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Some relations between L^p-spaces on locally compact group G and double coset $K \setminus G/H$

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Abstract. Let H and K be compact subgroups of locally compact group G. By considering the double coset space $K\setminus G/H$, which equipped with an N-strongly quasi invariant measure μ, for $1 \leqslant p \leqslant +\infty$, we make a norm decreasing linear map from $L^p(G)$ onto $L^p(K\setminus G/H, \mu)$ and demonstrate that it may be identified with a quotient space of $L^p(G)$. In addition, we illustrate that $L^p(K \setminus G/H, \mu)$ is isometrically isomorphic to a closed subspace of $L^p(G)$. These assist us to study the structure of the classical Banach space created on a double coset space by those produced on topological space.

Keywords: Double coset space, $L^p(K \setminus G/H, \mu)$, quotient space of $L^p(G)$, duality of $L^p(K/G, \mu)$.

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1. Introduction

Let G be a locally compact group and H be a closed subgroup of G and K a compact subgroup of G. Then the double coset space of G by H and K, respectively, is

$$K \setminus G/H = \{KxH \quad x \in G\},$$

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which is first introduced by Liu in [7]. When K is trivial, a double coset space changes to a homogeneous space G/H and when $H = K$, a double coset space becomes a hypergroup in which the homogeneous space G/H is a semi-hypergroup [5]. It is worth to mention that the hypergroups play important roles in physics. The function spaces on a locally compact group have particular structure, which may not be correct for function spaces on a locally compact Hausdorff space. For instance, it is well known that $L^1(G)$ is an involutive Banach algebra, where G is a locally compact topological group, while, $L^1(X)$ is just a Banach space where X is a locally compact Hausdorff space.

Among all of locally compact Hausdorff spaces, it is worthwhile to consider homogeneous spaces and double coset spaces and study some structures and properties of their function spaces. Recently, in [8], it is demonstrated a connection between $L^p(G)$ and $L^p(G/H)$, where H is compact. This results motivate us to extend that for double coset spaces. The rest of the paper is organized as follow.

Some preliminaries and notations about the double coset space $K \setminus G/H$ and related measures on it are stated in section 2. In section 3, we demonstrate that $L^p(K \setminus G/H, \mu)$ may be identified as a quotient space of $L^p(G)$, where $1 \leqslant p \leqslant +\infty$ and μ is a non zero Radon measure arises from a rho function on the triple (K, G, H). Next, in section 4, by limiting the domain of Q_p (for a definition see Section 2) to a special closed subspace of $L^p(G)$, we demonstrate that $L^p(K \setminus G/H, \mu)$ is isometrically isomorphic to this space.

2. Preliminaries and notation

Let G be a locally compact group and H be a closed subgroup of G and K a compact subgroup of G. Throughout this paper, we denote the left Haar measures on G, H, and K by m, ν_1, and ν_2, respectively and their modular functions by Δ_G, Δ_H, and Δ_K, respectively. The notion of double coset space is a natural generalization of that of coset space arising by two subgroups, simultaneously. Recall that if $K \setminus G/H$ is a double coset space of G by H and K, then the elements of $K \setminus G/H$ are given by KxH for $x \in G$. The canonical mapping $q : G \to K \setminus G/H$, defined by $q(x) = KxH$, is abbreviated by $\ddot{x}$ and is surjective. The double coset space $K \setminus G/H$, equipped with the quotient topology, the largest topology that makes q continuous. In this topology q is also an open mapping and **proper**, that is, for each compact set $F \subseteq K \setminus G/H$, there is a compact set $E \subseteq G$ with $q(E) = F$. With the above mentioned assumptions, $K \setminus G/H$ is a locally compact and Hausdorff space.

Let N be the normalizer of K in G, that is, $N = \{x \in G;\ xK = Kx\}$. Then there is a naturally defined mapping $\varphi : N \times K \setminus G/H \to K \setminus G/H$ given by $\varphi(n, q(x)) = KnxH$. It may be verified that φ is a well-defined continuous action of N to $K \setminus G/H$. Considering $K \setminus G/H$ with this action, we denote $\varphi(n, q(x))$ by $n \cdot q(x)$. Note that this action generally is not transitive. Also for a function g on G and $x \in G$, let $L_x g$ and $R_x g$ denote the left translation and the right translation of g by x, respectively, which are defined by $L_x g(y) = g(x^{-1}y)$ and $R_x g(y) = g(yx)$.

We define the mapping Q from $C_c(G)$ to $C_c(K \setminus G/H)$ by

$$Q(f)(KxH) = \int_K \int_H f(k^{-1}xh) d\nu_1(h) d\nu_2(k).$$

It is evident that Q is a well-defined continuous onto linear map, as well as $supp(Q(f)) \subseteq q(supp(f))$. Also $Q(L_n f) = L_n(Q(f))$ for all $n \in N$ and $f \in C_c(G)$.

Suppose that μ is a positive Radon measure on $K \setminus G/H$. Then μ is called N-relatively invariant if there is a positive real character χ on N such that

$$\int_{K\backslash G/H} Q(f)(n\ddot{x})d\mu(\ddot{x}) = \chi(n)\int_{K\backslash G/H} Q(f)(\ddot{x})d\mu(\ddot{x}),$$

for all $n \in N$ and $f \in C_c(G)$. The function χ is called the modular function of μ. An N-relatively invariant measure is said to be an N-invariant measure if its modular function is identically 1. For more results about relatively invariant measure on double coset space (see [3, 4, 9]).

