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Symmetry group analysis and similarity reduction of the thin film equation

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Abstract. In this article, by using the Lie symmetry method, we find the Lie symmetry group of the thin film equation. Also, the one-dimensional optimal system of Lie subalgebras is obtained. Then, we calculate the similarity reductions of the thin film equation and classify them by using the optimal system.

Keywords: Thin film equation, Lie symmetry group, similarity reduction, optimal system.

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1. Introduction and preliminaries

In recent decades, research on nonlinear equations has progressed significantly. These equations are more prevalent in physics and engineering and are often difficult to solve. Lie symmetry method is one of the useful and effective methods in obtaining exact solutions for such equations. This method was introduced by Lie [11]. The Lie symmetry group of the partial differential equation (PDE) is the largest local Lie group of transformations that acts on the PDE variables and keeps the solutions set invariant. Also, the Lie symmetry method provides a powerful tool for reducing the numbers of equation variables [5, 6]. There are many notable applications of Lie's symmetry groups in the study of differential equations, such as reduction of the order of ordinary differential equations (ODEs), find groups invariant solutions and classify them, constructing the conservation laws and so on [2–4, 13]. The main idea of the Lie symmetry method is the

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application of an invariant condition to the PDE for deriving similarity variables. With these similarity variables, we can construct the reduction equations and by solving these equations, group invariant solutions will be deduced [1, 7, 9, 10, 12]. In this paper, we obtain the group of symmetries for more general class of quadratic operators in the thin film equation that is given by

$$u_t + uu_{xxxx} - \beta u_x u_{xxx} - \gamma (u_{xx})^2 = 0, \tag{1}$$

where $u := u(x,t)$ is a real function for all $x, t \in \mathbb{R}$ and $\gamma, \beta \geq 0$. In physics and engineering, the thin-film equation is a partial differential equation that approximately predicts the time evolution of the thickness of a liquid film that lies on a surface [8].

This paper is organized as follows. Section 2 is devoted to determining the Lie symmetry group of the thin film equation. In Section 3, we construct the optimal system of one-dimensional subalgebras for the thin film equation. Finally, obtaining and classifying the similarity reductions of (1) is considered in Section 4.

2. Lie symmetry group of the thin film equation

In this section, at first, we review some definitions and previous studies of the Lie symmetry method that will be used later (see [13]) and then investigate the Lie symmetry of the thin film equation. Consider a system of PDEs of order n in p independent variables $x = (x^1, ..., x^p)$ and q dependent variables $u = (u^1, ..., u^q)$:

$$\Delta_v(x, u^{(n)}) = 0, \qquad v = 1, .., l, \tag{2}$$

where $u^{(n)}$ represents all the derivatives of u of order from 0 to n. The infinitesimal symmetry operator generally can be considered as,

$$V = \sum_{i=1}^{p} \xi^i(x,u)\partial_{x^i} + \sum_{j=1}^{q} \phi_j(x,u)\partial_{u^j}. \tag{3}$$

Let $V^{(n)}$ be the n-th order prolongation of V, then V is an infinitesimal generator for the symmetry group of system (2) whenever it justify the invariance criterion (Theorem 2.36 of [13]),

$$V^{(n)}[\Delta_v(x, u^{(n)})] = 0, \quad v = 1, ..., l, \quad \text{as} \quad \Delta_v(x, u^{(n)}) = 0. \tag{4}$$

Now, we compute the Lie symmetry group of (1). Suppose,

$$\begin{cases} \hat{x} &= x + \epsilon\xi(x,t,u) + o(\epsilon^2), \\ \hat{t} &= t + \epsilon\tau(x,t,u) + o(\epsilon^2), \\ \hat{u} &= u + \epsilon\phi(x,t,u) + o(\epsilon^2), \end{cases}$$

is the one-parameter Lie group of point transformations, where ϵ is a group parameter. The associated symmetry generator for this transformations group is of the form:

$$V = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u. \tag{5}$$

The fourth prolongation of V is given by

$$V^{(4)} = V + \phi^x \partial u_x + \phi^t \partial u_t + \phi^{x^2} \partial u_{x^2} + \phi^{xt} \partial u_{xt} + \phi^{t^2} \partial u_{t^2} + \phi^{x^3} \partial u_{x^3} \\ + \phi^{x^2 t} \partial u_{x^2 t} + \phi^{xt^2} \partial u_{xt^2} + \phi^{t^3} \partial u_{t^3} + \dots + \phi^{xt^4} \partial u_{xt^4}, \tag{6}$$

