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On the automorphism groups of 2-generator 2-groups of class two

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Abstract. In this paper, by using the relations and properties of some classes of two generator 2-groups of nilpotency class two, we find the order of automorphism group of these groups.

Keywords: Automorphism groups, 2-groups, nilpotency class.

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1. Introduction and preliminaries

Most of the authors that have been interested in studying automorphism groups, have considered the automorphism groups of p-groups. For example, Jamali in [3] considered some nonabelian 2-groups with abelian automorphism groups. Bidwell and Curran in [1] found the order, structure and presentation for the automorphism group of a split metacyclic p-group. Here, we calculate the order of automorphism groups of some classes of 2-groups. In [2], Hashemi found the order of automorphism groups of some classes of 2-generator nilpotent groups of nilpotncy class two.
Suppose that $N \lhd G$ and there is a subgroup H such that $G = NH$ and $H \cap N = \{e\}$, then G is said to be the semidirect product of N and H; in symbols $G = N \rtimes H$. Each element of G has a unique expression of the form ab where $a \in N$ and $b \in H$. Now, by using this notation, we state the following classification theorem without proof.

Theorem 1.1 [4] Let G be a finite nonabelian 2-generator 2-group of nilpotency class two. Then G is isomorphic to exactly one group of the following four types:

(1) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a,b] = c$, $[a,c] = [b,c] = 1$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|c| = 2^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geqslant \beta \geqslant \gamma$;

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(2) $G \cong \langle a \rangle \rtimes \langle b \rangle$, where $[a,b] = a^{2^{\alpha-\gamma}}$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|[a,b]| = 2^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geqslant 2\gamma$, $\beta \geqslant \gamma$, $\alpha + \beta > 3$;
(3) $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a,b] = a^{2^{\alpha-\gamma}}c$, $[c,b] = a^{-2^{2(\alpha-\gamma)}}c^{-2^{\alpha-\gamma}}$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|c| = 2^\sigma$, $|[a,b]| = 2^\gamma$, $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\beta \geqslant \gamma > \sigma$, $\alpha + \sigma \geqslant 2\gamma$;
(4) $G \cong (\langle c \rangle \times \langle a \rangle)\langle b \rangle$, where $|a| = |b| = 2^{\gamma+1}$, $|[a,b]| = 2^\gamma$, $|c| = 2^{\gamma-1}$, $[a,b] = a^2c$, $[c,b] = a^{-4}c^{-2}$, $a^{2^\gamma} = b^{2^\gamma}$, $\gamma \in \mathbb{N}$.

The following lemma establishes some properties of groups of nilpotency class two.

Lemma 1.2 If G is a group and $G' \subseteq Z(G)$, then the following hold for every integer k and $u, v, w \in G$:

(i) $[uv, w] = [u, w][v, w]$ and $[u, vw] = [u, v][u, w]$;
(ii) $[u^k, v] = [u, v^k] = [u, v]^k$;
(iii) $(uv)^k = u^k v^k [v, u]^{k(k-1)/2}$.

2. Main Results

In this section, we consider some classes of 2-generator 2-groups of class 2 and find the order of their automorphism groups. Also, we check the results by Group Algorithm Programming(GAP)[5].

Theorem 2.1 Let $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a,b] = c$, $[a,c] = [b,c] = 1$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|c| = 2^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geqslant \beta \geqslant \gamma$. Then

$$|Aut(G)| = \begin{cases} 2^{\alpha+3\beta+2\gamma-2} & \alpha > \beta \geqslant \gamma \\ 3 \times 2^{4\alpha+2\gamma-3} & \alpha = \beta > \gamma \\ 2^{6\alpha-3} & \alpha = \beta = \gamma. \end{cases}$$

