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Coincident and common fixed point theorems using comparison and admissible function in w-distance metric space

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Abstract. In this manuscript, the concept of generalized (η, χ, p) contractive mapping for two maps in the framework of w-distance is introduced and some fixed point results are established, which extend recent results of Lakzian and Rhoades [5] and many existing results in the literature. In addition, to validate the novelty of our findings, we give an illustrative example, which yields the main result. Moreover, as an application, we employ the achieved result to earn the existence criteria of the solution of a type of non-linear Fredholm integral equation.

Keywords: w-distance map, generalized (η, χ, p) contractive mapping, coincident point, common fixed point.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

Fixed point theory is an entrancing subject with tremendous number of utilizations in different field of mathematics and may be perceived as one of thrust areas of investigation in non linear analysis. One of the earliest and most important results in fixed point theory is Banach contraction principle, which states that every contraction mapping defined on a complete metric space possesses a unique fixed point. This principle has many applications in different domains, such as functional equations, medical science, economics, wild life and several others. Due to its applications in many disciplines within

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mathematics and outside it, several authors have improved, generalized and extended this principle in non-linear analysis (see [6]).

In 2012, Shahi et al. [8] introduced the perception of $\eta-$admissible with respect to S_2 function. Afterwards, Samet et al. [7] established some fixed point results for a new category of $(\alpha-\psi)$-contractive functions. In 2015, Lakzian [4] established some reults for $(\alpha-\psi)$-contractive functions in the frame of w-distance. In 2019, Lakzian and Rhoades [5] established few results for w-distance map with the aid of Meir-Keeler mapping in the framework of complete metric space. Very recently, Barootkoob et al. [1] investigated fixed point results involving generalized Meir-Keeler contractive function in the framework of w-distance metric space.
Now, we recollect some elementary results which are used in sequel.

Definition 1.1 [7] Let Ω be a family of functions $\chi : [0,\infty) \to [0,\infty)$ satisfying the following properties:
(Ω_1) χ is upper semi-continuous and strictly increasing;
(Ω_2) $\{\chi^n(t)\}_{n\in N}$ converges to 0 as $n \to \infty$ for all $t > 0$;
(Ω_3) $\chi(t) < t$, for every $t > 0$.
These functions are known as comparison functions.

Definition 1.2 [3] Let $f : \mathcal{H} \to \mathcal{H}$ and $\eta : \mathcal{H} \times \mathcal{H} \to [0,+\infty)$. Then, f is said to be $\eta-$admissible if $\eta(x,y) \geqslant 1 \Rightarrow \eta(fx,fy) \geqslant 1$ for each $x,y \in \mathcal{H}$.

Definition 1.3 [8] Let $S_1, S_2 : \mathcal{H} \to \mathcal{H}$ and $\eta : \mathcal{H} \times \mathcal{H} \to [0,+\infty)$. Then S_1 is $\eta-$admissible with respect to S_2 if $\eta(S_2x, S_2y) \geqslant 1 \Rightarrow \eta(S_1x,S_1y) \geqslant 1$ for each $x,y \in \mathcal{H}$.

Definition 1.4 [7] Let $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$ be a given self mapping in a metric space $(\mathcal{H},\sigma)$. Then, $\mathcal{Q}$ is termed as $\eta-\chi$ mapping of contraction if there occur two maps $\chi \in \Omega$ and $\eta : \mathcal{H} \times \mathcal{H} \to [0,+\infty)$ with the goal that

$$\eta(\Omega,\wp)\sigma(\mathcal{Q}\Omega,\mathcal{Q}\wp) \leqslant \psi(\sigma(\Omega,\wp))$$

for all $\Omega,\wp \in \mathcal{H}$.

Theorem 1.5 [7] Let $S : \mathcal{H} \to \mathcal{H}$ be (χ,β)-contractive mapping in $(\mathcal{H},\sigma)$ which is complete, one to one and onto. Also, S fulfils the accompanying conditions:
(i) S is continuous;
(ii) S is η-admissible;
(iii) There occur $\Omega_0 \in \mathcal{H}$ in a a manner that $\eta(\Omega_0, S\Omega_0) \geqslant 1$.
Then, S possess a fixed point in $\mathcal{H}$.

Theorem 1.6 [7] Let $S : \mathcal{H} \to \mathcal{H}$ be (χ,β)-contractive mapping in $(\mathcal{H},\sigma)$ which is complete, one to one and onto. Also, S fulfils the accompanying conditions:
(i) If $\{\Omega_n\}$ is an arrangement in $\mathcal{H}$ in a manner that $\eta(\Omega_n,\Omega_{n+1}) \geqslant 1$ and $\lim_{n\to\infty} \Omega_n = \Omega$, then $\beta(\Omega_n,\Omega) \geqslant 1$;
(ii) S is β-admissible; (iii) There occur $\Omega_0 \in \mathcal{H}$ in a a manner that $\eta(\Omega_0, S\Omega_0) \geqslant 1$.
Then, S possess a fixed point in $\mathcal{H}$.