where its coefficients are

$$\begin{cases} \phi^x \ = \ D_x(\phi - \xi u_x - \tau u_t) + \tau u_{xt} + \xi u_{x^2}, \\ \phi^t \ = \ D_t(\phi - \xi u_x - \tau u_t) + \tau u_{t^2} + \xi u_{xt}, \\ \quad \vdots \\ \phi^{t^4} = D_t^4(\phi - \xi u_x - \tau u_t) + \xi u_{xt^4} + \tau u_{t^5}. \end{cases} \tag{7}$$

For more details, see [13]. By applying the invariance condition (4), we have

$$V^{(4)}(u_t + u u_{xxxx} - \beta u_x u_{xxx} - \gamma\, (u_{xx})^2) = 0 \text{ whenever} \\ u_t + u u_{xxxx} - \beta u_x u_{xxx} - \gamma\, (u_{xx})^2 = 0. \tag{8}$$

After substituting (6) with its coefficients (7) in (8), we have

$$\phi_t - \xi_t \phi_x + 3\beta u_x u_{xxt} \tau_x + \cdots + \phi u_{4x} = 0. \tag{9}$$

By setting the individual coefficients equal to zero, determining equations can be generated as,

$$\phi_t = 0, \;\; \phi_x = 0, \;\; \phi_u = \frac{\phi}{u}, \;\; \tau_x = 0, \tau_u = 0, \\ \tau_{tt} = 0, \;\; \xi_t = 0, \;\; \xi_x = \frac{\phi + \tau_t u}{4u}, \;\; \xi_u = 0. \tag{10}$$

After solving this system, we have

$$\xi = \frac{1}{4}(c_1 + c_3)x + c_4, \;\; \tau = c_1 t + c_2, \;\; \phi = c_3 u, \tag{11}$$

where c_1, c_2, c_3 and c_4 are arbitrary real coefficients. So we proved the following theorem.

Theorem 2.1 The Lie algebra of point symmetry of (1) is generated by

$$V_1 = \partial_x, \quad V_2 = \partial_t, \quad V_3 = \frac{1}{4} x \partial_x + t \partial_t, \quad V_4 = \frac{1}{4} x \partial_x + u \partial_u. \tag{12}$$

This symmetry vector fields constitute a four-dimensional Lie algebra, which we denote by $\mathfrak{g}$;

$$[V_1, V_3] = \frac{1}{4} V_1, \quad [V_1, V_4] = \frac{1}{4} V_1, \quad [V_2, V_3] = V_2.$$

In the following, we obtain the commutator table of $\mathfrak{g}$.

Table 1.: The commutator table of $\mathfrak{g}$

[,]	V_1	V_2	V_3	V_4
V_1	0	0	$\frac{1}{4}V_1$	$\frac{1}{4}V_1$
V_2	0	0	V_2	0
V_3	$-\frac{1}{4}V_1$	$-V_2$	0	0
V_4	$-\frac{1}{4}V_1$	0	0	0

By exponentiating the symmetries in (12), one-parameter groups $E_r(\varepsilon)$, which are generated by V_r are obtained as $(r = 1, \dots, 4)$:

$$\begin{aligned} &E_1(\varepsilon) = (x,t,u) \longrightarrow (x+\varepsilon, t, u),\ E_2(\varepsilon) = (x,t,u) \longrightarrow (x, t+\varepsilon, u), \\ &E_3(\varepsilon) = (x,t,u) \longrightarrow (xe^{\frac{1}{4}\varepsilon}, te^{\varepsilon}, u),\ E_4(\varepsilon) = (x,t,u) \longrightarrow (xe^{\frac{1}{4}\varepsilon}, t, ue^{\varepsilon}). \end{aligned} \tag{13}$$

Therefore, we proved the following theorem.

Theorem 2.2 If $u = g(x,t)$ is a solution of (1), either are functions:

$$\begin{array}{ll} E_1(\varepsilon).g(x,t) = g(x-\varepsilon, t), & E_2(\varepsilon).g(x,t) = g(x, t-\varepsilon), \\ E_3(\varepsilon).g(x,t) = g(xe^{\frac{-1}{4}\varepsilon}, te^{-\varepsilon}), & E_4(\varepsilon).g(x,t) = g(xe^{\frac{-1}{4}\varepsilon}, t)e^{\varepsilon}, \end{array}$$

and any arbitrary combination of the above solutions is again a solution for (1).