Proof. Let $f \in Aut(G)$. Also let $f(a) = c^{r_1}a^{s_1}b^{t_1}$, $f(b) = c^{r_2}a^{s_2}b^{t_2}$, $f(c) = c^{r_3}a^{s_3}b^{t_3}$, where $0 \leqslant s_1, s_2, s_3 < 2^\alpha$, $0 \leqslant t_1, t_2, t_3 < 2^\beta$ and $0 \leqslant r_1, r_2, r_3 < 2^\gamma$. Then $|f(a)| = |a| = 2^\alpha$. So we have$(f(a))^{2^\alpha} = (c^{r_1}a^{s_1}b^{t_1})^{2^\alpha} = a^{2^\alpha s_1}b^{2^\alpha t_1}c^{2^\alpha r_1 - 2^{\alpha-1}(2^\alpha-1)t_1 s_1} = 1$. Therefore the below equations hold.

$$\begin{cases} 2^\alpha s_1 \equiv 0 \ (mod\ 2^\alpha) \\ 2^\alpha t_1 \equiv 0 \ (mod\ 2^\beta) \\ 2^\alpha r_1 - 2^{\alpha-1}(2^\alpha - 1)t_1 s_1 \equiv 0 \ (mod\ 2^\gamma). \end{cases}$$

Similarly, $(f(b))^{2^\beta} = 1$ and $(f(c))^{2^\gamma} = 1$, which imply the equations

$$\begin{cases} 2^\beta s_2 \equiv 0 \ (mod\ 2^\alpha) \\ 2^\beta t_2 \equiv 0 \ (mod\ 2^\beta) \\ 2^\beta r_2 - 2^{\beta-1}(2^\beta - 1)t_2 s_2 \equiv 0 \ (mod\ 2^\gamma) \end{cases} \quad \text{and} \quad \begin{matrix} 2^\gamma s_3 \equiv 0 \ (mod\ 2^\alpha) \\ 2^\gamma t_3 \equiv 0 \ (mod\ 2^\beta) \\ 2^\gamma r_3 - 2^{\gamma-1}(2^\gamma - 1)s_3 t_3 \equiv 0 \ (mod\ 2^\gamma). \end{matrix}$$

Now, by defining the relation $[a,b] = c$, we have $[f(a), f(b)] = f(c)$. It yields that $c^{s_1t_2 - s_2t_1} = c^{r_3}a^{s_3}b^{t_3}$. Thus

$$\begin{cases} s_3 \equiv 0 \ (mod\ 2^\alpha) \\ t_3 \equiv 0 \ (mod\ 2^\beta) \\ s_1t_2 - s_2t_1 \equiv r_3 \ (mod\ 2^\gamma). \end{cases}$$

Now, we claim that r_3 can not be even. Suppose the contrary. So there exists $t \in \mathbb{N}$ such that $r_3 = 2t$. It follows that $(f(c))^{2^{\gamma-1}} = a^{2^{\gamma-1}s_3}b^{2^{\gamma-1}t_3}c^{2^{\gamma-1}r_3-2^{\gamma-2}(2^{\gamma-1}-1)t_3s_3} = 1$, which is contradiction. So r_3 must be odd. This, together with all of the above systems yield

$$\begin{cases} 2^{\alpha-1}(2^{\alpha}-1)t_1s_1 \equiv 0 \ (mod\ 2^{\gamma}) \\ 2^{\beta}s_2 \equiv 0 \ (mod\ 2^{\alpha}) \\ 2^{\beta-1}(2^{\beta}-1)t_2s_2 \equiv 0 \ (mod\ 2^{\gamma}) \\ 2^{\gamma-1}(2^{\gamma}-1)t_3s_3 \equiv 0 \ (mod\ 2^{\gamma}) \\ s_3 \equiv 0 \ (mod\ 2^{\alpha}) \\ t_3 \equiv 0 \ (mod\ 2^{\beta}) \\ s_1t_2 - s_2t_1 \equiv r_3 \ (mod\ 2^{\gamma}) \\ (r_3, 2) = 1. \end{cases} \qquad (*)$$

Now, for solving the above system, we consider the following four cases:

(1) $\alpha > \beta > \gamma$;
(2) $\alpha > \beta = \gamma$;
(3) $\alpha = \beta > \gamma$;
(4) $\alpha = \beta = \gamma$.

