and Olver [1].

## 2. Main results

Let $V_r$ be an $r$-dimensional subspace and the linear independent set $\{f_1, \cdots, f_r\}$ form a basis for $V_r$. Then the $(r-1)$-order Wronskian matrix is

$$W_{r-1} = \begin{pmatrix} f_1(x) & \cdots & f_1^{(r-1)}(x) \\ \vdots & \ddots & \vdots \\ f_r(x) & \cdots & f_r^{(r-1)}(x) \end{pmatrix}$$

and hence $V_r$ is regular of order $r-1$.
Consider the following well defined $r$-parametric abelian linear group action

$$(x, u) \mapsto (x, u + \sum_{i=1}^{r} t_i f_i(x)).$$

There is a natural correspondence between the group elements and the coefficients of the linear combinations of elements in linear space $V_r$. If we assume the $f_i$'s are solutions of an homogeneous linear ODE, then any linear combination of them is also a solution. So this group maps the solutions to solutions and will be a symmetry group.
Also, the $n$-prolonged group action denoted by

$$(x, u^{(n)}) \mapsto (x, u + \sum_{i=1}^{r} t_i f_i(x), \cdots, u^n + \sum_{i=1}^{r} t_i {f_i}^{(n)}(x)).$$

We consider the problem of one variable real valued functions linear space, so let $E = X \times U$, the jet space of order $n$ of $E$ (which is denoted by $J^{(n)}E = X \times U^n$) is the Euclidean space of dimension $(n+2)$, whose coordinates consists of one independent variable $x$ and the dependent variable $u$ and the derivative coordinates $u^\alpha$, $\alpha = 1, \cdots, n$. Thus, the coordinates of a typical point $z^{(n)} \in J^{(n)}$ are denoted by $(x, u^{(n)})$. The dimension of the orbits in our prolonged group action contained in the $n$-th order jet fiber $J^{(n)}E|_z$ equals the Wronskian rank $\rho_n(z)$ and since the rank of Wronskian at any point is constant, the group action is semi-regular. Also, since the Wronskian of $\{f_i\}$ is of order $r-1$, then $G^{(r)}$ acts freely and regularly and this properties guarantee the existence of moving frame. For above prolonged group action, we choose a regular cross section $\mathcal{K} = \{u = \cdots = u^{(n)} = 0\}$ on jet space. Then, by using the moving frame method, we achieve the following system of equations:

$$\begin{cases} u + \sum_{i=1}^{r} t_i f_i(x) = 0, \\ \vdots \\ u^{(r-1)} + \sum_{i=1}^{r} t_i f_i^{(r-1)}(x) = 0. \end{cases}$$

Then, by Cramer's formula, we find the group elements $t_i$'s as follow:

$$t_i = \frac{|W_{r-1}[f_1, \cdots, f_{i-1}, u, f_{i+1}, \cdots, f_r]|}{|W_{r-1}[f_1, \cdots, f_r]|}.$$

In the next theorem, we show that these group elements are the first integrals of our ODE.

**Theorem 2.1** Suppose that $V_r = \mathcal{L}\{f_1, \cdots, f_r\}$ is the solution space of the linear ODE

$$L[u] \equiv a_0(x)u^{(r)} + a_1(x)u^{(r-1)} + \cdots + a_{r-1}(x)u^{'} + a_r(x)u = 0. \tag{1}$$

Also, consider the prolonged $r$-parametric group action by

$$(x, u^{(n)}) \mapsto (x, u + \sum_{i=1}^{r} t_i f_i(x), \cdots, u^n + \sum_{i=1}^{r} t_i {f_i}^{(n)}(x)).$$

Then the group parameters $t_i$'s are the first integrals of $L[u]$.

**Proof.** We consider the condition $Dt_i|_{L[u]=0} \equiv 0$ for all $t_i$'s. By total derivative formula

$$Dt_i = \frac{\partial t_i}{\partial x} + \frac{\partial t_i}{\partial u}u_x + \frac{\partial t_i}{\partial u_x}u_{xx} + \cdots + \frac{\partial t_i}{\partial u^{(r-1)}}u^{(r)}$$

and

$$t_i = \frac{\begin{vmatrix} f_1(x) & \cdots & f_{i-1}(x) & u & f_{i+1}(x) & \cdots & f_r(x) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1^{(r-1)}(x) & \cdots & f_{i-1}^{(r-1)}(x) & u^{(r-1)} & f_{i+1}^{(r-1)}(x) & \cdots & f_r^{(r-1)}(x) \end{vmatrix}}{\begin{vmatrix} f_1(x) & \cdots & f_r(x) \\ \vdots & \ddots & \vdots \\ f_1^{(r-1)}(x) & \cdots & f_r^{(r-1)}(x) \end{vmatrix}} = \frac{\Delta_i}{\Delta}$$

and the fact that the determinant is a multilinear function of its rows, if we write the rows of $\Delta$ as $\delta_1, ..., \delta_r$ and the rows of $\Delta_i$ as $\delta_1^i, ..., \delta_r^i$, we have

$$\Delta = f(\delta_1, \delta_2, ..., \delta_r),$$
$$\Delta_i = f(\delta_1^i, \delta_2^i, ..., \delta_r^i).$$

The derivative of $\Delta$ is

$$D\Delta = f(D\delta_1, \delta_2, ..., \delta_r) + f(\delta_1, D\delta_2, ..., \delta_r) + ... + f(\delta_1, \delta_2, ..., D\delta_r),$$

but all of elements of the above sum except the latter term are zero. Similarly, for any $\Delta_i$, the only non-zero part of $D\Delta_i$ is the derivative of last row. Thus,

$$D\Delta = f(\delta_1, \delta_2, ..., D\delta_r),$$
$$D\Delta_i = f(\delta_1^i, \delta_2^i, ..., D\delta_r^i)$$

A direct calculation for any $f_j(x) \in V_r$ shows that $\{\Delta D\Delta_i - \Delta_i D\Delta\}|_{u=f_j(x)} = 0$. Therefore, $Dt_i|_{u=f_j(x)} = 0$. ■

We define our group action linearly so the first integrals are also linear, but the operators that are made in the following by them (first integrals) are not essentially linear. As interesting application of these operators, these nonlinear operators used to generalize separation of variables method for evolution equations (see [2]).

