Journal of
Linear and Topological Algebra
Vol. 09, *No.* 03, 2020, 221- 230

Some improvements of numerical radius inequalities via Specht's ratio

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Received 31 May 2020; Revised 14 September 2020; Accepted 20 September 2020.
Communicated by Mohammad Sadegh Asgari

Abstract. We obtain some inequalities related to the powers of numerical radius inequalities of Hilbert space operators. Some results that employ the Hermite–Hadamard inequality for vectors in normed linear spaces are also obtained. We improve and generalize some inequalities with respect to Specht's ratio. Among them, we show that, if $A, B \in \mathcal{B}(\mathcal{H})$ satisfy in some conditions, it follows that

$$\omega^2(A^*B) \leqslant \frac{1}{2S(\sqrt{h})}\left\||A|^4 + |B|^4\right\| - \inf_{\|x\|=1} \frac{1}{4S(\sqrt{h})}\left(\left\langle\left(A^*A - B^*B\right)x, x\right\rangle\right)^2$$

for some $h > 0$, where $\|\cdot\|$, $\omega(\cdot)$ and $S(\cdot)$ denote the usual operator norm, numerical radius and the Specht's ratio, respectively.

Keywords: Positive operators, numerical radius, Specht's ratio, Hermite–Hadamard inequality.

2010 AMS Subject Classification: 47A12, 47A30, 47A63.

1. Introduction and preliminaries

Let $B(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle ., . \rangle$. We recall some definitions and concepts from [11].

An operator A in $B(\mathcal{H})$ is positive, denoted by $A \geqslant 0$, if A is self-adjoint ($A = A^*$) and $\langle Ax, x\rangle \geqslant 0$ for every $x \in \mathcal{H}$; equivalently, A is positive if and only if $A = B^*B$ for some operator $B \in B(\mathcal{H})$. In particular, for some scalars m and M, we write $mI \leqslant A \leqslant MI$ if $m \leqslant \langle Ax, x\rangle \leqslant M$ for every $x \in \mathcal{H}$, $\|x\| = 1$, where I stands for the identity operator of

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

$\mathcal{B}(\mathcal{H})$. The absolute value of A is denoted by $|A| = (A^*A)^{\frac{1}{2}}$. Note that for a self-adjoint operator A, $mI \leqslant A \leqslant MI$ if and only if $sp(A) \subset [m, M]$. Also the set of all positive invertible operators is denoted by $\mathcal{B}^+(\mathcal{H})$.

For an operator $A \in \mathcal{B}(\mathcal{H})$, the usual operator norm is defined by $\|A\| = \sup \|Ax\|$ for every $x \in \mathcal{H}$, $\|x\| = 1$ and the numerical radius of A is given by $\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$. The numerical radius satisfies

$$\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|. \tag{1}$$

The second inequality in (1) has been improved in [7, Theorem 1] as follows:

$$\omega(A) \leqslant \frac{1}{2}\||A| + |A^*|\| \leqslant \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}) \tag{2}$$

for every operator $A \in \mathcal{B}(\mathcal{H})$. The left hand of inequality (2) was extended in [4, Theorem 1] as follows:

$$\omega^r(A) \leqslant \frac{1}{2}\left\||A|^{2r\nu} + |A^*|^{2r(1-\nu)}\right\|, \quad r \geqslant 1, 0 < \nu < 1, \tag{3}$$

which this inequality will be improved in the end of this paper.

Dragomir in [2, Theorem 1], proved the following inequality by the product of two operators:

$$\omega^r(B^*A) \leqslant \frac{1}{2}\left\||A|^{2r} + |B|^{2r}\right\|, \quad r \geqslant 1. \tag{4}$$

By using of operator inequality, we improve the inequality (4).

Let $A \in \mathcal{B}^+(\mathcal{H})$ and let B be a positive operator in $B(\mathcal{H})$. The operator ν-weighted geometric mean of A and B is defined by $A\natural_\nu B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}$, which $\nu \in [0, 1]$.