For a positive Radon measure μ, let μ_n denote its translation by $n \in N$, that is, $\mu_n(E) = \mu(n \cdot E)$ for any Borel set E in $K \setminus G/H$. The measure μ is called N-strongly quasi-invariant if there exists a positive continuous function λ from $N \times K \setminus G/H$ such that $d\mu_n(\ddot{y}) = \lambda(n, \ddot{y})d\mu(\ddot{y})$. A rho-function, for the triple (K, G, H), is a positive locally integrable function ρ on G such that

$$\rho(kxh) = \frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(h)}\rho(x),$$

for all $x \in G$, $h \in H$ and $k \in K$. It is known that, for each triple (K, G, H), there exists a rho-function ρ and a corresponding N-strongly quasi-invariant measure μ on $K \setminus G/H$ such that

$$\int_{K\backslash G/H} Q(f)(\ddot{x})d\mu(\ddot{x}) = \int_G f(x)\rho(x)dm(x), \tag{1}$$

for all $f \in C_c(G)$. In this case, we have

$$\lambda(n, \ddot{y}) = \frac{\rho(ny)}{\rho(y)} \qquad n \in N,\ \ddot{y} \in K \setminus G/H. \tag{2}$$

A partially converse of this also holds (for more details see [4, Theorems 3.7 and 4.3]). From now on, we consider the double coset space $K \setminus G/H$ with a N-strongly invariant measure μ.

Among other things, in the following theorem is verified that (1) holds for all $f \in L^1(G)$.

Theorem 2.1 If μ is an N-strongly quasi-invariant measure on $K \setminus G/H$, then the following hold:

(i) There is a measurable set $A \subseteq K \setminus G/H$ such that $\mu(A) = 0$, and for each $f \in L^1(G)$ the function $(h, k) \longmapsto f(k^{-1}xh)$ is in $L^1(H \times K)$, where $(h, k) \in H \times K$, and $x \in G$ with $q(x) \notin A$.

(ii) The function $\ddot{x} \longmapsto \int_{H\times K} f(k^{-1}xh)d(\nu_1 \times \nu_2)(h, k)$ defined almost everywhere on $K \setminus G/H$, is integrable.

(iii) If $f \in L^1(G)$, then

$$\int_{K\backslash G/H}\int_{H\times K} f(k^{-1}xh)d(\nu_1 \times \nu_2)(h, k)d\mu(\ddot{x}) = \int_G f(x)\rho(x)dm(x)$$

or

$$\int_{K\backslash G/H}\int_{H\times K} \frac{f(k^{-1}xh)}{\rho(k^{-1}xh)}d(\nu_1 \times \nu_2)(h, k)d\mu(\ddot{x}) = \int_G f(x)dm(x).$$

Proof. The proof is similar to [2]. ■

3. $L^p(K \setminus G/H, \mu)$ as a quotient space of $L^p(G)$

In this section, by taking H and K as compact subgroups of G and $1 \leqslant p \leqslant +\infty$, we make a bounded linear map Q_p of $L^p(G)$ onto $L^p(K \setminus G/H, \mu)$ and illustrate that $L^p(K\setminus G/H, \mu)$ may be considered as a quotient space of $L^p(G)$, where $K\setminus G/H$ equipped with an N-strongly measure arising from a rho-function.

In [3, 7] has been shown that, there is a surjective linear map $Q : C_c(G) \to C_c(K\setminus G/H)$ such that

$$Q(f)(KxH) = \int_{H\times K} f(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k), \qquad (f \in C_c(G),\ x \in G).$$

By using the fact that H and K are compact, $1 \leqslant p < +\infty$, and also applying Minkowski's inequality for integrals,for $f \in C_c(G)$, one may investigate that

$$\begin{aligned}\int_{K\setminus G/H} &| \int_{H\times K} f(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k)|^p d\mu(\ddot{x}) \\ &\leqslant \int_{K\setminus G/H} \int_{H\times K} |f(k^{-1}xh)|^p d(\nu_1 \times \nu_2)(h,k)d\mu(\ddot{x}),\end{aligned}$$

for all $x \in G$. By applying (1), we get

$$\begin{aligned}\|Q(f)\|_p &= \left(\int_{K\setminus G/H} |Q(f)(\ddot{x})|^p d\mu(\ddot{x})\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_{K\setminus G/H} \int_{H\times K} |f(k^{-1}xh)|^p d(\nu_1 \times \nu_2)(h,k)d\mu(\ddot{x})\right)^{\frac{1}{p}} \\ &= \left(\int_G |f(x)|^p \rho(x)dm(x)\right)^{\frac{1}{p}} = \|f \cdot \rho^{\frac{1}{p}}\|_p.\end{aligned}$$

Therefore, the surjective linear map $Q_p : C_c(G) \to C_c(K \setminus G/H)$ defined by

$$Q_p(f)(\ddot{x}) = \int_{H\times K} \frac{f(k^{-1}xh)}{\rho(k^{-1}xh)^{\frac{1}{p}}} d(\nu_1 \times \nu_2)(h,k) \qquad (x \in G),$$

is norm decreasing with respect to $\|\cdot\|_p$, that is,

$$\|Q_p(f)\|_p = \|Q\left(\frac{f}{\rho^{\frac{1}{p}}}\right)\|_p \leqslant \|f\|_p. \tag{3}$$

Hence Q_p is bounded and linear with respect to $\|\cdot\|_p$ and $\|Q_p\| \leqslant 1$.

The next proposition states some properties of this linear map.