First, let $\alpha > \beta > \gamma$. Then, the system reduces to the following system

$$\begin{cases} 2^{\beta}s_2 \equiv 0 \ (mod\ 2^{\alpha}) \\ s_1t_2 - s_2t_1 \equiv r_3 \ (mod\ 2^{\gamma}) \\ s_3 \equiv 0 \ (mod\ 2^{\alpha}) \\ t_3 \equiv 0 \ (mod\ 2^{\beta}) \\ (r_3, 2) = 1. \end{cases}$$

Since $2^{\beta}s_2 \equiv 0 \ (mod\ 2^{\alpha})$ and $0 \leqslant s_2 < 2^{\alpha}$, we can choose s_2 in 2^{β} ways. Also we have $s_3 \equiv 0 \ (mod\ 2^{\alpha})$ and $0 \leqslant s_3 < 2^{\alpha}$. Thus $s_3 = 0$. Similarly, from $t_3 \equiv 0 \ (mod\ 2^{\beta})$ and $0 \leqslant t_3 < 2^{\beta}$, we get $t_3 = 0$. Now, since $s_1t_2 - s_2t_1 \equiv r_3 \ (mod\ 2^{\gamma})$, $(r_3, 2) = 1$ and $2^{\alpha-\beta}|s_2t_1$, we conclude that s_1t_2 must be odd. This implies that s_1 and t_2 are odd. Therefore we can choose s_1 and t_2 in $2^{\alpha-1}$ and $2^{\beta-1}$ ways, respectively. By calculating s_1,s_2,t_1 and t_2, the parameter r_3 can be only chosen in one way. Now, since t_1 is arbitrary and $0 \leqslant t_1 < 2^{\beta}$, so can be chosen in 2^{β} ways. In this system r_1 and r_2 are free, so each of them can be chosen in 2^{γ} ways. Consequently, the number of the solutions of this system will be $2^{\alpha+3\beta+2\gamma-2}$. Each of the three other cases can be solved similarly. ■

In the following, we check the number of the solutions of the system(*), for some values of α, β and γ, by GAP.

α	β	γ	*The number of solutions*
3	2	1	2^9
4	3	1	2^{13}
3	2	2	2^{11}
4	1	1	2^7
4	3	3	2^{17}
2	2	1	3×2^7
4	4	3	3×2^{19}
1	1	1	2^3
2	2	2	2^9
3	3	3	2^{15}

Theorem 2.2 Let $G \cong \langle a\rangle \rtimes \langle b\rangle$, where $[a,b] = a^{2^{\alpha-\gamma}}$, $|a| = 2^\alpha$, $|b| = 2^\beta$, $|[a,b]| = 2^\gamma$, $\alpha,\beta,\gamma \in \mathbb{N}$, $\alpha \geqslant 2\gamma$, $\beta \geqslant \gamma$, $\alpha+\beta > 3$. Then

$$|Aut(G)| = \begin{cases} 2^{\alpha+3\beta-\gamma-1} & if\ \alpha > \beta \geqslant \gamma, \alpha \geqslant 2\gamma, \alpha-\gamma \geqslant \beta;\\ 2^{2\alpha+2\beta-2\gamma-1} & if\ \alpha > \beta \geqslant \gamma, \alpha \geqslant 2\gamma, \alpha-\gamma < \beta;\\ 2^{4\alpha-2\gamma-1} & if\ \alpha = \beta \geqslant 2\gamma;\\ 2^{3\alpha+\beta-2\gamma-1} & if\ \beta > \alpha \geqslant 2\gamma. \end{cases}$$