Recall that the Specht's ratio [5, 13] was defined by $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ for a positive real number $h \neq 1$, and it has some properties as follows:

(i) $S(1) = 1$ and $S(h) = S(\frac{1}{h}) > 1$ for $h > 0$.
(ii) $S(h)$ is a monotone increasing function on $(1, \infty)$.
(iii) $S(h)$ is a monotone decreasing function on $(0, 1)$.

Lemma 1.1 [6, Theorem 1] For $a, b > 0$ and $\nu \in [0, 1]$, it follows that $(1-\nu)a + \nu b \geqslant S((\frac{b}{a})^r)a^{1-\nu}b^\nu$, where $r = \min\{\nu, 1-\nu\}$ and $S(.)$ is the Specht's ratio.

Theorem 1.2 [6, Theorem 2] Let A and B be two positive operators and let m, m', M, M' be positive real numbers satisfying the following conditions (i) or (ii):

(i) $0 < m'I \leqslant A \leqslant mI < MI \leqslant B \leqslant M'I$,
(ii) $0 < m'I \leqslant B \leqslant mI < MI \leqslant A \leqslant M'I$,

with $h = \frac{M}{m}$. Then

$$\begin{aligned}(1-\nu)A + \nu B \geqslant S(h^r)A\natural_\nu B \geqslant A\natural_\nu B &\geqslant S(h^r)\{(1-\nu)A^{-1} + \nu B^{-1}\}^{-1} \\ &\geqslant \{(1-\nu)A^{-1} + \nu B^{-1}\}^{-1},\end{aligned}$$

where $\nu \in [0, 1]$, $r = \min\{\nu, 1-\nu\}$, and $S(.)$ is the Specht's ratio.

Remark 1 *Note that if $A = aI$, $B = bI$, $\nu = \frac{1}{2}$, and $r = \frac{1}{2}$ in Theorem 1.2, then*

$$S(\sqrt{h})\sqrt{ab} \leqslant \tfrac{a+b}{2},$$

where $S(.)$ is the Specht's ratio.

The goal of this paper is to establish considerable generalizations of these inequalities that are based on some classical convexity inequalities for non–negative real numbers and some operator inequalities. Also, by using of operator inequality and Specht's ratio, we improve some numerical radius inequalities.

2. Main results

In this section, we state some useful lemmas that we need them for improving and generalizing some inequalities. The first lemma is a generalized form of the mixed Schwarz inequality, which was proved by Kittaneh [8, Theorem 1].

Lemma 2.1 Let A be an operator in $B(\mathcal{H})$ and let f and g be nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$|\langle Ax, y\rangle| \leqslant \|f(|A|)x\|\|g(|A^*|)y\|$$

for all $x, y \in \mathcal{H}$.

The well-known Hermite–Hadamard inequalities state that for a convex function $f : J \to \mathbb{R}$, it follows that

$$f\Big(\frac{a+b}{2}\Big) \leqslant \int_0^1 f(ta + (1-t)b)dt \leqslant \frac{f(a) + f(b)}{2}, \tag{5}$$

for every a, b in real interval J (see [1]).

Let f be a convex function on a real interval J containing sp(A), where A is a self-adjoint operator. Then for every $x \in \mathcal{H}$, $\|x\| = 1$ the inequality

$$f(\langle Ax, x\rangle) \leqslant \langle f(A)x, x\rangle \tag{6}$$

is an operator version of the Jensen inequality due to Mond and Pečarić [9, Theorem 1].

Utilizing the following lemma, leads to the improvement of some inequalities that prove by other mathematicians.