In 1996, Kada et al. [2] introduced the perception of w distance on $\mathcal{H}$ as follows:

Definition 1.7 Let $\mathcal{H}$ be a metric space associated with the metric σ. A map $p : \mathcal{H} \times \mathcal{H} \to [0,\infty)$ is named as w-distance on $\mathcal{H}$ if it fulfils the accompanying properties for each $\ell, s, r \in \mathcal{H}$:

(1) $p(\ell,s) \leqslant p(\ell,r) + p(r,s)$;

(2) p is lower semi-continuous, that is, if $\ell \in \mathcal{X}$ and $r_n \to r$, then $p(\ell, r) \leqslant \liminf p(\ell, r_n)$ when $n \to \infty$;
(3) For every $\gamma > 0$, we can find $\rho > 0$ in such a manner that $p(s, \ell) \leqslant \rho$ and $p(s, r) \leqslant \rho$ implies that $\sigma(\ell, r) \leqslant \gamma$.

For proving the main theorem in this paper, we need the following Lemma proved by Kada et al. [2].

Lemma 1.8 Let $(\mathcal{H}, \sigma)$ be a metric space, p be a w-distance on $\mathcal{H}$. Let $\{v_n\}$ and $\{w_n\}$ be sequences in $\mathcal{H}$, $\{c_n\}$ and $\{d_n\}$ be sequences in $[0, \infty)$ tending to 0, and $y, z, t \in \mathcal{H}$. Then, the accompanying assertions hold.

(1) If $p(y_n, z) \leqslant c_n$ and $p(y_n, t) \leqslant d_n$ for every $n \in \mathbf{Z}_+$, then $z = t$.
(2) If $p(y_n, z_n) \leqslant c_n$ and $p(y_n, z) \leqslant d_n$ for every $n \in \mathbf{Z}_+$, then $\{z_n\}$ tends to z.
(3) If $p(y_n, y_{n_1}) \leqslant c_n$ for every $n, n_1 \in \mathbf{Z}_+$ with $n_1 > n$, then $\{y_n\}$ is a Cauchy sequence.

In this paper, inspired by the concept of Lakzian and Rhoades [5], we introduce new perception of generalized (η, χ, p) contractive mapping in the edge of w-distance and establish some common fixed point results, which generalize many existing results in the literature.

2. Main results

Let $S_1, S_2 : \mathcal{H} \to \mathcal{H}$ be two maps. We identify the set of coincidence and common fixed points of S_1 and S_2 by $\mathcal{C}(S_1, S_2)$ and $\mathcal{CF}(S_1, S_2)$, where $\mathcal{C}(S_1, S_2) = \{z \in \mathcal{H} : S_1 z = S_2 z\}$ and $\mathcal{CF}(S_1, S_2) = \{z \in \mathcal{H} : S_1 z = S_2 z = z\}$.

Definition 2.1 Let $(\mathcal{H}, \sigma)$ be a metric space, p be a w-distance and S_1, S_2 be the self maps. Then, (S_1, S_2) is generalized (η, χ, p) contractive map, if there exists two maps $\eta : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ and $\chi \in \Omega$ such that

$$\eta(S_2 x, S_2 y) p(S_1 x, S_1 y) \leqslant \chi(P(S_2 x, S_2 y), \tag{1}$$

where

$$P(S_2 x, S_2 y) = \max\left\{p(S_2 x, S_2 y), \frac{p(S_2 x, S_1 x) + p(S_2 y, S_1 y)}{2}, \frac{p(S_2 x, S_1 y) + p(S_2 y, S_1 x)}{2}\right\}.$$

Theorem 2.2 Let $(\mathcal{H}, \sigma)$ be a complete metric space, p be a w-distance and S_1, S_2 be the self maps such that $S_1 \mathcal{H} \subseteq S_2 \mathcal{H}$. Let (S_1, S_2) is generalized (η, χ, p) contractive map which fulfils the following conditions:

(i) There exists $y_0 \in \mathcal{H}$ such that $\eta(S_2 y_0, S_1 y_0) \geqslant 1$;
(ii) S_1 is $\eta-$admissible with respect to S_2;
(iii) If $\{S_2 y_n\}$ is a sequence in $\mathcal{H}$ such that $\eta(S_2 y_n, S_2 y_{n+1}) \geqslant 1$ for all n and $S_2 y_n \to S_2 u \in S_2(\mathcal{H})$ as $n \to \infty$, then there exists a subsequence $\{S_2 y_{n(j)}\}$ of $\{S_2 y_n\}$ such that $\eta(S_2 y_{n(j)}, S_2 u) \geqslant 1$ for all k.
(iv) $p(x, x) = 0$.