Proof. Let $f \in Aut(G)$. Also let $f(a) = a^{r_1}b^{s_1}$, $f(b) = a^{r_2}b^{s_2}$, where $0 \leqslant r_1, r_2 < 2^\alpha$, and $0 \leqslant s_1, s_2 < 2^\beta$. Similar to the proof of the last theorem, $|f(a)|^{2^\alpha} = 1$ yields:

$$\begin{cases} 2^\alpha r_1 - 2^{2\alpha-\gamma-1}(2^\alpha-1)r_1s_1 \equiv 0\ (mod\ 2^\alpha)\\ 2^\alpha s_1 \equiv 0\ (mod\ 2^\beta)\end{cases}$$

and $|f(b)|^{2^\beta} = 1$ implies:

$$\begin{cases} 2^\beta r_2 - 2^{\alpha+\beta-\gamma-1}(2^\beta-1)r_2s_2 \equiv 0\ (mod\ 2^\alpha)\\ 2^\beta s_2 \equiv 0\ (mod\ 2^\beta).\end{cases}$$

Moreover, we have $[f(a), f(b)] = f(a)^{2^{\alpha-\gamma}}$, which yields

$$\begin{cases} 2^{\alpha-\gamma} s_1 \equiv 0\ (mod\ 2^\beta)\\ 2^{\alpha-\gamma}(r_1s_2 - r_2s_1) \equiv 2^{\alpha-\gamma}(r_1 - 2^{\alpha-\gamma-1}(2^{\alpha-\gamma}-1)r_1s_1)\ (mod\ 2^\alpha).\end{cases}$$

Consequently, we have

$$\begin{cases} 2^\alpha s_1 \equiv 0\ (mod 2^\beta)\\ 2^\beta r_2 - 2^{\alpha+\beta-\gamma-1}(2^\beta-1)r_2s_2 \equiv 0\ (mod\ 2^\alpha)\\ 2^{\alpha-\gamma}s_1 \equiv 0\ (mod\ 2^\beta)\\ 2^{\alpha-\gamma}(r_1s_2 - r_2s_1) \equiv 2^{\alpha-\gamma}(r_1 - 2^{\alpha-\gamma-1}(2^{\alpha-\gamma}-1)r_1s_1) \equiv 0\ (mod\ 2^\alpha).\end{cases}$$

Now, we consider the following three different cases:
(i) $\alpha > \beta$; (ii) $\alpha = \beta$; (iii) $\beta > \alpha$.
First, we consider $\alpha > \beta$. In this case, if r_1 is even we get that $(f(a))^{2^{\alpha-1}} = 1$ which is contradiction. Consequently r_1 must be odd. So we have

$$\begin{cases} (r_1, 2) = 1\\ 2^\beta r_2 \equiv 0\ (mod\ 2^\alpha)\\ 2^{\alpha-\gamma}s_1 \equiv 0\ (mod\ 2^\beta)\\ r_1s_2 - r_2s_1 \equiv r_1 - 2^{\alpha-\gamma-1}(2^{\alpha-\gamma}-1)r_1s_1 \equiv 0\ (mod\ 2^\gamma).\end{cases}$$

Now, we have two subcases $\beta > \gamma$ and $\beta = \gamma$. Let $\beta > \gamma$, then the above system will be reduced again and we encounter two subcases $\alpha > 2\gamma$ and $\alpha = 2\gamma$. If $\alpha > 2\gamma$, then we get the following congruence system

$$\begin{cases} (r_1, 2) = 1\\ 2^\beta r_2 \equiv 0\ (mod\ 2^\alpha)\\ 2^{\alpha-\gamma}s_1 \equiv 0\ (mod\ 2^\beta)\\ r_1s_2 - r_2s_1 \equiv r_1\ (mod\ 2^\gamma).\end{cases}$$

Since r_1 is odd and r_2 is even, s_2 must be odd. Now, subcases $\alpha-\gamma \geqslant \beta$ or $\alpha-\gamma<\beta$ occur. Let $\alpha-\gamma \geqslant \beta$. Then the system will be reduced to the following system

$$\begin{cases}(r_1,2)=1\\ 2^{\beta}r_2\equiv 0\ (mod\ 2^{\alpha})\\ r_1(s_2-1)\equiv 0\ (mod\ 2^{\gamma}).\end{cases} \qquad (**)$$