3. One-dimensional optimal system of the thin film equation

Here, we obtain the one-dimensional optimal system of Lie symmetry subalgebras of (1).

Definition 3.1 Suppose G is a Lie group. An optimal system of s-parameter subgroups is a list of conjugacy inequivalent s-parameter subgroups with the property that each other subgroup is conjugated exactly to one subgroup of this list. Also, a list of s-dimensional subalgebras forms an optimal system if every s-dimensional subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{\mathfrak{h}} = \mathrm{Ad}g(\mathfrak{h})$ [13].

We can construct infinite number of one-dimensional subalgebras for a PDE, by considering an arbitrary linear combination of infinitesimal symmetries. So, to know that which subgroups offer a different type of solution is important. For this aim, we must know which invariant solutions are not related by the symmetry group transformations. This classification problem is in fact classification of orbits for the adjoint representation that is solved by a simple method [13, 14]. In this method, a general element of the Lie algebra is taken and is simplified as far as possible by subjecting it to different adjoint transformations. Optimal system of subalgebras is constructed by choosing one representative from every equivalence class. Adjoint representation is computed due to the Lie series:

$$\mathrm{Ad}(\exp(\varepsilon.V_i).V_j) = V_j - \varepsilon.[V_i, V_j] + \frac{\varepsilon^2}{2}.[V_i, [V_i, V_j]] - \cdots, \tag{14}$$

where $[V_i, V_j]$ is the commutator of $\mathfrak{g}$ and ε is a parameter $i, j = 1, ..., 4$. So we get the Table 2 that its (i,j)-th entry imply $\mathrm{Ad}(\exp(\varepsilon.V_i).V_j)$.

Table 2.: Adjoint representation for the infinitesimal generators of $\mathfrak{g}$.

Ad	V_1	V_2	V_3	V_4
V_1	V_1	V_2	$-\frac{1}{4}\varepsilon V_1 + V_3$	$-\frac{1}{4}\varepsilon V_1 + V_4$
V_2	V_1	V_2	$-\varepsilon V_2 + V_3$	V_4
V_3	$e^{\frac{\varepsilon}{4}} V_1$	$e^{\varepsilon} V_2$	V_3	V_4
V_4	$e^{\frac{\varepsilon}{4}} V_1$	V_2	V_3	V_4

Theorem 3.2 An optimal system of one-dimensional Lie algebras of equation (1) can be obtained from

$$\begin{array}{lll} 1) V_1, & 2) \pm V_1 + V_2, & 3) V_4, \\ 4) \pm V_2 + V_4, & 5) V_3 + bV_4, & 6) \pm V_1 + V_3 - V_4, \end{array} \tag{15}$$

where $b \in \mathbb{R}$ is an arbitrary value.

Proof. Let $L_i^{\varepsilon} : \mathfrak{g} \to \mathfrak{g}$ denote the adjoint transformation $V \mapsto \mathrm{Ad}(\exp(\varepsilon.V_i).V)$. We can find the matrix of L_i^{ε} with respect to the basis V_i, $i = 1, ..., 4$, as follow:

$$M_1^s = \begin{bmatrix} 1 & 0 & -\frac{\varepsilon}{4} & -\frac{\varepsilon}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; M_2^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\varepsilon & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_3^s = \begin{bmatrix} e^{\frac{\varepsilon}{4}} & 0 & 0 & 0 \\ 0 & e^{\varepsilon} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_4^s = \begin{bmatrix} e^{\frac{1}{4}\varepsilon} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If $V = \sum_{i=1}^{4} a_i V_i$ be a general element of $\mathfrak{g}$, then we have

$$\begin{aligned} L_4^{\varepsilon_4} \circ L_3^{\varepsilon_3} \circ L_2^{\varepsilon_2} \circ L_1^{\varepsilon_1} : V \mapsto & \left(a_1 + e^{\varepsilon_1} \varepsilon_2 a_2\right) V_1 + \left(a_2 - e^{\varepsilon_2} s_1 a_1 + a_2 - \tfrac{1}{4} e^{\varepsilon_2} \varepsilon_4 a_4\right) V_2 \\ & + \left(a_3 - \tfrac{1}{4} e^{\varepsilon_3} \varepsilon_4 a_4\right) V_3 + \left(a_4 \tfrac{1}{4} e^{\varepsilon_4} \varepsilon_2 a_4 + a_4 + \tfrac{1}{4} e^{\varepsilon_4} \varepsilon_3 a_4\right) V_4. \end{aligned}$$