Proposition 3.1 Suppose that H and K are compact subgroups of G and $1 \leqslant p < +\infty$. Then Q_p is onto and for all $\varphi \in C_c(K \setminus G/H)$, we have

$$\|\varphi\|_p = \inf\{\|f\|_p \ \ f \in C_c(G), \ \varphi = Q_p(f)\}.$$

Proof. Take $\varphi \in C_c(K \setminus G/H)$. Since $Q : C_c(G) \to C_c(K \setminus G/H)$ given by $Q(f)(\ddot{x}) = \int_{H\times K} f(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k)$ is onto, there exists $g \in C_c(G)$ such that $Q(g) = \varphi$. Now, by choosing $f = \rho^{\frac{1}{p}} \cdot g$, we get $Q_p(f) = \varphi$. Now, according to norm decreasing of Q_p, we get

$$\|\varphi\|_p \leqslant \inf\{\|f\|_p \ \ f \in C_c(G), \ \varphi = Q_p(f)\}.$$

By assuming $f_0 = \rho^{\frac{1}{p}}(\varphi o q)$, it is easy to check that $f_0 \in C_c(G)$, $Q_p(f) = \varphi$. Now, by applying the generalized Weil's formula (Theorem 3·8 in [4]), one may get

$$\begin{aligned}\|f_0\|_p^p &= \int_G \rho(x)|(\varphi o q)(x)|^p dm(x) = \int_{K\setminus G/H} Q(|\varphi o q|^p)(\ddot{x})d\mu(\ddot{x})\\ &= \int_{K\setminus G/H} |\varphi(\ddot{x})|^p d\mu(\ddot{x}) = \|\varphi\|_p^p,\end{aligned}$$

where μ is the N-strongly quasi invariant measure. This transmits that $\|\varphi\|_p = \inf\{\|f\|_p \ \ f \in C_c(G), \ \varphi = Q_p(f)\}$. ■

It is an easy investigation that if X and Y are dense subspaces of Banach spaces $\tilde{X}$ and $\tilde{Y}$, respectively, then every linear map $T : X \to Y$ with the property that $\|T(x)\| = \inf\{\|z\| \ \ z \in X, \ T(z) = T(x)\}$ $(x \in X)$ has a unique extension $\tilde{T} : \tilde{X} \to \tilde{Y}$ such that

$$\|\tilde{T}(x)\| = \inf\{\|z\| \ \ z \in \tilde{X}, \ \tilde{T}(z) = \tilde{T}(x)\} \quad (x \in \tilde{X}).$$

Hence, by using Proposition 3.1, there is a surjective norm decreasing linear map $Q_p : L^p(G) \to L^p(K \setminus G/H)$ such that, for all $\varphi \in L^p(K \setminus G/H, \mu)$, we have $\|\varphi\|_p = \inf\{\|f\|_p \ \ f \in L^p(G), \ \varphi = Q_p(f)\}$. It is important to mention that Q_p induces an isometrically isomorphism between $L^p(G)/\ker Q_p$ and $L^p(K \setminus G/H, \mu)$, where $L^p(G)/\ker Q_p$ is equipped with the usual quotient norm. The following theorem shows that for all $f \in L^p(G)$,

$$Q_p(f)(\ddot{x}) = \int_{H\times K} \frac{f(k^{-1}xh)}{\rho(k^{-1}xh)^{\frac{1}{p}}} d(\nu_1 \times \nu_2)(h,k), \quad (\mu-\text{almost everywhere } \ \ddot{x} \in K \setminus G/H).$$

Theorem 3.2 With the above notation, for all $f \in L^p(G), 1 < p < \infty$ and μ-a.e. $\ddot{x} \in K \setminus G/H$, we have

$$Q_p(f)(\ddot{x}) = \int_{H\times K} \frac{f(k^{-1}xh)}{\rho(k^{-1}xh)^{\frac{1}{p}}} d(\nu_1 \times \nu_2)(h,k).$$

Proof. Let $f \in L^p(G)$. Select $\{f_n\}_{n\in\mathbb{N}} \subseteq C_c(G)$ such that $\|f_n - f\|_p^p < 2^{-n}$. Since $|f|^p \in L^1(G)$, then by applying part (i) of Theorem 2.1, there is a null set A_0 in $K \setminus G/H$

such that for each $x \in G$ with $\ddot{x} \notin A_0$, we have

$$\int_{H\times K} \left(\frac{|f|^p}{\rho}\right)(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k) \in L^1(K \setminus G/H, \mu).$$

Also $(h,k) \longmapsto \dfrac{|f|^p}{\rho}(k^{-1}xh) \in L^1(H \times K)$ for each $x \in G$ for which $\ddot{x} \notin A_0$. Since H and K are compact, then according to Hölder's inequality, for each $x \in G$ with $\ddot{x} \notin A_0$, we have

$$\begin{aligned}
\left|\int_{H\times K} \frac{f}{\rho^{\frac{1}{p}}}(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k)\right| &\leqslant \int_{H\times K} \frac{|f|}{\rho^{\frac{1}{p}}}(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k)\\
&\leqslant \left(\int_{H\times K} (\frac{|f|^p}{\rho})(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k)\right)^{\frac{1}{p}}\\
&\quad \times \left(\int_{H\times K} d(\nu_1 \times \nu_2)(h,k)\right)^{\frac{1}{p'}}\\
&= \left(\int_{H\times K} (\frac{|f|^p}{\rho})(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k)\right)^{\frac{1}{p}},
\end{aligned}$$