For solving the system, it is sufficient to find the number of solutions for r_1, r_2, s_1 and s_2. Since r_1 is odd and $0\leqslant r_1<2^{\alpha}$, it can be chosen in $2^{\alpha-1}$ ways. Also since $2^{\alpha-\beta}$ divides r_2, we can choose it in 2^{β} ways. Furthermore, by solving $r_1(s_2-1)\equiv 0\ (mod\ 2^{\gamma})$, we get that s_2 can be chosen in $2^{\beta-\gamma}$ ways and finally since s_1 is free, the number of its values is 2^{β}. Consequently, when $\alpha>\beta>\gamma$, $\alpha>2\gamma$ and $\alpha-\gamma\geqslant\beta$ we have $|Aut(G)|=2^{\alpha+3\beta-\gamma-1}$. So, when $\alpha>\beta$ we get the following cases:

$$\begin{cases}\beta>\gamma\begin{cases}\alpha>2\gamma\begin{cases}\alpha-\gamma\geqslant\beta\\ \alpha-\gamma<\beta\end{cases}\\ \alpha=2\gamma\end{cases}\\ \beta=\gamma\begin{cases}\alpha>2\gamma\\ \alpha=2\gamma\end{cases}\end{cases}$$

Similarly we can solve the systems in other subcases. The proof of parts (ii) and (iii) are similar. ■

The following table shows some of the results that have been obtained by GAP.

α	β	γ	*The number of solutions*
3	2	1	2^7
7	5	3	2^{17}
6	4	3	2^{13}
5	2	2	2^8
6	3	3	2^{11}
5	5	2	2^{15}
5	7	2	2^{17}

Theorem 2.3 Let $G\cong(\langle c\rangle\times\langle a\rangle)\rtimes\langle b\rangle$, where $[a,b]=a^{2^{\alpha-\gamma}}c$, $[c,b]=a^{-2^{2(\alpha-\gamma)}}c^{-2^{\alpha-\gamma}}$, $|a|=2^{\alpha}$, $|b|=2^{\beta}$, $|c|=2^{\sigma}$, $|[a,b]|=2^{\gamma}$, $\alpha,\beta,\gamma,\sigma\in\mathbb{N}$, $\beta\geqslant\gamma>\sigma$, $\alpha+\sigma\geqslant 2\gamma$. Then $|Aut(G)|=2^{\alpha+\beta+\sigma-\gamma-1}$, where $\alpha-\gamma>\beta>\gamma$.

Proof. Let $f\in Aut(G)$. Also let $f(a)=c^{r_1}a^{s_1}b^{t_1}$, $f(b)=c^{r_2}a^{s_2}b^{t_2}$, $f(c)=c^{r_3}a^{s_3}b^{t_3}$, where $0\leqslant s_1,s_2,s_3<2^{\alpha}$, $0\leqslant t_1,t_2,t_3<2^{\beta}$ and $0\leqslant r_1,r_2,r_3<2^{\sigma}$. Then since f is an automorphism, we obtain

$$\begin{cases}|f(a)|=2^{\alpha};\\ |f(b)|=2^{\beta};\\ |f(c)|=2^{\sigma};\\ f(a)^{-1}f(b)^{-1}f(a)f(b)=f(a)^{2^{\alpha-\gamma}}f(c);\\ f(c)^{-1}f(b)^{-1}f(c)f(b)=f(a)^{-2^{2(\alpha-\gamma)}}f(c)^{-2^{\alpha-\gamma}}.\end{cases}$$