Now, we can simplify V by acting suitable adjoint representation $L_i^{\varepsilon_i}$ on it:

- If $a_2 = a_3 = a_4 = 0$ and $a_1 \neq 0$, then by scaling V if required, V reduces to the case (1).
- If $a_3 = a_4 = 0$ and $a_2 \neq 0$, then by setting $\varepsilon_3 = -4 \ln |a_2|$ in $L_3^{\varepsilon_3}$, we can make the coefficient of V_1, ± 1. Scaling V if required, we can consider $a_2 = 1$. So V reduces to the case (2).
- If $a_2 = a_3 = 0$ and $a_4 \neq 0$, then by setting $\varepsilon_1 = \frac{4a_1}{a_4}$ in $L_1^{\varepsilon_1}$, we can vanish the coefficient of V_1. Scaling V if required, we can consider $a_4 = 1$. So V reduces to the case (3).
- If $a_3 = 0$, $a_2 \neq 0$ and $a_4 \neq 0$, by setting $\varepsilon_1 = \frac{4a_1}{a_4}$ and $\varepsilon_2 = -\ln |a_2|$ in $L_1^{\varepsilon_1}$ and $L_2^{\varepsilon_2}$ respectively, we can vanish the coefficient of V_1 and make the coefficient of V_2, ± 1. Scaling V if required, we can consider $a_4 = 1$. So V reduces to the case (4).
- If $a_1 = 0$, $a_4 = -1$ and $a_3 \neq 0$, by setting $\varepsilon_2 = \frac{a_2}{a_3}$ in $L_2^{\varepsilon_2}$, we can vanish the coefficient of V_2. Scaling V if required, we can consider $a_3 = 1$. So V reduces to the case (5).
- If $a_1 \neq 0$, $a_4 = -1$ and $a_3 \neq 0$, by setting $\varepsilon_2 = \frac{a_2}{a_3}$ and $\varepsilon_4 = -2 \ln |a_1|$ in $L_2^{\varepsilon_2}$ and $L_4^{\varepsilon_4}$ respectively, we can vanish the coefficient of V_2 and make the coefficient of V_1, ± 1.

Scaling V if required, we can consider $a_3 = 1$. So V reduces to the case (6).

- If $a_3 \neq 0$ and $a_4 \neq -1$, by setting $\varepsilon_2 = \frac{a_2}{a_3}$ and $\varepsilon_1 = \frac{4a_1}{1+a_4}$ in $L_2^{\varepsilon_2}$ and $L_1^{\varepsilon_1}$ respectively, we can vanish the coefficient of V_2 and V_1. Scaling V if required, we can consider $a_3 = 1$. So again V reduces to the case (5).

There is not any more possible case for investigating and the proof is complete. ■

4. Reductions of the thin film equation

In this section, we reduce the order of the thin film equation using the new coordinates. The thin film equation is introduced by coordinates x, t and u. To reduce the order of (1), we use new and appropriate coordinates (z, f). Then, using the chain rule, the reduced form of the equation is obtained. We explain one of the infinitesimal generators in the optimal system of the thin film equation and list the rest of the results in the Table 3. For example, we describe the second case of Theorem 3.2, $V := V_1 + V_2$. To determine the independent invariant k, we must solve the PDE, $V(k) = 0$. That's mean

$$(V_1 + V_2)\, k = (\partial_x + \partial_t)\, k = \frac{\partial k}{\partial x} + \frac{\partial k}{\partial t} = 0.$$

For solving this partial differential equation, we must solve the following characteristic ODE system.

$$\frac{dx}{1} = \frac{dt}{1}.$$

Thus two independent invariant functions $z = -x + t$ and $f = u$ are obtained. We can gain the derivatives of u with respect to x and t in terms of f and z. Using the chain rule, we have

$$u_t = -f'(z) \qquad , \qquad u_x = f'(z)$$

After substituting the above relations in (1), we obtain

$$u_t + u u_{xxxx} + \beta u_x u_{xxx} + \gamma\,(u_{xx})^2 = f' + f f^{(4)} - \beta f' f^{(3)} - \gamma f''^2 = 0.$$

So the reduced equation is

$$f' + f f^{(4)} - \beta f' f^{(3)} - \gamma (f")^2 = 0.$$

This equation has an independent variable z and a dependent variable f. In a similar way, we can compute all the similarity reductions of infinitesimal generators in Theorem 3.2. The rest of the similarity reductions are listed in Table **??**.