where p' is the conjugate exponent of p. Now, by using Theorem 2.1, part (iii) and the above inequality, we have

$$\begin{aligned}
&\int_{K\backslash G/H} |\int_{H\times K} \frac{f}{\rho^{\frac{1}{p}}}(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k)|^p d\mu(\ddot{x})\\
&\quad \leqslant \int_{K\backslash G/H} (\int_{H\times K} \frac{|f|^p}{\rho}(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k))d\mu(\ddot{x})\\
&\quad = \int_G |f|^p(x)dm(x) = \|f\|_p^p,
\end{aligned}$$

which implies that

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_{K\backslash G/H} |\int_{H\times K} \frac{(f-f_n)}{\rho^{\frac{1}{p}}}(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k)|^p d\mu(\ddot{x}) &\leqslant \sum_{n=1}^{\infty} \|f - f_n\|_p^p\\
&\leqslant \sum_{n=1}^{\infty} 2^{-n} < +\infty.
\end{aligned}$$

Therefore, almost everywhere with respect to μ

$$\left|\int_{H\times K} (\frac{f}{\rho^{\frac{1}{p}}}(k^{-1}xh) - \frac{f_n}{\rho^{\frac{1}{p}}}(k^{-1}xh))d(\nu_1 \times \nu_2)(h,k)\right|^p \to 0.$$

In other words

$$\lim_n Q_p(f_n)(\ddot{x}) = \int_{H\times K} \frac{f}{\rho^{\frac{1}{p}}}(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k),$$

almost everywhere with respect to μ. Consequently, almost everywhere with respect to μ, we may write

$$Q_p(f)(\ddot{x}) = \int_{H\times K} \frac{f}{\rho^{\frac{1}{p}}}(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k).$$

■

Remark 1 *By applying Theorem 3.2, it is straight-forward to verify that Q_p from $L^p(G)$ onto $L^p(K \setminus G/H, \mu)$ is bounded, linear, onto, and $\|Q_p\| \leqslant 1$.*

Note that Theorem 3.2 also holds for $p = +\infty$, by considering $\frac{1}{+\infty} = 0$. We first require the following lemma and proposition.

Lemma 3.3 If H and K are compact subgroups of the locally compact group G, then a subset E of $K \setminus G/H$ is a null set if and only if $q^{-1}(E)$ is a null set of G.

Proof. The proof is similar to Lemma 2·22 in [6]. ■

Proposition 3.4 Let the assumptions of Proposition 3.1 be hold and let $f \in L^p(G)$. Then $\rho^{\frac{1}{p}}(Q_p(f)oq) \in L^p(G)$ and $\|\rho^{\frac{1}{p}}(Q_p(f)oq)\|_p = \|Q_p(f)\|_p$.

Proof. For all $f \in L^p(G)$ by applying Theorem 3.2, we may get

$$Q_p(f)(\ddot{x}) = \int_{H\times K} \frac{f}{\rho^{\frac{1}{p}}}(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k),$$

for μ-almost all $\ddot{x} \in K\setminus G/H$ and hence by Lemma 3.3 for almost all $x \in G$. Furthermore, according to Minkowski's inequality for integrals, we have

$$\begin{aligned}
&\left(\int_G |\rho^{\frac{1}{p}}(x)(Q_p(f)oq)(x)|^p dm(x)\right)^{\frac{1}{p}} \\
&= \left(\int_G \rho(x)| \int_{H\times K} (\frac{f}{\rho^{\frac{1}{p}}})(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k)|^p dm(x)\right)^{\frac{1}{p}} \\
&\leqslant \int_{H\times K} \left(\int_G |f(k^{-1}xh)|^p dm(x)\right)^{\frac{1}{p}} d(\nu_1 \times \nu_2)(h,k) \\
&= \|f\|_p.
\end{aligned}$$

Hence, $\rho^{\frac{1}{p}}(Q_p(f)oq) \in L^p(G)$ and

$$\begin{aligned}
\|\rho^{\frac{1}{p}}(Q_p(f)oq)\|_p^p &= \int_G |\rho^{\frac{1}{p}}(x)(Q_p(f)oq)(x)|^p dm(x) \\
&= \int_G \rho(x)|(Q_p(f)\circ q)(x)|^p dm(x) \\
&= \int_{K\setminus G/H}\int_{H\times K} |(Q_p(f)oq)(k^{-1}xh)|^p d(\nu_1\times\nu_2)(h,k)d\mu(x) \\
&= \|Q_p(f)\|_p^p.
\end{aligned}$$

■

Theorem 3.5 Suppose that H and K are compact subgroups of G. Then there is a surjective norm decreasing linear map $Q_\infty : L^\infty(G) \to L^\infty(K\setminus G/H,\mu)$ such that, for all $f \in L^\infty(G)$, we have

$$Q_\infty(f)(\ddot{x}) = \int_{H\times K} f(k^{-1}xh)d(\nu_1\times\nu_2)(h,k) \quad (\mu\text{-a.e.}\ddot{x} \in K\setminus G/H),$$

and for all $\varphi \in L^\infty(K\setminus G/H,\mu)$,

$$\|\varphi\|_\infty = \inf\{\|f\|_\infty,\ f\in L^\infty(G),\ \varphi = Q_\infty(f)\}.$$