Lemma 2.2 [10, page 5] Let f be a twice differentiable on $[a, b]$. If f is convex such that $f'' \geqslant \lambda := \min_{x\in[a,b]} f(x) > 0$. Then

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{f(a) + f(b)}{2} - \frac{1}{8}\lambda(b-a)^2. \tag{7}$$

Theorem 2.3 Let $A, B, X \in \mathcal{B}(\mathcal{H})$, let the continuous functions f and g be non-negative functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, and let k be a non-negative increasing convex function on $[0, \infty)$ and twice differentiable such that $k'' \geqslant \lambda > 0$, with $k(0) = 0$. Also let the positive real numbers m, m', M, M' satisfy one of the following conditions:

(i) $0 < m' \leqslant \langle B^* f^2(|X|)Bx, x\rangle \leqslant m < M \leqslant \langle A^* g^2(|X^*|)Ax, x\rangle \leqslant M'$,
(ii) $0 < m' \leqslant \langle A^* f^2(|X|)Ax, x\rangle \leqslant m < M \leqslant \langle B^* g^2(|X^*|)Bx, x\rangle \leqslant M'$,

with $h = \frac{M}{m}$. Then

$$k\big(\omega(A^*XB)\big) \leqslant \frac{1}{2S(\sqrt{h})}\big\|k(B^* f^2(|X|)B) + k(A^* g^2(|X^*|)A)\big\| - \inf_{\|x\|=1} \xi(x), \tag{8}$$

whenever

$$\xi(x) = \frac{1}{8S(\sqrt{h})}\lambda\big(\big\langle\big(A^* g^2(|X^*|)A - B^* f^2(|X|)B\big)x, x\big\rangle\big)^2,$$

where $S(.)$ is the Specht's ratio and $\lambda > 0$.

Proof. Using Lemma 2.1, we get

$$|\langle A^*XBx, x\rangle| = |\langle XBx, Ax\rangle| \leqslant \sqrt{\langle B^* f^2(|X|)Bx, x\rangle\langle A^* g^2(|X^*|)Ax, x\rangle}. \tag{9}$$

Now, Remark *1* implies that

$$\begin{aligned}\sqrt{\langle B^* f^2(|X|)Bx, x\rangle\langle A^* g^2(|X^*|)Ax, x\rangle} &\leqslant \frac{1}{2S(\sqrt{h}))}\Big(\langle B^* f^2(|X|)Bx, x\rangle + \langle A^* g^2(|X^*|)Ax, x\rangle\Big)\\ &= \frac{1}{2S(\sqrt{h})}\Big(\langle(B^* f^2(|X|)B + A^* g^2(|X^*|)A)x, x\rangle\Big).\end{aligned}$$

It follows from the last inequality and (9) that

$$|\langle A^*XBx, x\rangle| \leqslant \tfrac{1}{2S(\sqrt{h})}\Big(\big\langle(B^* f^2(|X|)B + A^* g^2(|X^*|)A)x, x\big\rangle\Big).$$

Then we have

$$\begin{aligned}k(|\langle A^*XBx, x\rangle|) &\leqslant k\Big(\frac{1}{2S(\sqrt{h})}\big(\langle B^* f^2(|X|)Bx, x\rangle + \langle A^* g^2(|X^*|)Ax, x\rangle\big)\Big)\\ &\leqslant \frac{1}{S(\sqrt{h})}k\Big(\frac{\langle B^* f^2(|X|)Bx, x\rangle + \langle A^* g^2(|X^*|)Ax, x\rangle}{2}\Big) \qquad (10)\\ &\leqslant \frac{1}{S(\sqrt{h})}\Big[\frac{k(\langle B^* f^2(|X|)Bx, x\rangle) + k(\langle A^* g^2(|X^*|)Ax, x\rangle)}{2}\\ &\quad - \frac{1}{8}\lambda\big(\langle A^* g^2(|X^*|)Ax, x\rangle - \langle B^* f^2(|X|)Bx, x\rangle\big)^2\Big] \quad \text{(by Lemma 2.2)}\\ &\leqslant \frac{1}{2S(\sqrt{h})}\Big[\big(\langle k(B^* f^2(|X|)B)x, x\rangle\big) + \big(\langle k(A^* g^2(|X^*|)A)x, x\rangle\big)\Big]\\ &\quad - \frac{1}{8S(\sqrt{h})}\lambda\big(\langle A^* g^2(|X^*|)Ax, x\rangle - \langle B^* f^2(|X|)Bx, x\rangle\big)^2 \text{ (by Lemma 2.1)}\\ &= \frac{1}{2S(\sqrt{h})}\Big[\big\langle\big(k(B^* f^2(|X|)B) + k(A^* g^2(|X^*|)A)\big)x, x\big\rangle\Big]\\ &\quad - \frac{1}{8S(\sqrt{h})}\lambda\big(\big\langle\big(A^* g^2(|X^*|)A - B^* f^2(|X|)B\big)x, x\big\rangle\big)^2.\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we reach