Then, S_1 and S_2 have a coincidence point.

Proof. Let y_0 be any arbitrary point on $\mathcal{H}$. Owing to first assumption, we have

$\eta(S_2y_0, S_1y_0) \geqslant 1$. Also, $S_1\mathcal{H} \subseteq S_2\mathcal{H}$. So, we can find a point $y_1 \in \mathcal{H}$ so that $S_1y_0 = S_2y_1$ and $y_2 \in \mathcal{X}$ so that $S_1y_1 = S_2y_2$. With the assistance of mathematical induction, we can find a sequence $z_n \in \mathcal{H}$ so that

$$z_n = S_1y_n = S_2y_{n+1}. \tag{2}$$

Owing to second assumption, we have $\eta(S_2y_0, S_1y_0) = \eta(S_2y_0, S_2y_1) \geqslant 1$, which implies that $\eta(S_1y_0, S_1y_1) = \eta(S_2y_1, S_2y_2) \geqslant 1$. Again applying the process of induction, we acquire

$$\eta(S_2y_n, S_2y_{n+1}) \geqslant 1. \tag{3}$$

Therefore, $\eta(z_{n-1}, z_n) \geqslant 1$. If there exist $n \in \mathcal{N}$ such that $z_{n-1} = z_n$, then (2) yields that $S_2y_{n+1} = S_1y_{n+1}$. Thus, S_1 and S_2 have a coincident point at y_{n+1}, which completes the proof.

Let $z_{n-1} \neq z_n$ for every $n \in \mathcal{N}$. On account of (1) and (3), we obtain

$$\begin{aligned}
p(z_n, z_{n+1}) &= p(S_2y_{n+1}, S_2y_{n+2}) \\
&= p(S_1y_n, S_1y_{n+1}) \\
&\leqslant \eta(S_2y_n, S_2y_{n+1})p(S_1y_n, S_1y_{n+1}) \\
&\leqslant \chi(\max\left\{p(S_2y_n, S_2y_{n+1}), \frac{p(S_2y_n, S_1y_n) + p(S_2y_{n+1}, S_1y_{n+1})}{2},\right. \\
&\qquad \left.\frac{p(S_2y_n, S_1y_{n+1}) + p(S_2y_{n+1}, S_1y_n)}{2}\right\}) \\
&= \chi(\max\left\{p(z_{n-1}, z_n), \frac{p(z_{n-1}, z_n) + p(z_n, z_{n+1})}{2},\right. \\
&\qquad \left.\frac{p(z_{n-1}, z_{n+1}) + p(z_n, z_n)}{2}\right\}).
\end{aligned}$$

But,

$$\frac{p(z_{n-1}, z_{n+1})}{2} \leqslant \max\{p(z_{n-1}, z_n), p(z_n, z_{n+1})\}.$$

Therefore,

$$p(z_n, z_{n+1}) \leqslant \chi(\max\{p(z_{n-1}, z_n), p(z_n, z_{n+1})\}). \tag{4}$$

If $\max\{p(z_{n-1}, z_n), p(z_n, z_{n+1})\} = p(z_n, z_{n+1})$, then $p(z_{n-1}, z_n) \leqslant p(z_n, z_{n+1})$. Using (4), we get $p(z_n, z_{n+1}) \leqslant \chi(p(z_n, z_{n+1}))$. Using property of comparison function, $\chi(p(z_n, z_{n+1})) < p(z_n, z_{n+1})$. So, $p(z_n, z_{n+1}) < p(z_n, z_{n+1})$, which is a contradiction. So,

$$\max\{p(z_{n-1}, z_n), p(z_n, z_{n+1})\} = p(z_{n-1}, z_n).$$

Thus,

$$p(z_n, z_{n+1}) < \chi(p(z_{n-1}, z_n)). \tag{5}$$

Using property of comparison function, $\chi(p(z_{n-1}, z_n)) < p(z_{n-1}, z_n)$. Therefore, $p(z_n, z_{n+1}) < p(z_{n-1}, z_n)$. Hence, we conclude that the sequence $\{p(z_{n-1}, z_n)\}$ is non-decreasing and bounded from below by zero. Consequently, there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} p(z_{n-1}, z_n) = r \geqslant 0.$$