Proof. Let $f \in L^\infty(G)$ be arbitrary and from now be fixed and for $\varphi \in L^1(K\setminus G/H,\mu)$, put $\varphi_\rho = \rho(\varphi\circ q)$. Note that $\varphi_\rho \in L^1(G)$. We define the linear functional S_f on $L^1(K\setminus G/H,\mu)$ as follows:

$$S_f(\varphi) = \int_G \varphi_\rho(x)f(x)dm(x).$$

Hence S_f is a continuous function. Indeed, by Theorem 2.1, we have

$$\begin{aligned}
|S_f(\varphi)| &= \Big|\int_G \varphi_\rho(x)f(x)dm(x)\Big| \\
&\leqslant \int_G |\varphi_\rho(x)||f(x)|dm(x) \\
&\leqslant \|f\|_\infty \int_G |\varphi_\rho(x)|dm(x) \\
&= \|f\|_\infty \int_{K\setminus G/H}\int_{H\times K} \frac{|\varphi_\rho(k^{-1}xh)|}{\rho(k^{-1}xh)} d(\nu_1,\nu_2)(h,k)d\mu(\ddot{x}) \\
&= \|f\|_\infty \int_{K\setminus G/H} |\varphi(\ddot{x})|d\mu(\ddot{x})\Big(\int_{H\times K} d(\nu_1\times\nu_2)(h,k)\Big) \\
&= \|f\|_\infty\|\varphi\|_1.
\end{aligned}$$

Hence $|S_f(\varphi)| \leqslant \|f\|_\infty\|\varphi\|_1$. Now, since $L^\infty(K\setminus G/H,\mu)$ is the dual of $L^1(K\setminus G/H,\mu)$, so there is $\psi_f \in L^\infty(K\setminus G/H,\mu)$ such that $S_f = \psi_f$. Now, according to Theorem 2.1

and by using the fact that H and K are compact, for each $\varphi \in L^1(K \setminus G/H, \mu)$, we have

$$\begin{aligned}\langle \psi_f, \varphi \rangle &= \int_G \varphi_\rho(x) f(x) dm(x) \\ &= \int_{K\setminus G/H} \int_{H\times K} \frac{\varphi_\rho(k^{-1}xh) f(k^{-1}xh)}{\rho(k^{-1}xh)} d(\nu_1 \times \nu_2)(h,k) d\mu(\ddot{x}) \\ &= \int_{K\setminus G/H} \varphi(\ddot{x}) \Big(\int_{H\times K} f(k^{-1}xh) d(\nu_1 \times \nu_2)(h,k) \Big) d\mu(\ddot{x}).\end{aligned}$$

Therefore, for μ-a.e. $\ddot{x} \in K \setminus G/H$, we have

$$\psi_f(\ddot{x}) = \int_{K\times K} f(k^{-1}xh) d(\nu_1 \times \nu_2)(h,k).$$

Consequently, $Q_\infty : L^\infty(G) \to L^\infty(K \setminus G/H, \mu)$ given by

$$Q_\infty(f)(\ddot{x}) = \int_{K\setminus G/H} f(k^{-1}xh) d(\nu_1 \times \nu_2)(h,k),$$

for μ-a.e. $\ddot{x} \in K \setminus G/H$, is a well-defined linear mapping. Furthermore, for all $f \in L^\infty(G)$

$$\begin{aligned}\|f\|_\infty &= \sup\{|\langle f, g\rangle|;\ g \in L^1(G),\ \|g\| \leqslant 1\} \\ &\geqslant \sup\{|\langle f, \varphi_\rho\rangle|;\ \varphi \in L^1(K \setminus G/H),\ \|\varphi\| \leqslant 1\} \\ &= \sup\{|\langle Q_\infty f, \varphi\rangle|;\ \varphi \in L^1(K \setminus G/H),\ \|\varphi\| \leqslant 1\} = \|Q_\infty f\|_\infty.\end{aligned}$$

In addition, by taking $\varphi \in L^\infty(K \setminus G/H, \mu)$, we have $\varphi oq \in L^\infty(G)$, $Q_\infty(\varphi oq) = \varphi$ and $\|\varphi oq\|_\infty = \|\varphi\|_\infty$. Therefore, Q_∞ is onto and

$$\|\varphi\|_\infty = \inf\{\|f\|_\infty;\ f \in L^\infty(G),\ \varphi = Q_\infty f\}.$$

■

As mentioned at the beginning of this section and by Theorem 3.5, there is an isometrically isomorphism between $L^\infty(K \setminus G/H, \mu)$ and $L^\infty(G)/\ker Q_\infty$.

Remark 2 *By using the fact that $L^p(K \setminus G/H, \mu) \cong L^p(G)/\ker Q_p$ for all $1 \leqslant p \leqslant +\infty$, every left modular structure of $L^p(G)$ induces a left module structure on $L^p(K \setminus G/H, \mu)$, whenever $\ker Q_p$ is an invariant subspace of $L^p(G)$ under the module action. Specially, it is well known that $L^p(G)$ is a left Banach $L^1(G)$ module for $1 \leqslant p \leqslant +\infty$, using the convolution of function as the left action.*
Note that if $K \lhd G$, $f \in L^1(G)$, and $g \in \ker Q_p$, then $\rho^{\frac{1}{p}}(Q_p g \circ q) = 0$ in $L^p(G)$. So, for

almost all $x \in G$ and hence for μ-almost all $\ddot{x} \in K \setminus G/H$, we have