5. Conclusions

In this paper by using the invariance criterion of the equation under the prolonged infinitesimal generators, we find the Lie point symmetry group of the thin film equation. Using the adjoint representation, we obtained the one-dimensional optimal system of

Table 3.: Lie invariants, reduced equations

j	$\mathfrak{h}_j$	z_j, w_j	u_j	reduced equations
1	V_4	$\left\{t, \frac{u}{x^4}\right\}$	$f(z)x^4$	$f' + (24 - 96\beta - 144\gamma)f^2 = 0$
2	V_1	$\{t, u\}$	$f(z)$	$f' = 0$
3	$V_1 + V_2$	$\{-x + t, u\}$	$f(z)$	$f' + ff^4 - \beta f' f^3 - \gamma f''^2 = 0$
4	$-V_1 + V_2$	$\{x + t, u\}$	$f(z)$	$f' + ff^4 - \beta f' f^3 - \gamma f''^2 = 0$
5	$V_2 + V_4$	$\left\{-4\ln(x) + t, \frac{u}{x^4}\right\}$	$f(z)x^4$	$\left(\beta - \frac{1}{4}\right) f^2 - \left(-\frac{16}{3}\beta + \frac{25}{12}\right) f'f +$ $\left(6\beta - \frac{35}{6}\right) f^2 f - \left(-\frac{8\beta}{3} + \frac{20}{3}\right) f^3$ $-\frac{8}{3} f^4 f + \left(\beta f' - \frac{18}{13}\beta f^2 + \frac{8}{13}\beta f^3 - \frac{1}{416}\right) = 0$
6	$-V_2 + V_4$	$\left\{4\ln(x) + t, \frac{u}{x^4}\right\}$	$f(z)x^4$	$\left(\beta - \frac{1}{4}\right) f^2 + \left(-\frac{16}{3}\beta + \frac{25}{12}\right) f'f +$ $\left(6\beta - \frac{35}{6}\right) f^2 f + \left(-\frac{8\beta}{3} + \frac{20}{3}\right) f^3$ $-\frac{8}{3} f^4 f + \left(\beta f' - \frac{18}{13}\beta f^2 + \frac{8}{13}\beta f^3 - \frac{1}{416}\right) = 0$
7	$V_3 + bV_4$	$\left\{tx^{\frac{-4}{b+1}}, ux^{\frac{-4b}{b+1}}\right\}$	$f(z)x^{\frac{4b}{b+1}}$	$\left(3\left(\beta + \frac{5}{2}\right) + 2(\beta + 5) f + 2z^2 f^2\right) f^2 f'$ $+ \left(1 - 24\beta z^3 f^2 - z^4 \beta f^3\right) f = 0$
8	$V_1 + V_3 - V_4$	$\left\{\frac{t}{(x+2)^2}, \frac{u}{(x+2)^2}\right\}$	$f(z)(x+2)^2$	$24\left(\beta + \frac{5}{2}\right) z^2 f^2 + \left(\left((\beta + 5) f^3 + f^4\right) 16 z^2 f'\right)$ $- \left(24 z^3 \beta f^2 - \beta z^4 f^3 - 1\right) f' = 0$
9	$-V_1 + V_3 - V_4$	$\left\{\frac{t}{(x-2)^2}, \frac{u}{(x-2)^2}\right\}$	$f(z)(x-2)^2$	$24\left(\beta + \frac{5}{2}\right) z^2 f^2 - \left(\left((\beta + 5) f^3 + f^4\right) 16 z^2 f'\right)$ $+ \left(24 z^3 \beta f^2 - \beta z^4 f^3 - 1\right) f' = 0$

Lie subalgebras for the Lie symmetry group. Also, the classification of group-invariant solutions is obtained. Finally, we obtain the reduced equations for each element of optimal system of the thin film equation.

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