We claim that $r = 0$. Repeating the process inductively in (5), we acquire

$$p(z_n, z_{n+1}) < \chi^n(p(z_0, z_1)) \tag{6}$$

for each $n \in \mathcal{N}$. Taking Ω_2 into account, we obtain $\lim_{n\to\infty} \chi^n(p(z_0, z_1)) = 0$. Therefore, $\lim_{n\to\infty} p(z_{n-1}, z_n) = r = 0$. Next, we want to show that $\{z_n\}$ is a cauchy sequence. Let $n, q \in \mathcal{N}$ such that $q > n$. With the assistance of (p_1), we have

$$\begin{aligned} p(z_n, z_q) &\leqslant p(z_n, z_{n+1}) + p(z_{n+1}, z_{n+2}) + \ldots + p(z_{q-1}, z_q) \\ &\leqslant \sum_{p=n}^{q-1} \chi^p(p(z_0, z_1)) \\ &\leqslant \sum_{p=n}^{\infty} \chi^p(p(z_0, z_1)). \end{aligned}$$

Taking Ω_2 into account, we obtain

$$\lim_{n\to\infty} \sum_{p=n}^{\infty} \chi^p(p(z_0, z_1)) = 0. \tag{7}$$

On account of Lemma 1.8, we get $\{z_n\}$ is a cauchy sequence in $(\mathcal{X}, \sigma)$. Due to completeness of $(\mathcal{H}, \sigma)$, we can find a point $\rho \in \mathcal{H}$ such that $\lim_{n\to\infty} S_2 y_n = S_2\rho$. We assert that S_1 and S_2 have a coincident point. Now, from (i) of Theorem 2.2, we acquire

$$\eta(S_2 y_n, S_2\rho) \geqslant 1. \tag{8}$$

Using Definition 1.7, we have

$$p(S_2 y_n, S_2\rho) \leqslant \lim_{r\to\infty} \inf p(S_2 y_n, S_2 y_r) = \kappa_n.$$

With the assistance of (7), we have $\lim_{r\to\infty} \kappa_n = 0$. Thus,

$$\lim_{n\to\infty} p(S_2 y_n, S_2\rho) = 0. \tag{9}$$

Using Definition 1.7, we get

$$p(S_2 y_{n+1}, S_1\rho) \leqslant \liminf p(S_2 y_{n+1}, S_1 y_n) < \varepsilon.$$

Thus, $\lim_{n\to\infty}(S_2 y_{n+1}, S_1\rho) = 0$. On account of triangle inequality, we acquire

$$p(S_2 y_n, S_1\rho) \leqslant p(S_2 y_n, S_2 y_{n+1}) + p(S_2 y_{n+1}, S_1\rho) = p(z_{n-1}, z_n) + p(S_2 y_{n+1}, S_1\rho).$$

On account of (5), we acquire

$$\lim_{n\to\infty} p(S_2 y_n, S_1\rho) = 0. \tag{10}$$

On account of (9), (10) and Lemma 1.8, we have

$$S_1\rho = S_2\rho, \tag{11}$$

which indicates that $\rho \in \mathcal{C}(S_1, S_2)$. ■

***Example* 2.3** Consider $\mathcal{H} = [0, +\infty)$ associated with the metric

$$\sigma(\Omega, \mho) = \begin{cases} 0, & \text{if } \Omega = \mho, \\ \max\{\Omega, \mho\}, & \text{otherwise,} \end{cases}$$

with $p(\Omega, \mho) = \mho$ for all $\Omega, \mho \in \mathcal{H}$. Define the self mappings S_1 and S_2 by $S_1(\ell) = \ell$ and $S_2(\ell) = 4\ell$ for $\ell \in \mathcal{H}$ with $\chi(t) = \frac{t}{2}$. Now, we formalize the mapping $\eta : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ as

$$\eta(\Theta_1, \Theta_2) = \begin{cases} 1, & \text{if } (\Theta_1, \Theta_2) \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Now, we exhibit that (S_1, S_2) is generalized (η, χ, p) contractive mapping.
If $\Omega, \mho \in [0, 1]$, then we get

$$\eta(\Omega, \mho) p(S_1\Omega, S_1\mho) = p(S_1\Omega, S_1\mho) = S_1\mho = \mho.$$

Now,

$$\begin{aligned} \chi(P(S_2\Omega, S_2\mho)) &= \frac{P(S_2\Omega, S_2\mho)}{2} \\ &= \frac{1}{2}\max\left\{p(S_2\Omega, S_2\mho), \frac{p(S_2\Omega, S_1\Omega) + p(S_2\mho, S_1\mho)}{2},\right. \\ &\left.\frac{p(S_2\Omega, S_1\mho) + p(S_2\mho, S_1\Omega)}{2}\right\} \\ &= \frac{1}{2}\max\left\{S_2\mho, \frac{S_1\Omega + S_1\mho}{2}, \frac{S_1\mho + S_1\Omega}{2}\right\} \\ &= \frac{1}{2}\max\left\{4\mho, \frac{\Omega + \mho}{2}\right\} = 2\mho. \end{aligned}$$