$$\begin{aligned}
Q_p(f*g)(KxH) &= \int_{H\times K} \frac{(f*g)(k^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}xh)} d(\nu_1\times\nu_2)(h,k) \\
&= \int_{H\times K} \Big(\int_G \frac{f(y)g(y^{-1}k^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}xh)} dm(y)\Big) d(\nu_1\times\nu_2)(h,k) \\
&= \int_G \Big(\int_{H\times K} \frac{f(y)g(y^{-1}k^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}xh)} d(\nu_1\times\nu_2)(h,k) dm(y) \\
&= \frac{1}{\rho^{\frac{1}{p}}(x)} \int_{H\times K} \Big(\int_G f(y)g(ky^{-1}xh)dm(y)\Big) d(\nu_1\times\nu_2)(h,k) \\
&= \frac{1}{\rho^{\frac{1}{p}}(x)} \int_G f(y)\Big(\int_{H\times K} \frac{g(k^{-1}y^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}y^{-1}xh)} \rho^{\frac{1}{p}}(k^{-1}y^{-1}xh)\Big) d(\nu_1\times\nu_2)(h,k)dm(y) \\
&= \frac{1}{\rho^{\frac{1}{p}}(x)} \int_G f(y)\rho^{\frac{1}{p}} Q_p(g\circ q)(y^{-1}x)dm(y) \\
&= \frac{1}{\rho^{\frac{1}{p}}(x)} f*\rho^{\frac{1}{p}} Q_p(g\circ q)(x) = 0.
\end{aligned}$$

This shows that $\ker Q_p$ is invariant under the module action of $L^1(G)$ on $L^p(G)$. This makes $L^p(K\setminus G/H,\mu)$ a Banach left $L^1(G)$-module, where the action is defined by

$$\begin{aligned}
L^1(G)\times L^p(K\setminus G/H,\mu) &\to L^p(K\setminus G/H,\mu), \\
(f,\varphi) &\mapsto Q_p(f*g)
\end{aligned}$$

in which $g \in L^p(G)$ and $\psi = Q_p(g)$.

4. $L^p(K\setminus G/H,\mu)$ as a closed subspace of $L^p(G)$

In the last section, we introduce the linear mapping Q_p as a connection between two spaces $L^p(G)$ and $L^p(K\setminus G/H,\mu), (1\leqslant p\leqslant\infty)$, and we found some properties of this mapping. In this section, we demonstrate that $L^p(K\setminus G/H,\mu)$ is isometrically isomorphic to a closed subspace of $L^p(G)$.

For closed subgroups H and K of G, put

$$C_c(K:G:H) = \{f\in C_c(G)\ \ R_hL_kf = f,\ h\in H,\ k\in K\},$$

which is a subalgebra of $C_c(G)$ with pointwise multiplication. Note that if $K \lhd G$, then $C_c(K:G:H)$ with the convolution on $C_c(G)$ is also a subalgebra of $C_c(G)$.
We ascertain by $L^p(K:G:H)$ the closure of $C_c(K:G:H)$ in $L^p(G)$ for all $1\leqslant p<+\infty$, and we set

$$L^\infty(K:G:H) = \{f\in L^\infty(G);\ R_hL_kf = f,\ h\in H,\ k\in K\},$$

where L_kf, R_hf are the left and right translations on f by k and h, respectively. If H

and K are compact and $1 \leqslant p < +\infty$, then $Q_p(f) \in C_c(K \setminus G/H, \mu)$, and we have

$$\rho(x)^{\frac{1}{p}}(Q_p(f)oq)(x) = \int_{H\times K} f(k^{-1}xh)d(\nu_1 \times \nu_2)(h,k) = f(x),$$

for all $f \in C_c(K : G : H)$. Therefore, $C_c(K : G : H) = \{\rho^{\frac{1}{p}}(\varphi oq);\ \varphi \in C_c(K \setminus G/H)\}$.

Theorem 4.1 Suppose that H and K are compact subgroups of G and $1 \leqslant p < +\infty$. Then $L^p(K : G : H) = \{\rho^{\frac{1}{p}}(\varphi oq)\ \ \varphi \in L^p(K \setminus G/H)\}$. In particular, $f = \rho^{\frac{1}{p}}(Q_p(f)oq)$ for all $f \in L^p(K : G : H)$.

Proof. Assume that $1 \leqslant p < +\infty$ and $f \in L^p(K : G : H)$. Select a sequence $\{f_n\} \subseteq C_c(K : G : H)$ in such a way that $\|f_n - f_n\|_p \to 0$. Then by using Lemma 3.4, we may write

$$\begin{aligned}\|f - \rho^{\frac{1}{p}}(Q_p(f)oq)\|_p &\leqslant \|f - f_n\|_p + \|\rho^{\frac{1}{p}}(Q_p(f_n - f)oq)\|_p \\ &\leqslant \|f - f_n\|_p + \|Q_p(f_n - f)\|_p \leqslant 2\|f - f_n\|_p.\end{aligned}$$

Hence, we have $f = \rho^{\frac{1}{p}}(Q_p(f)oq)$ in $L^p(G)$. Therefore, by density of $C_c(K \setminus G/H)$ in $L^p(K \setminus G/H, \mu)$ the proof is complete. ■

Remark 3 *Since H and K are compact, it is easy to check that Theorem 4.1, for $p = \infty$, also holds.*

Next, we illustrate that $L^p(K \setminus G/H, \mu)$ is isometrically isomorphic to $L^p(K : G : H)$.

Theorem 4.2 Let the assumptions of Theorem 4.1 be hold. Then $L^p(K \setminus G/H, \mu)$ is isometrically isomorphic to $L^p(K : G : H)$. More precisely, the restriction of Q_p on $L^p(K : G : H)$ is an isometrically isomorphism.