These equations give the following congruence system:

$$\begin{cases}
(1)\ \ 2^{\alpha}s_1+2^{3\alpha-2\gamma-1}(2^{\alpha}-1)t_1r_1-2^{2\alpha-\gamma-1}(2^{\alpha}-1)t_1s_1\equiv 0 \pmod{2^{\alpha}}\\
(2)\ \ 2^{\alpha}t_1\equiv 0 \pmod{2^{\beta}}\\
(3)\ \ 2^{\alpha}r_1+2^{2\alpha-\gamma-1}(2^{\alpha}-1)t_1r_1-2^{\alpha-1}(2^{\alpha}-1)t_1s_1\equiv 0 \pmod{2^{\sigma}}\\
(4)\ \ 2^{\beta}s_2+2^{2\alpha+\beta-2\gamma-1}(2^{\beta}-1)t_2r_2-2^{\alpha+\beta-\gamma-1}(2^{\beta}-1)t_2s_2\equiv 0 \pmod{2^{\alpha}}\\
(5)\ \ 2^{\beta}t_2\equiv 0 \pmod{2^{\beta}}\\
(6)\ \ 2^{\beta}r_2+2^{\alpha+\beta-\gamma-1}(2^{\beta}-1)t_2r_2-2^{\beta-1}(2^{\beta}-1)t_2s_2\equiv 0 \pmod{2^{\sigma}}\\
(7)\ \ 2^{\sigma}s_3+2^{2\alpha+\sigma-2\gamma-1}(2^{\sigma}-1)t_3r_3-2^{\alpha+\sigma-\gamma-1}(2^{\sigma}-1)t_3s_3\equiv 0 \pmod{2^{\alpha}}\\
(8)\ \ 2^{\sigma}t_3\equiv 0 \pmod{2^{\beta}}\\
(9)\ \ 2^{\sigma}r_3+2^{\alpha+\sigma-\gamma-1}(2^{\sigma}-1)t_3r_3-2^{\sigma-1}(2^{\sigma}-1)t_3s_3\equiv 0 \pmod{2^{\sigma}}\\
(10)\ \ (s_1t_2-t_1s_2)-2^{\alpha-\gamma}(r_1t_2-r_2t_1)\equiv 2^{\alpha-\gamma}r_1+2^{2\alpha-2\gamma-1}(2^{\alpha-\gamma}-1)t_1r_1-\\
\qquad 2^{\alpha-\gamma-1}(2^{\alpha-\gamma}-1)t_1s_1+r_3+2^{2(\alpha-\gamma)}t_1r_3-2^{\alpha-\gamma}t_1s_3 \pmod{2^{\sigma}}\\
(11)\ \ 2^{\alpha-\gamma}(s_1t_2-t_1s_2)-2^{2(\alpha-\gamma)}(r_1t_2-r_2t_1)\equiv 2^{\alpha-\gamma}s_1+2^{3\alpha-3\gamma-1}(2^{\alpha-\gamma}-1)t_1r_1-\\
\qquad 2^{2\alpha-2\gamma-1}(2^{\alpha-\gamma}-1)t_1s_1+s_3+2^{3(\alpha-\gamma)}t_1r_3-2^{2(\alpha-\gamma)}t_1s_3 \pmod{2^{\alpha}}\\
(12)\ \ 2^{\alpha-\gamma}t_1+t_3\equiv 0 \pmod{2^{\beta}}\\
(13)\ \ (s_2t_3-t_2s_3)-2^{\alpha-\gamma}(r_3t_2-r_2t_3)\equiv 2^{\alpha-\gamma}r_3+2^{2\alpha-2\gamma-1}(2^{\alpha-\gamma}-1)t_3r_3-\\
\qquad 2^{\alpha-\gamma-1}(2^{\alpha-\gamma}-1)t_3s_3+2^{2(\alpha-\gamma)}r_1+2^{3(\alpha-\gamma)}(2^{2(\alpha-\gamma)}-1)t_1r_1-\\
\qquad 2^{2(\alpha-\gamma)-1}(2^{2(\alpha-\gamma)}-1)t_1s_1+2^{4(\alpha-\gamma)}r_1t_3-2^{3(\alpha-\gamma)}s_1t_3 \pmod{2^{\sigma}}\\
(14)\ \ 2^{\alpha-\gamma}(s_2t_3-t_2s_3)+2^{2(\alpha-\gamma)}(r_3t_2-r_2t_3)\equiv 2^{2(\alpha-\gamma)}s_1+2^{4(\alpha-\gamma)-1}(2^{2(\alpha-\gamma)}-1)t_1r_1-\\
\qquad 2^{3(\alpha-\gamma)-1}(2^{2(\alpha-\gamma)}-1)t_1s_1-2^{4(\alpha-\gamma)}s_1t_3+2^{\alpha-\gamma}s_3\\
\qquad +2^{3(\alpha-\gamma)-1}(2^{\alpha-\gamma}-1)t_3r_3-2^{2(\alpha-\gamma)-1}(2^{\alpha-\gamma}-1)t_3s_3+2^{5(\alpha-\gamma)}r_1t_3 \pmod{2^{\alpha}}\\
(15)\ \ 2^{2(\alpha-\gamma)}t_1+2^{\alpha-\gamma}t_3\equiv 0 \pmod{2^{\beta}}.
\end{cases}$$