Thus,

$$\eta(S_2\Omega, S_2\mho) p(S_1\Omega, S_1\mho) \leqslant \chi(P(S_2\Omega, S_2\mho).$$

If $\Omega \notin [0, 1]$ or $\mho \notin [0, 1]$, then $\eta(\Omega, \mho) = 0$. Consequently,

$$\eta(S_2\Omega, S_2\mho) p(S_1\Omega, S_1\mho) \leqslant \chi(P(S_2\Omega, S_2\mho)$$

holds trivially. Hence, (S_1, S_2) is generalized (η, χ, p) contractive mapping. All the assertions of Theorem 2.2 are fulfilled. Also, $0 \in \mathcal{CF}(S_1, S_2)$.

Theorem 2.4 In addition to first three assumptions of Theorem 2.2, imagine that $\eta(S_2\rho, S_2\rho) \geqslant 1$ for every $\rho \in \mathcal{C}(S_1, S_2)$. Then, $p(S_2\rho, S_2\rho) = 0$.

Proof. Let us imagine that there exist $\rho \in \mathcal{C}(S_1, S_2)$ such that $p(S_2\rho, S_2\rho) > 0$. Also,

$$\eta(S_2\rho, S_2\rho) \geqslant 1. \tag{12}$$

With the aid of (1), (11) and (12), we acquire

$$\begin{aligned} p(S_2\rho, S_2\rho) &\leqslant \eta(S_2\rho, S_2\rho)p(S_1\rho, S_1\rho) \\ &\leqslant \chi(\max\left\{p(S_2\rho, S_2\rho), \frac{p(S_2\rho, S_1\rho) + p(S_2\rho, S_1\rho)}{2}, \right. \\ &\left. \frac{p(S_2\rho, S_1\rho) + p(S_2\rho, S_1\rho)}{2}\right\}) \\ &= \chi(\max\{p(S_2\rho, S_2\rho), p(S_2\rho, S_1\rho)\} \\ &= \chi(p(S_2\rho, S_2\rho) \\ &< p(S_2\rho, S_2\rho), \end{aligned}$$

which is a counterstatement. Consequently, $p(S_2\rho, S_2\rho) = 0$. ■

Theorem 2.5 Let $(\mathcal{H}, \sigma)$ be a metric space, p be a w-distance and S_1, S_2 be the self maps such that $S_1\mathcal{H} \subseteq S_2\mathcal{H}$. Also, let (S_1, S_2) be generalized (η, χ, p) contractive map which fulfils the following conditions:

(i) there exists $x_0 \in \mathcal{H}$ such that $\alpha(S_2x_0, S_1x_0) \geqslant 1$;
(ii) S_1 is $\eta-$admissible with respect to S_2;
(iii) S_1 is continuous.

Then, S_1 and S_2 have a coincidence point.

Proof. Mimicking the steps of Theorem 2.2, we can find a point $\rho \in \mathcal{X}$ such that $\lim_{n\to\infty} S_2y_n = S_2\rho$. Since S_1 is continuous, we acquire

$$S_2\rho = \lim_{n\to\infty} S_2y_{n+1} = \lim_{n\to\infty} S_1y_n = S_1(\lim_{n\to\infty} y_n) = S_1\rho,$$

which indicates that $\rho \in \mathcal{C}(S_1, S_2)$. ■

Theorem 2.6 Let $(\mathcal{H}, \sigma)$ be a complete metric space, p be a w-distance and S_1, S_2 be the self maps such that $S_1\mathcal{H} \subseteq S_2\mathcal{H}$. Also, let (S_1, S_2) be generalized (η, χ, p) contractive map which fulfils the following conditions:

(i) There exists $y_0 \in \mathcal{H}$ such that $\eta(S_2y_0, S_1y_0) \geqslant 1$;
(ii) S_1 is $\eta-$admissible with respect to S_2;
(iii) For all $s, y \in \mathcal{H}$ with $s \neq S_1s$, $\inf\{p(y, s) + p(y, S_1y)\} > 0$.

Then, S_1 and S_2 have a coincidence point. Further, if $\eta(S_2\rho, S_2\rho) \geqslant 1$ for every $\rho \in \mathcal{C}(S_1, S_2)$, then $p(S_2\rho, S_2\rho) = 0$.