$$k\big(\omega(A^*XB)\big) \leqslant \tfrac{1}{2S(\sqrt{h})}\big\|k(B^*f^2(|X|)B) + k(A^*g^2(|X^*|)A)\big\| - \inf_{\|x\|=1} \xi(x),$$

whenever

$$\xi(x) = \frac{1}{8S(\sqrt{h})}\lambda\big(\big\langle\big(A^*g^2(|X^*|)A - B^*f^2(|X|)B\big)x, x\big\rangle\big)^2.$$

This inequality completes proof and imlies inequality (8).
Note that inequality (10) follows from that if k is a non-negative increasing convex function, with $k(0) = 0$ and $\alpha = \frac{1}{S(\sqrt{h})} \leqslant 1$ then $k(\alpha t) \leqslant \alpha k(t)$. ■

Shebrawi and Albadawi [12, Remark 2.10], for each $A, B, X \in \mathcal{B}(\mathcal{H})$, proved the following general numerical radius inequality:

$$\omega^r(A^*XB) \leqslant \frac{1}{2}\big\|\big(A^*|X^*|A\big)^r + \big(B^*|X|B\big)^r\big\|, \quad r \geqslant 1. \tag{11}$$

From inequality (11) and Theorem 2.3, we obtain the following inequality.

Corollary 2.4 Let the assumptions of Theorem 2.3 hold. By taking $k(t) = t^2$ on $[0, \infty)$, thus the required λ would be $'2'$.

(i) If $0 < m'I < B^*|X|B \leqslant mI < MI \leqslant A^*|X^*|A < M'I$ or $0 < m'I < A^*|X|A \leqslant mI < MI \leqslant B^*|X|B < M'I$ for positive real numbers m, m', M, M', then

$$\omega^2(A^*XB) \leqslant \frac{1}{2S(\sqrt{h})}\big\|\big(A^*|X^*|A\big)^2 + \big(B^*|X|B\big)^2\big\| - \inf_{\|x\|=1}\frac{1}{4S(\sqrt{h})}\big(\big\langle\big(A^*|X^*|A - B^*|X|B\big)x, x\big\rangle\big)^2,$$

which improves inequality (11) in especial conditions.

(ii) If $X = I$ holds in conditions of (i), then

$$\omega^2(A^*B) \leqslant \frac{1}{2S(\sqrt{h})}\Big\||A|^4 + |B|^4\Big\| - \inf_{\|x\|=1}\frac{1}{4S(\sqrt{h})}\big(\big\langle\big(A^*A - B^*B\big)x, x\big\rangle\big)^2,$$

which improves inequality (4) in especial conditions.

(iii) If $A = B = I$ holds in conditions of (i), then

$$\omega^2(X) \leqslant \frac{1}{2S(\sqrt{h})}\Big\||X^*|^2 + |X|^2\Big\| - \inf_{\|x\|=1}\frac{1}{4S(\sqrt{h})}\big(\big\langle\big(|X^*| - |X|\big)x, x\big\rangle\big)^2,$$

where $S(.)$ is the Specht's ratio and $\lambda > 0$.

In order to prove the following theorem, we need Lemmas 2.1 and 2.2. Recall that Dragomir provides an extension of Furuta's inequality as follows:

$$|\langle DCBAx, y\rangle|^2 \leqslant \langle A^*|B|^2Ax, x\rangle\langle D|C^*|^2D^*y, y\rangle, \tag{12}$$

for every $A, B, C, D \in B(\mathcal{H})$ and any vectors $x, y \in \mathcal{H}$. In (12), the equality holds if and only if the vectors BAx and C^*D^*y are linearly dependent in $\mathcal{H}$ (see [3]).