Now, we state the main theorem of invariant subspaces that determine the general structure of the invariant operators.

**Theorem 2.2** [3] Every operator $F[u] = F(x, u, u', \cdots, u^{(p)})$ possessing the invariant space $V_r = \mathcal{L}\{f_1(x), \cdots, f_r(x)\}$ is given by

$$F[u] = \sum_{i=1}^{r} A^i(I_1, \cdots, I_r) f_i(x),$$

where $A^i(I_1, \cdots, I_r)$ are arbitrary of the first integrals of the corresponding equation

$$L[u] \equiv a_0(x)u^{(r)} + a_1(x)u^{(r-1)} + \cdots + a_{r-1}(x)u' + a_r(x)u = 0.$$

Indeed, in linear space $V_r = \mathcal{L}\{f_1, \cdots, f_r\}$, if we assume that $f_i$'s are solutions of the homogeneous linear ODE (1), then $F[V_r] \subseteq V_r$ means that $L[F[u]]|_{L[u]=0} = 0$ and it is actually the invariance criterion for this linear differential equation with respect to the Lie-Bäcklund operator $X = F[u]\frac{\partial}{\partial u}$. Therefore, all results can be interpreted in terms of symmetry of linear ODEs (see more details in [3, 5]). We use our method to resolve the problem of finding the general invariant operators for subspaces generated by some elementary functions.

***Example* 2.3** Let subspace $V_3 = \mathcal{L}\{1, x, x^2\}$ be the solution space to the linear ordinary differential equation $u_{xxx} = 0$. We find all operators which preserve this subspace invariant. The Wronskian matrix of order 2 of base elements of $V_3$ is

$$W_2 = \begin{pmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{pmatrix}.$$

Since the determinant of $W_2$ is non-zero, so $V_3$ will be a regular subspace. Then we define the group action $(x, u) \mapsto (x, u + t_1 + t_2x + t_3x^2)$ and by prolongation of it on $J^{(2)}$, we have

$$(x, u, u_x, u_{xx}) \mapsto (x, u + t_1 + t_2x + t_3x^2, u_x + t_2 + 2t_3x, u_{xx} + 2t_3).$$

We choose the regular cross section $\mathcal{K} = \{u = u_x = u_{xx} = 0\}$. Then, with moving frame, we have the following system of equations:

$$\begin{cases} u + t_1 + t_2x + t_3x^2 = 0, \\ u_x + t_2 + 2t_3x = 0, \\ u_{xx} + 2t_3 = 0, \end{cases}$$

and the group elements obtained as follow:

$$t_1 = \frac{-1}{2} x^2 u_{xx} + x u_x - u, \quad t_2 = x u_{xx} - u_x, \quad t_3 = \frac{-1}{2} u_{xx}.$$

Also, these elements are first integrals of differential equation $u_{xxx} = 0$. Thus, according to the main theorem, we can use them to make up invariant operators

$$F[u] = A^1(t_1, t_2, t_3) + A^2(t_1, t_2, t_3)x + A^3(t_1, t_2, t_3)x^2,$$

where $A^i$ are arbitrary smooth functions.

***Example* 2.4** As a solution space of differential equation $u_{xxx} = -u_x$, let $V_3 = \mathcal{L}\{1, \sin x, \cos x\} = \{t_1 + t_2 \sin x + t_3 \cos x | t_1, t_2, t_3 \in \mathbb{R}\}$ be a three dimensional linear subspace. We find all operators which preserve this subspace invariant. The Wronskian matrix of the base functions of $V_3$ is

$$W_2 = \begin{pmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{pmatrix}.$$

Since $|W_2| = -1$ is non-zero at every point, then $V_3$ is a regular subspace. By cross section $\mathcal{K} = \{u = u_x = u_{xx} = 0\}$, the system of equations takes the form

$$\begin{cases} u + t_1 + t_2 \sin x + t_3 \cos x = 0, \\ u_x + t_2 \cos x - t_3 \sin x = 0, \\ u_{xx} - t_2 \sin x - t_3 \cos x = 0. \end{cases}$$

Solving this system of equations for group elements $t_1$, $t_2$ and $t_3$, we have

$$t_1 = -(u + u_{xx}), \quad t_2 = u_{xx} \sin x - u_x \cos x, \quad t_3 = u_{xx} \cos x + u_x \sin x.$$

Also, these elements are first integrals of differential equation $u_{xxx} = -u_x$. According to the main theorem, we state the general form of all operators by arbitrary smooth functions $A^i$ in the general form

$$F[u] = A^1(t_1, t_2, t_3) + A^2(t_1, t_2, t_3) \cos x + A^3(t_1, t_2, t_3) \sin x.$$

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