Proof. In the light of Theorem 2.2, we get that $\{S_2y_n\}$ is Cauchy sequence. But $(\mathcal{H}, \sigma)$ is complete. Thus, we can find a point $\rho \in \mathcal{H}$ such that $S_2y_n \to S_2\rho$, when $n \to \infty$. Let us imagine that $S_2\rho \neq S_1\rho$. Now, utilizing the given assumption (iii), we acquire

$$\begin{aligned} 0 &\leqslant \inf\{p(S_2y, S_2\rho) + p(S_2y, S_1y) \\ &\leqslant \inf\{p(S_2y_n, S_2\rho) + p(S_2y_n, S_2y_{n+1}) \\ &= 0, \end{aligned}$$

which is a counterstatement. Consequently, $S_2\rho = S_1\rho$. Proceeding with the same strategy of Theorem 2.2, we confirm that $p(S_2\rho, S_2\rho) = 0$. ■

Theorem 2.7 In conjuction with the assumptions of Theorem 2.2, assume that for all $\rho_1, \rho_2 \in \mathcal{C}(S_1, S_2)$, there occur $\rho_3 \in \mathcal{H}$ in a manner that $\eta(S_2\rho_1, S_2\rho_3) \geqslant 1, \eta(S_2\rho_2, S_2\rho_3) \geqslant 1$ and S_1, S_2 commute at $\rho \in \mathcal{C}(S_1, S_2)$. Then, there occur a unique $\rho \in \mathcal{H}$ such that $\rho \in \mathcal{CF}(S_1, S_2)$.

Proof. We assert that if $\rho_1, \rho_2 \in \mathcal{C}(S_1, S_2)$, then $S_2\rho_1 = S_2\rho_2$.
With the assistance of given assumption, there occur $\rho_3 \in \mathcal{H}$ such that

$$\eta(S_2\rho_3, S_2\rho_1) \geqslant 1, \eta(S_2\rho_3, S_2\rho_2) \geqslant 1. \tag{13}$$

With the aid of η-admissibility of (S_1, S_2) and (13), we acquire

$$\eta(S_1^n\rho_3, S_2\rho_1) \geqslant 1, \eta(S_1^n\rho_3, S_2\rho_2) \geqslant 1. \tag{14}$$

In the light of (1) and (14), we acquire

$$\begin{aligned} p(S_1^{n+1}\rho_3, S_2\rho_1) &= p(S_1(S_1^n\rho_3), S_1\rho_1) \\ &\leqslant \eta(S_2(S_1^n\rho_3), S_2\rho_1)p(S_1(S_1^n\rho_3), S_1\rho_1) \\ &\leqslant \chi \max\{p(S_2(S_1^n\rho_3), S_2\rho_1), \frac{p(S_2(S_1^n\rho_3), S_1(S_1^n\rho_3) + p(S_2\rho_1, S_1\rho_1)}{2}, \\ &\frac{p(S_2(S_1^n\rho_3), S_1\rho_1) + p(S_2\rho_1, S_1(S_1^n\rho_3)}{2}\}, \end{aligned}$$

for every $n \in \mathcal{N}$. With the assistance of induction process, we have

$$p(S_1^{n+1}\rho_3, S_2\rho_1) \leqslant \chi^n(p(S_2(S_1\rho_3), S_2\rho_1)). \tag{15}$$

Likewise, we confirm that

$$p(S_1^{n+1}\rho_3, S_2\rho_2) \leqslant \chi^n(p(S_2(S_1\rho_3), S_2\rho_2)). \tag{16}$$

From Definition 1.1, we acquire

$$\lim_{n\to\infty} \chi^n(p(S_2(S_1\rho_3), S_2\rho_1)) = 0 \tag{17}$$

and

$$\lim_{n\to\infty} \chi^n(p(S_2(S_1\rho_3), S_2\rho_2)) = 0. \tag{18}$$

With the aid of (15)-(18) and Lemma 1.8, we acquire $S_2\rho_1 = S_2\rho_2$. Now, we exhibit the presence of a unique $\mathcal{CF}$. Let $\rho_1 \in \mathcal{C}(S_1, S_2)$, which indicates that $S_2\rho_1 = S_1\rho_1$. Due to commutativity of S_1 and S_2 at $\rho_1 \in \mathcal{C}(S_1, S_2)$, we acquire

$$S_2^2\rho_1 = S_2S_1\rho_1 = S_1S_2\rho_1. \tag{19}$$

Let $S_2\rho_1 = \rho$. With the aid of (19), we acquire $S_2\rho = S_1\rho$. Consequently, $\rho \in \mathcal{C}(S_2, S_1)$. Thus, $S_2\rho_1 = S_2\rho = \rho = S_1\rho$.Hence, $\rho \in \mathcal{CF}(S_2, S_1)$. Next, we exhibit that $\mathcal{CF}$ is unique. Let us imagine that ρ^* is another common fixed point of S_1 and S_2. Thus, $\rho^* \in \mathcal{C}(S_1, S_2)$. Now, we conclude that $\rho^* = S_2\rho^* = S_2\rho = \rho$, which indicates that $\mathcal{CF}$ is unique. ■