Theorem 2.5 Suppose that A, B, C, D in $B(\mathcal{H})$ are operators that f is a positive increasing operator convex function on $\mathbb{R}$ and also that f is twice differentiable such that $f'' \geqslant \lambda > 0$, with $f(0) = 0$. Let the positive real numbers m, m', M, M' satisfy one of the following conditions:

(i) $0 < m'I \leqslant A^*|B|^2A \leqslant mI \leqslant MI \leqslant D|C^*|^2D^* \leqslant M'I$,
(ii) $0 < m'I \leqslant D|C^*|^2D^* \leqslant mI \leqslant MI \leqslant A^*|B|^2A \leqslant M'I$,

with $h = \frac{M}{m}$. Then for every $x, y \in \mathcal{H}$, it follows that

$$f(|\langle DCBAx, y\rangle|) \leqslant \frac{1}{2S(\sqrt{h})}\big[\langle f(A^*|B|^2A)x, x\rangle + \langle f(D|C^*|^2D^*)y, y\rangle - \frac{1}{4}\lambda(\langle A^*|B|^2Ax, x\rangle - \langle D|C^*|^2D^*y, y\rangle)^2\big], \tag{13}$$

where $S(.)$ is the Specht's ratio and $\lambda > 0$.

Proof. Utilizing the monotonicity and convexity of increasing function f for the inequality (12), implies that

$$\begin{aligned}
f(|\langle DCBAx, y\rangle|) &\leqslant f\Big(\langle A^*|B|^2Ax, x\rangle^{\frac{1}{2}}\langle D|C^*|^2D^*y, y\rangle^{\frac{1}{2}}\Big)\\
&\leqslant f\Big(\frac{1}{2S(\sqrt{h})}\big(\langle A^*|B|^2Ax, x\rangle + \langle D|C^*|^2D^*y, y\rangle\big)\Big) \quad \text{(by Remark 1)}\\
&\leqslant \frac{1}{S(\sqrt{h})}f\Big(\frac{\langle A^*|B|^2Ax, x\rangle + \langle D|C^*|^2D^*y, y\rangle}{2}\Big) \qquad (14)\\
&\leqslant \frac{1}{S(\sqrt{h})}\Big[\frac{f(\langle A^*|B|^2Ax, x\rangle) + f(\langle D|C^*|^2D^*y, y\rangle)}{2}\\
&\quad - \frac{1}{8}\lambda(\langle A^*|B|^2Ax, x\rangle - \langle D|C^*|^2D^*y, y\rangle)^2)\Big] \quad \text{(by Lemma 2.2)}\\
&\leqslant \frac{1}{2S(\sqrt{h})}\big[\langle f(A^*|B|^2A)x, x\rangle + \langle f(D|C^*|^2D^*)y, y\rangle\\
&\quad - \frac{1}{4}\lambda(\langle A^*|B|^2Ax, x\rangle - \langle D|C^*|^2D^*y, y\rangle)^2\big] \quad \text{(by Lemma 2.1)}
\end{aligned}$$

for every vectors $x, y \in \mathcal{H}$, which proves the desired inequality. ∎

Corollary 2.6 Suppose that T in $B(\mathcal{H})$ that f is a positive increasing operator convex function on $\mathbb{R}$ and also that f is twice differentiable such that $f'' \geqslant \lambda > 0$. Let the positive real numbers m, m', M, M' satisfy one of the following conditions:

(i) $0 < m'I \leqslant |T|^{2\alpha} \leqslant mI \leqslant MI \leqslant |T^*|^{2\beta} \leqslant M'I$,
(ii) $0 < m'I \leqslant |T^*|^{2\beta} \leqslant mI \leqslant MI \leqslant |T|^{2\alpha} \leqslant M'I$,

with $h = \frac{M}{m}$. Then for every $x, y \in \mathcal{H}$ and $\alpha, \beta \in [0, 1]$ (with $\alpha + \beta \geqslant 1$), it follows that

$$f\Big(\Big|\Big\langle T|T|^{\alpha+\beta-1}x, y\Big\rangle\Big|\Big) \leqslant \frac{1}{2S(\sqrt{h})}\Big[\big\langle f(|T|^{2\alpha})x, x\big\rangle + \big\langle f\big(|T^*|^{2\beta}\big)y, y\big\rangle - \frac{1}{4}\lambda\Big(\langle |T|^{2\alpha}x, x\rangle - \langle |T^*|^{2\beta}y, y\rangle\Big)^2\Big], \tag{15}$$

where $S(.)$ is the Specht's ratio and $\lambda > 0$.