Proof. Let $1 \leqslant p \leqslant +\infty$. By the definition of $L^p(K : G : H)$ and the density of $C_c(K \setminus G/H)$ in $L^p(K \setminus G/H, \mu)$, it is enough to show that the mapping $Q_p : C_c(K : G : H) \to C_c(K \setminus G/H)$ is an isometrically isomorphism, where $C_c(K : G : H)$ and $C_c(K \setminus G/H)$ are equipped with $\|\cdot\|_p$. For this, first we note that if $\varphi \in C_c(K \setminus G/H)$, then $\rho^{\frac{1}{p}}(\varphi oq) \in C_c(K : G : H)$ and $Q_p(\rho^{\frac{1}{p}}(\varphi oq)) = \varphi$. Now, by taking $f \in C_c(K : G : H)$, we may write,

$$\begin{aligned}\|Q_p(f)\|_p^p &= \int_{K\setminus G/H} |Q_p(f)(\ddot{x})|^p d\mu(\ddot{x}) \\ &= \int_{K\setminus G/H} \Big| \int_{H\times K} \frac{f(k^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}xh)} d(\nu_1 \times \nu_2)(h,k) \Big|^p d\mu(\ddot{x}) \\ &= \int_{K\setminus G/H} \frac{|f(x)|^p}{\rho(x)} d\mu(\ddot{x}) \\ &= \int_{K\setminus G/H} \int_{H\times K} \frac{|f(k^{-1}xh)|^p}{\rho(k^{-1}xh)} d(\nu_1 \times \nu_2)(h,k) d\mu(\ddot{x}) \\ &= \int_G |f(x)|^p dm(x) = \|f\|_p^p.\end{aligned}$$

The consequence, for the case $p = +\infty$, is obtained by applying Theorem 4.1, that is,

$$\|Q_\infty(f)\|_\infty = \|f\|_\infty,$$

for all $f \in L^\infty(K : G : H)$. ■

In the sequel, we aim to find an adjoint of Q_p for all $1 \leqslant p < +\infty$.

Theorem 4.3 Suppose that H and K are compact subgroups of G. Then if $Q_p^* : L^{p'}(K \setminus G/H, \mu) \to L^{p'}(G)$ is the adjoint of Q_p, we have $Q_p^*(\psi) = \rho^{\frac{1}{p'}}(\psi \circ q)$ for all $\psi \in L^{p'}(K \setminus G/H, \mu)$.

Proof. First suppose that $1 < p < +\infty$ and $\psi \in L^{p'}(K \setminus G/H, \mu)$, then by using the fact that H and K are compact and according to Theorem 2.1 and Proposition 3.4, for each $f \in L^p(G)$, we have

$$\begin{aligned}
\langle Q_p^*(\psi), f\rangle &= \langle \psi, Q_p(f)\rangle \\
&= \int_{K\setminus G/H} \psi(\ddot{x}) Q_p(f)(\ddot{x}) d\mu(\ddot{x}) \\
&= \int_{K\setminus G/H} \int_{H\times K} \frac{\psi \circ q(k^{-1}xh) f(k^{-1}xh)}{\rho^{\frac{1}{p}}(k^{-1}xh)} d(\nu_1 \times \nu_2)(h,k) d\mu(\ddot{x}) \\
&= \int_{K\setminus G/H} \int_{H\times K} \frac{\rho^{\frac{1}{p'}}(\psi \circ q)(k^{-1}xh) f(k^{-1}xh)}{\rho(k^{-1}xh)} d(\nu_1 \times \nu_2)(h,k) d\mu(\ddot{x}) \\
&= \int_G \rho^{\frac{1}{p'}}(\psi \circ q)(x) f(x) dm(x) \\
&= \langle \rho^{\frac{1}{p'}}(\psi \circ q), f\rangle.
\end{aligned}$$

Therefore $Q_p^*(\psi) = \rho^{\frac{1}{p'}}(\psi \circ q)$. Now suppose that $p = 1$, $\psi \in L^\infty(K \setminus G/H, \mu)$. Then, for each $f \in L^1(G)$, by using Theorem 2.1 and the compactness of H and K, we have

$$\begin{aligned}
\langle Q_1^*(\psi), f\rangle &= \langle \psi, Q_1(f)\rangle \\
&= \int_{K\setminus G/H} \psi(\ddot{x}) Q_1(f)(\ddot{x}) d\mu(\ddot{x}) \\
&= \int_{K\setminus G/H} \int_{H\times K} \frac{\psi \circ q(k^{-1}xh) f(k^{-1}xh)}{\rho(k^{-1}xh)} d(\nu_1 \times \nu_2)(h,k) d\mu(\ddot{x}) \\
&= \langle \psi \circ q, f\rangle.
\end{aligned}$$

Therefore assuming $\frac{1}{\infty} = 0$, we have $Q_1^*(\psi) = \psi \circ q$. ■

Corollary 4.4 For each $1 \leqslant p \leqslant +\infty$, the adjoint of Q_p, that is, Q_p^* which introduced in Theorem 4.3 is isometric.