Corollary 2.8 [4] Let $(\mathcal{H}, \sigma)$ be a complete metric space associated with a w-distance p. Let $S_1 : \mathcal{H} \to \mathcal{H}, \eta : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ and $\chi \in \Omega$ fulfils the following assertions:

(i) $\eta(x, y)p(S_1x, S_1y) \leqslant \chi(p(x, y)$;
(ii) There exists $y_0 \in \mathcal{H}$ such that $\eta(y_0, S_1y_0) \geqslant 1$;
(iii) S_1 is $\eta-$admissible function;
(iv) If S_1 is continuous or $\{y_n\}$ is a sequence in $\mathcal{H}$ such that $\eta(y_n, y_{n+1}) \geqslant 1$ for all n and $y_n \to u \in \mathcal{H}$ as $n \to \infty$, then $\eta(y_n, u) \geqslant 1$ for all n.

Then, there exist $u \in \mathcal{H}$ such that $S_1u = u$.

Proof. Result follows from Theorem 2.2 by inserting S_2 as identity map. ■

Corollary 2.9 Let $(\mathcal{H}, \sigma)$ be a complete metric space. $S_1 and S_2$ be the self maps such that $S_1\mathcal{H} \subseteq S_2\mathcal{H}$, $\eta : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ and $\chi \in \Omega$. Let (S_1, S_2) is contractive map, which fulfils the following condition:

$$\eta(S_2x, S_2y)d(S_1x, S_1y) \leqslant \chi(M(S_2x, S_2y),$$

where

$$M(S_2x, S_2y) = \max\left\{d(S_2x, S_2y), \frac{d(S_2x, S_1x) + d(S_2y, S_1y)}{2}, \frac{d(S_2x, S_1y) + d(S_2y, S_1x)}{2}\right\}.$$

Suppose that

(i) There exists $y_0 \in \mathcal{H}$ such that $\eta(S_2y_0, S_1y_0) \geqslant 1$;
(ii) S_1 is $\eta-$admissible with respect to S_2;
(iii) If $\{S_2y_n\}$ is a sequence in $\mathcal{H}$ such that $\eta(S_2y_n, S_2y_{n+1}) \geqslant 1$ for all n and $S_2y_n \to S_2u \in S_2(\mathcal{H})$ as $n \to \infty$, then there exists a subsequence $\{S_2y_{n(j)}\}$ of $\{S_2y_n\}$such that $\eta(S_2y_{n(j)}, S_2u) \geqslant 1$ for all k.

Then, S_1 and S_2 have a coincidence point.

Proof. Result follows from Theorem 2.2 by inserting p = d. ■

3. Application to the integral equation

In this section, we give an application of the integral equation. Let

$$C[a, b] = \{f | f : [a, b] \to R \,\text{is a continuous function}\}.$$

Theorem 3.1 Let us consider the non-linear Fredholm integral equation

$$S_1x(t) = S_2(t) + \int_a^b F(t, s, x(s))ds, \tag{20}$$

for some $a, b \in R$ with $a < b, S_2 : [a, b] \to R$ and $H : [a, b]^2 \times R \to R$ be two continuous maps. Also, imagine that the subsequent properties hold:
(i) $S_1 : C[a, b] \to C[a, b]$ is a continuous mapping;
(ii) There exists $\chi \in \Omega$ satisfying

$$\begin{aligned} \mid F(t,s,x(s)) \mid + \mid F(t,s,y(s)) \mid \leqslant \frac{1}{b-a}\chi(\max\Big\{ & \mid S_2x(t) \mid + \mid S_2y(t) \mid, \\ & \frac{(\mid S_2x(t) \mid + \mid S_1x(t) \mid) + (\mid S_2y(t) \mid + \mid S_1y(t) \mid)}{2}, \\ & \frac{(\mid S_2x(t) \mid + \mid S_1y(t) \mid) + (\mid S_2y(t) \mid + \mid S_1x(t) \mid)}{2}\Big\} \\ & - 2 \mid S_2(t) \mid \end{aligned}$$

for all $t, s \in [a, b]$.
Then, the non-linear Fredholm integral equation (20) owns a unique solution in $C[a, b]$.