Proof. Replacing D by U, B by $1_{\mathcal{H}}$, C by $|T|^{\beta}$ and A by $|T|^{\alpha}$ in (13), implies that

$$DCBA = U|T|^{\beta}|T|^{\alpha} = U|T||T|^{\alpha+\beta-1} = T|T|^{\alpha+\beta-1}.$$

Then, by utilizing

$$A^*|B|^2A = |T|^{2\alpha} \text{ and } D|C^*|^2D^* = U|T|^{2\beta}U^* = |T|^{2\beta},$$

we reach the interest inequality (15). ■

In resumption, we introduce improvement of inequality (3), in the following theorem.

Theorem 2.7 Let $T \in \mathcal{B}(\mathcal{H})$ and let f be a positive increasing operator convex function on $\mathbb{R}$ and twice differentiable such that $f'' \geqslant \lambda > 0$. Also let positive real numbers m, m', M, M' satisfy in the following conditions (i) or (ii):

(i) $0 < m'I \leqslant |T|^{2s} \leqslant mI < MI \leqslant |T^*|^{2t} \leqslant M'I$,
(ii) $0 < m'I \leqslant |T^*|^{2t} \leqslant mI < MI \leqslant |T|^{2s} \leqslant M'I$,

with $h = \frac{M}{m}$ and $s + t = 1$. Then for every $x \in \mathcal{H}$, we have

$$\begin{aligned} f(\langle Tx, x\rangle) \leqslant \frac{1}{2S(\sqrt{h})}\Big(&\langle f(|T|^{2s})x, x\rangle) + \langle f(|T^*|^{2t})x, x\rangle) \\ &- \frac{1}{4}\lambda(\langle |T^*|^{2t}x, x\rangle - \langle |T|^{2s}x, x\rangle)^2\Big). \end{aligned} \tag{16}$$

Moreover, for every $r \geqslant 1$, it follows that

$$\omega^r(T) \leqslant \frac{1}{2S(\sqrt{h})}\big\||T|^{2rs} + |T^*|^{2rt}\big\|, \tag{17}$$

where $S(.)$ is the Specht's ratio.

Proof. Suppose that $T = U|T|$ is the polar decomposition of T. Utilizing Schwarz inequality in the Hilbert space, Remark *1* and convexity of the function $h(t) = t^r$ for $r \geqslant 1$ imply that

$$\begin{aligned} |\langle Tx, x\rangle| &= |\langle |T|^s x, |T|^t U^* x\rangle| && (18) \\ &\leqslant \big\||T|^s x\big\|.\big\||T|^t U^* x\big\| \\ &= \langle |T|^{2s}x, x\rangle^{\frac{1}{2}} \langle |T^*|^{2t}x, x\rangle^{\frac{1}{2}} \\ &\leqslant \frac{\langle |T|^{2s}x, x\rangle + \langle |T^*|^{2t}x, x\rangle}{2S(\sqrt{h})} \\ &\leqslant \Big(\frac{\langle |T|^{2s}x, x\rangle^r + \langle |T^*|^{2t}x, x\rangle^r}{2S(\sqrt{h})}\Big)^{\frac{1}{r}} && (19) \end{aligned}$$