After simplification we obtain

$$\begin{cases}
2^{\alpha}t_1\equiv 0 \pmod{2^{\beta}}\\
2^{\beta}s_2\equiv 0 \pmod{2^{\alpha}}\\
2^{\sigma}s_3-2^{\alpha+\sigma-\gamma-1}(2^{\sigma}-1)t_3s_3\equiv 0 \pmod{2^{\alpha}}\\
2^{\sigma}t_3\equiv 0 \pmod{2^{\beta}}\\
(s_1t_2-t_1s_2)-2^{\alpha-\gamma}(r_1t_2-r_2t_1)\equiv 2^{\alpha-\gamma}r_1+2^{2\alpha-2\gamma-1}(2^{\alpha-\gamma}-1)t_1r_1-\\
\qquad 2^{\alpha-\gamma-1}(2^{\alpha-\gamma}-1)t_1s_1+r_3+2^{2(\alpha-\gamma)}t_1r_3-2^{\alpha-\gamma}t_1s_3 \pmod{2^{\sigma}}\\
\ \ 2^{\alpha-\gamma}(s_1t_2-t_1s_2)-2^{2(\alpha-\gamma)}(r_1t_2-r_2t_1)\equiv 2^{\alpha-\gamma}s_1+2^{3\alpha-3\gamma-1}(2^{\alpha-\gamma}-1)t_1r_1-\\
\qquad 2^{2\alpha-2\gamma-1}(2^{\alpha-\gamma}-1)t_1s_1+s_3+2^{3(\alpha-\gamma)}t_1r_3-2^{2(\alpha-\gamma)}t_1s_3 \pmod{2^{\alpha}}\\
2^{\alpha-\gamma}t_1+t_3\equiv 0 \pmod{2^{\beta}}\\
(s_2t_3-t_2s_3)+2^{\alpha-\gamma}(r_3t_2-r_2t_3)\equiv 2^{\alpha-\gamma}r_3+2^{2\alpha-2\gamma-1}(2^{\alpha-\gamma}-1)t_3r_3-\\
\qquad 2^{\alpha-\gamma-1}(2^{\alpha-\gamma}-1)t_3s_3+2^{2(\alpha-\gamma)}r_1+2^{3(\alpha-\gamma)-1}(2^{2(\alpha-\gamma)}-1)t_1r_1-\\
\qquad 2^{2(\alpha-\gamma)-1}(2^{2(\alpha-\gamma)}-1)t_1s_1+2^{4(\alpha-\gamma)}r_1t_3-2^{3(\alpha-\gamma)}s_1t_3 \pmod{2^{\sigma}}\\
2^{\alpha-\gamma}(s_2t_3-t_2s_3)+2^{2(\alpha-\gamma)}(r_3t_2-r_2t_3)\equiv 2^{2(\alpha-\gamma)}s_1+2^{4(\alpha-\gamma)-1}(2^{2(\alpha-\gamma)}-1)t_1r_1-\\
\qquad 2^{3(\alpha-\gamma)-1}(2^{2(\alpha-\gamma)}-1)t_1s_1-2^{4(\alpha-\gamma)}s_1t_3+2^{\alpha-\gamma}s_3\\
\qquad +2^{3(\alpha-\gamma)-1}(2^{\alpha-\gamma}-1)t_3r_3-2^{2(\alpha-\gamma)-1}(2^{\alpha-\gamma}-1)t_3s_3+2^{5(\alpha-\gamma)}r_1t_3 \pmod{2^{\alpha}}
\end{cases}$$