Proof. We know that $C[a, b]$ is complete with respect to the metric $\sigma : C[a, b] \times C[a, b] \longrightarrow R^+$ defined as $\sigma(x, y) = \sup_{t \in [a,b]} \mid x(t) - y(t) \mid$, where $x, y \in C[a, b]$. Let $p : C[a, b] \times C[a, b] \longrightarrow R^+$ be defined by $p(x, y) = \sup_{t \in [a,b]} \mid x(t) \mid + \sup_{t \in [a,b]} \mid y(t) \mid$, where $x, y \in C[a, b]$. Clearly, p is a w-distance on $C[a, b]$. Now,

$$\begin{aligned} \mid S_1x(t) \mid + \mid S_1y(t) \mid &= \mid S_2(t) + \int_a^b F(t,s,x(s))ds \mid + \mid S_2(t) + \int_a^b F(t,s,y(s))ds \mid \\ &\leqslant \mid S_2(t) \mid + \mid \int_a^b F(t,s,x(s))ds \mid + \mid S_2(t) \mid + \mid \int_a^b F(t,s,y(s))ds \mid \\ &\leqslant 2 \mid S_2(t) \mid + \mid \int_a^b F(t,s,x(s))ds \mid + \mid \int_a^b F(t,s,y(s))ds \mid \\ &\leqslant 2 \mid S_2(t) \mid + \int_a^b \mid F(t,s,x(s)) \mid ds + \int_a^b \mid F(t,s,y(s)) \mid ds \\ &\leqslant 2 \mid S_2(t) \mid + \int_a^b (\mid F(t,s,x(s)) \mid + \mid F(t,s,y(s)) \mid)ds \\ &\leqslant 2 \mid S_2(t) \mid + \int_a^b \Big(\frac{1}{b-a}\chi(\max\Big\{ \mid S_2x(t) \mid + \mid S_2y(t) \mid, \\ &\frac{(\mid S_2x(t) \mid + \mid S_1x(t) \mid) + (\mid S_2y(t) \mid + \mid S_1y(t) \mid)}{2}, \\ &\frac{(\mid S_2x(t) \mid + \mid S_1y(t) \mid) + (\mid S_2y(t) \mid + \mid S_1x(t) \mid)}{2}\Big\} - 2 \mid S_2(t) \mid \Big)ds \\ &= 2 \mid S_2(t) \mid + \Big(\frac{1}{b-a}\chi(\max\Big\{ \mid S_2x(t) \mid + \mid S_2y(t) \mid, \end{aligned}$$

$$\frac{(\mid S_2x(t)\mid + \mid S_1x(t)\mid) + (\mid S_2y(t)\mid + \mid S_1y(t)\mid)}{2},$$
$$\frac{(\mid S_2x(t)\mid + \mid S_1y(t)\mid) + (\mid S_2y(t)\mid + \mid S_1x(t)\mid)}{2}\Bigg\} - 2\mid S_2(t)\mid \Bigg) \int_a^b ds$$
$$= \chi\Bigg(\max\Bigg\{ \mid S_2x(t)\mid + \mid S_2y(t)\mid,$$
$$\frac{(\mid S_2x(t)\mid + \mid S_1x(t)\mid) + (\mid S_2y(t)\mid + \mid S_1y(t)\mid)}{2},$$
$$\frac{(\mid S_2x(t)\mid + \mid S_1y(t)\mid) + (\mid S_2y(t)\mid + \mid S_1x(t)\mid)}{2}\Bigg\}\Bigg)$$
$$\leqslant \chi\Bigg(\max\Bigg\{ p(S_2x, S_2y), \frac{p(S_2x, S_1x) + p(S_2y, S_1y)}{2},$$
$$\frac{p(S_2x, S_1y) + p(S_2y, S_1x)}{2}\Bigg\}\Bigg)$$
$$= \chi(P(S_2x, S_2y))$$

for all $x, y \in C[a, b]$ and $t \in [0, \infty]$. Consequently,

$$\sup_{t\in[a,b]} \mid S_1x(t)\mid + \sup_{t\in[a,b]} \mid S_1y(t)\mid \leqslant \chi(P(S_2x, S_2y)),$$

which indicates that $p(S_1x, S_1y) \leqslant \chi(P(S_2x, S_2y))$. Now, we formalize the mapping $\eta : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ as

$$\eta(\Theta_1, \Theta_2) = \begin{cases} 1, & \text{if } (\Theta_1, \Theta_2) \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Thus, (S_1, S_2) is generalized (η, χ, p) contractive map. Therefore, by Theorem 2.1, the non-linear Fredholm integral equation (20) owns a solution. ■

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