for each $x \in \mathcal{H}$. Also, by using Remark *1*, we get

$$\begin{aligned} f\Big(\langle |T|^{2s}x,x\rangle^{\frac{1}{2}}\langle |T^*|^{2t}x,x\rangle^{\frac{1}{2}}\Big) &\leqslant f\Big(\frac{\langle |T|^{2s}x,x\rangle+\langle |T^*|^{2t}x,x\rangle}{2S(\sqrt{h})}\Big) \\ &\leqslant \frac{1}{S(\sqrt{h})} f\Big(\frac{\langle |T|^{2s}x,x\rangle+\langle |T^*|^{2t}x,x\rangle}{2}\Big) \qquad (20) \\ &\leqslant \frac{1}{2S(\sqrt{h})}\Big(f(\langle |T|^{2s}x,x\rangle)+f(\langle |T^*|^{2t}x,x\rangle) \\ &\quad -\frac{1}{4}\lambda(\langle |T^*|^{2t}x,x\rangle-\langle |T|^{2s}x,x\rangle)^2\Big) \quad \text{(by Lemma 2.2)} \\ &\leqslant \frac{1}{2S(\sqrt{h})}\Big(\langle f(|T|^{2s})x,x\rangle)+\langle f(|T^*|^{2t})x,x\rangle) \qquad (21) \\ &\quad -\frac{1}{4}\lambda(\langle |T^*|^{2t}x,x\rangle-\langle |T|^{2s}x,x\rangle)^2\Big). \quad \text{(by Lemma 2.1)} \end{aligned}$$

Therefore, by combining inequalities (18) and (21), we imply the desired inequality (16).

From (19) and applying Holder–McCarthy inequality for the positive operator $|T|^{2s}$ and $|T^*|^{2t}$ and the convexity of the function $f(t) = t^r$ for $r \geqslant 1$ imply that

$$\begin{aligned} \Big(\frac{\langle |T|^{2s}x,x\rangle^r+\langle |T^*|^{2t}x,x\rangle^r}{2S(\sqrt{h})}\Big)^{\frac{1}{r}} &\leqslant \Big(\frac{\langle |T|^{2rs}x,x\rangle+\langle |T^*|^{2rt}x,x\rangle}{2S(\sqrt{h})}\Big)^{\frac{1}{r}} \qquad (22) \\ &= \Big(\frac{\langle (|T|^{2rs}+|T^*|^{2rt})x,x\rangle}{2S(\sqrt{h})}\Big)^{\frac{1}{r}} \end{aligned}$$

for each $x \in \mathcal{H}$ with $\|x\| = 1$. Now, combining inequalities (18) and (22) implies that

$$|\langle Tx,x\rangle|^r \leqslant \frac{1}{2S(\sqrt{h})}\big\langle (|T|^{2rs}+|T^*|^{2rt})x,x\big\rangle$$

for each $x \in \mathcal{H}$ with $\|x\| = 1$. By taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ and this fact the operator $|T|^{2rs} + |T^*|^{2tr}$ is self-adjoint, we have the desired inequality (17), which this improves (3). ■

Theorem 2.8 Let $T \in \mathcal{B}(\mathcal{H})$ and let positive real numbers m, m', M, M' satisfy in the following conditions (i) or (ii):

(i) $0 < m'I \leqslant |T|^{\frac{2s}{\alpha}} \leqslant mI < MI \leqslant |T^*|^{\frac{2t}{1-\alpha}} \leqslant M'I$,

(ii) $0 < m'I \leqslant |T^*|^{\frac{2t}{1-\alpha}} \leqslant mI < MI \leqslant |T|^{\frac{2s}{\alpha}} \leqslant M'I$,

with $h = \frac{M}{m}$, $0 < \alpha < 1$ and $s + t = 1$. Then for every $r \geqslant 1$, we have

$$\omega^{2r}(T) \leqslant \Big\| \frac{\alpha}{S(h^r)}|T|^{\frac{2rs}{\alpha}} + \frac{1-\alpha}{S(h^r)}|T^*|^{\frac{2rt}{1-\alpha}} \Big\|, \qquad (23)$$

where $S(.)$ is the Specht's ratio.