Let $\alpha-\gamma>\beta>\gamma$. First, we claim that s_1 must be odd. Suppose the contrary. Then there exist integer s such that $s_1=2s$. Thus

$$(f(a))^{2^{\alpha-1}}=c^{2^{\alpha-1}r_1+2^{2\alpha-\gamma-1}(2^{\alpha}-1)t_1r_1-2^{\alpha-1}(2^{\alpha}-1)t_1s_1}$$
$$a^{2^{\alpha-1}s_1+2^{3\alpha-2\gamma-1}(2^{\alpha}-1)t_1r_1-2^{2\alpha-\gamma-1}(2^{\alpha}-1)t_1s_1}b^{2^{\alpha-1}t_1}=a^{2^{\alpha}s}=1$$

which is a contradiction to $|f(a)| = 2^{\alpha}$. This together with condition $\alpha - \gamma > \beta > \gamma$ reduce the above system to the following system

$$\begin{cases} (s_1, 2) = 1 \\ t_3 = 0 \\ 2^{\beta} s_2 \equiv 0 \quad (mod\ 2^{\alpha}) \\ 2^{\sigma} s_3 \equiv 0 \quad (mod\ 2^{\alpha}) \\ s_1(t_2 - 1) \equiv s_3/2^{\alpha-\gamma} \quad (mod\ 2^{\gamma}) \\ s_1 t_2 \equiv r_3 \quad (mod\ 2^{\sigma}). \end{cases} \qquad (***)$$

Since s_1 is odd and $0 \leqslant s_1 < 2^{\alpha}$, it can choose its values in $2^{\alpha-1}$ ways. Also since $2^{\alpha-\beta}|s_2$ and $0 \leqslant s_2 < 2^{\alpha}$, the number of choices for s_2 is equal to 2^{β}. Furthermore we have $2^{\alpha-\sigma}|s_3$ and $0 \leqslant s_3 < 2^{\alpha}$. This implies that s_3 has 2^{σ} ways for choosing its values. Now, by putting s_1 and s_3, the parameter t_2 will be obtained. So, t_2 has one choice when $0 \leqslant t_2 < 2^{\gamma}$. But $0 \leqslant t_2 < 2^{\beta}$, therefore we must multiply the number of solutions to $2^{\beta-\gamma}$. Then by putting s_1 and the obtained t_2, we get r_3. Hence r_3 can just have one choice. Also, it is clear that t_3 has one choice too. Now by multiplying the number of choices of each parameter, we get that the number of solutions of the system$(***)$ is $2^{\alpha+2\beta+\sigma-\gamma-1}$. All that remains to be done is to multiply the number of choices of free parameters of this system which are t_2, r_1 and r_2. Consequently, we have

$$|Aut(G)| = 2^{\alpha+3\beta+3\sigma-\gamma-1}.$$

■

In the following table, we bring the number of solutions of system$(***)$ for some values of σ, γ, β and α.

σ	γ	β	α	*The number of solutions*
1	2	3	7	2^{11}
1	2	3	8	2^{12}
1	2	4	8	2^{14}
1	3	4	8	2^{13}
2	3	4	8	2^{14}
2	4	5	10	2^{17}
1	3	5	10	2^{17}

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