Proof. Suppose that $T = U|T|$ is the polar decomposition of T. By using Schwarz inequality, we get

$$|\langle Tx,x\rangle|^2 \leqslant \langle |T|^{2s}x,x\rangle\langle |T^*|^{2t}x,x\rangle = \Big\langle (|T|^{\frac{2s}{\alpha}})^{\alpha}x,x\Big\rangle\Big\langle (|T^*|^{\frac{2t}{1-\alpha}})^{1-\alpha}x,x\Big\rangle \tag{24}$$

for every $x \in \mathcal{H}$. We note that, for positive operator Q and $0 < w < 1$ and unit vector $u \in \mathcal{H}$, $\langle Q^w u,u\rangle \leqslant \langle Qu,u\rangle^w$. If we apply this property for positive operators $|T|^{\frac{2s}{\alpha}}$ and $|T^*|^{\frac{2t}{1-\alpha}}$ (for $0 < \alpha < 1$), then

$$\Big\langle (|T|^{\frac{2s}{\alpha}})^{\alpha}x,x\Big\rangle\Big\langle (|T^*|^{\frac{2t}{1-\alpha}})^{1-\alpha}x,x\Big\rangle \leqslant \Big\langle (|T|^{\frac{2s}{\alpha}})x,x\Big\rangle^{\alpha}\Big\langle (|T^*|^{\frac{2t}{1-\alpha}})x,x\Big\rangle^{1-\alpha} \tag{25}$$

for every $x \in \mathcal{H}$ with $\|x\| = 1$. Now, using Theorem 1.2, it implies that

$$\Big\langle (|T|^{\frac{2s}{\alpha}})^{\alpha}x,x\Big\rangle\Big\langle (|T^*|^{\frac{2t}{1-\alpha}})^{1-\alpha}x,x\Big\rangle \leqslant \frac{\alpha}{S(h^r)}\Big\langle |T|^{\frac{2s}{\alpha}}x,x\Big\rangle + \frac{1-\alpha}{S(h^r)}\Big\langle |T^*|^{\frac{2t}{1-\alpha}}x,x\Big\rangle \tag{26}$$

for every $x \in \mathcal{H}$ with $\|x\| = 1$.

On the other hand, we have the elementary inequality from the convexity of $h(u) = u^r$ (for $r \geqslant 1$) in the following:

$$\alpha a + (1-\alpha)b \leqslant (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}, \quad \alpha \in (0,1), a, b \geqslant 0.$$

Using this inequality leads to

$$\begin{aligned}
&\frac{\alpha}{S(h^r)}\Big\langle |T|^{\frac{2s}{\alpha}}x,x\Big\rangle + \frac{1-\alpha}{S(h^r)}\Big\langle |T^*|^{\frac{2t}{1-\alpha}}x,x\Big\rangle \\
&\leqslant \Big(\frac{\alpha}{S(h^r)}\Big\langle |T|^{\frac{2s}{\alpha}}x,x\Big\rangle^r + \frac{1-\alpha}{S(h^r)}\Big\langle |T^*|^{\frac{2t}{1-\alpha}}x,x\Big\rangle^r\Big)^{\frac{1}{r}} \\
&\leqslant \Big(\frac{\alpha}{S(h^r)}\Big\langle |T|^{\frac{2rs}{\alpha}}x,x\Big\rangle + \frac{1-\alpha}{S(h^r)}\Big\langle |T^*|^{\frac{2rt}{1-\alpha}}x,x\Big\rangle\Big)^{\frac{1}{r}}
\end{aligned} \tag{27}$$

for every $x \in \mathcal{H}$ with $\|x\| = 1$. Now, by using inequalities (24), (25), (26) and (27) we reach

$$|\langle Tx,x\rangle|^{2r} \leqslant \Big\langle \Big(\frac{\alpha}{S(h^r)}|T|^{\frac{2rs}{\alpha}} + \frac{1-\alpha}{S(h^r)}|T^*|^{\frac{2rt}{1-\alpha}}\Big)x,x\Big\rangle \tag{28}$$

for every $x \in \mathcal{H}$ with $\|x\| = 1$. At the end, we take the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ in the inequality (28) and we get the interest inequality (23). ■

Acknowledgments

The authors would like to thank the referees for their useful comments.

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