Proof. Suppose that $\psi \in L^{p'}(K\setminus G/H, \mu)$. Then, by using Theorem 2.1 and compactness of H and K, we have

$$\begin{aligned}\|Q_p^*(\psi)\|_{p'}^{p'} &= \int_G |\rho^{\frac{1}{p'}}(\psi\circ q)(x)|^{p'}dm(x)\\ &= \int_{K\backslash G/H}\int_{H\times K}\frac{\rho(k^{-1}xh)|\psi\circ q|^{p'}(k^{-1}xh)}{\rho(k^{-1}xh)}d(\nu_1\times\nu_2)(h,k)d\mu(\ddot{x})\\ &= \int_{K\backslash G/H}|\psi(\ddot{x})^{p'}d\mu(\ddot{x})\\ &= \|\psi\|_{p'}^{p'}.\end{aligned}$$

Hence $\|Q_p^*(\psi)\|_{p'} = \|\psi\|_{p'}$. That is Q_p^* is isometric. ■

In the following, we give another characterization of $L^p(K:G:H)$.

Proposition 4.5 Suppose that H and K are compact subgroups of G. Then, for all $1 \leqslant p \leqslant +\infty$, we have

$$L^p(K:G:H) = \{f \in L^p(G),\ L_kR_hf = f \text{ in } L^p(G),\ h\in H,\ k\in K\}.$$

Proof. First, assume that $f \in L^p(K:G:H)$. Then by using Theorem 4.1, we can write

$$\begin{aligned}L_kR_hf(x) &= L_kR_h(\rho^{\frac{1}{p}}(Q_p(f)oq))(x) = \rho^{\frac{1}{p}}(k^{-1}xh)(Q_p(f)oq)(k^{-1}xh)\\ &= \rho^{\frac{1}{p}}(x)(Q_p(f)oq)(x) = f(x),\end{aligned}$$

for μ-almost all $\ddot{x} \in K\setminus G/H$ and m-almost all $x\in G$ (using Lemma 3.3). Hence, $L_kR_hf = f$ as elements of $L^p(G)$. Now, suppose that $f\in L^p(G)$ and $L_kR_hf = f$ for all $h\in H$ and $k\in K$. By the duality of $L^p(K\setminus G/H)$ and $L^{p'}(K\setminus G/H)$, for all $g\in L^{p'}(G)$, we may write

$$\begin{aligned}\langle\rho^{\frac{1}{p}}(Q_p(f)oq), g\rangle &= \int_G \rho^{\frac{1}{p}}Q_p(f)(\ddot{x})g(x)dm(x)\\ &= \int_G\left(\int_{H\times K} f(k^{-1}xh)d(\nu_1\times\nu_2)(h,k)\right)g(x)dm(x)\\ &= \int_{H\times K}\left(\int_G L_kR_hf(x)g(x)dm(x)\right)d(\nu_1\times\nu_2)(h,k)\\ &= \int_{H\times K}\langle L_kR_hf, g\rangle d(\nu_1\times\nu_2)(h,k)\\ &= \int_{H\times K}\langle f,g\rangle d(\nu_1\times\nu_2)(h,k) = \langle f,g\rangle,\end{aligned}$$

where p and p' are conjugate exponents. Hence, $f = \rho^{\frac{1}{p}}(Q_p(f)oq) \in L^p(K:G:H)$. ■

Remark 4 *Note that for each $1 \leqslant p < +\infty$ with conjugate exponent p', by using the fact that $L^{p'}(K\setminus G/H,\mu) \longrightarrow L^p(K\setminus G/H,\mu)^*$ we get an isometric isomorphism between $L^{p'}(K:G:H)$ and $L^p(K:G:H)^*$ regarding the following commutative diagram:*

$$\begin{array}{ccc}
L^{p'}(K:G:H) & \longrightarrow & L^p(K:G:H)^* \\
\Big\downarrow {\scriptstyle Q_{p'}} & & \Big\downarrow {\scriptstyle Q_p^*} \\
L^{p'}(K\setminus G/H,\mu) & \longrightarrow & L^p(K\setminus G/H,\mu)^*
\end{array}$$

Therefore for each $f \in L^p(K:G:H)$ *and* $g \in L^{p'}(K:G:H)$*, we have* $\langle f, g\rangle = \langle Q_p(f), Q_{p'}(g)\rangle$*. Now using the facts* $Q_p(f)\circ q = \frac{f}{\rho^{\frac{1}{p}}}$ *and* $Q_{p'}(g)\circ q = \frac{g}{\rho^{\frac{1}{p'}}}$*, we get,*

$$\begin{aligned}
\langle Q_p(f), Q_{p'}(g)\rangle &= \int_{K\setminus G/H} Q_p(f)(\ddot{x})Q_{p'}(g)(\ddot{x})d\mu(\ddot{x}) \\
&= \int_{K\setminus G/H} \frac{g}{\rho^{\frac{1}{p'}}}(x)\Big(\int_{H\times K} d(\nu_1\times\nu_2)(h,k)\Big) \\
&\quad \times \frac{1}{\rho^{\frac{1}{p}}(x)}\int_{H\times K} f(k_1^{-1}xh_1)d(\nu_1\times\nu_2)(k_1,h_1)\Big)d\mu(\ddot{x}) \\
&= \int_{K\setminus G/H} \frac{g(x)}{\rho^{\frac{1}{p}+\frac{1}{p'}}(x)}\int_{H\times K} f(k_1^{-1}xh_1)d(\nu_1\times\nu_2)(h_1,k_1)d\mu(\ddot{x}) \\
&= \int_{K\setminus G/H}\int_{H\times K} \frac{f\cdot g(k_1^{-1}xh_1)}{\rho(k_1^{-1}xh_1)}d(\nu_1\times\nu_2)(h_1,k_1)d\mu(\ddot{x}) \\
&= \int_G f(x)g(x)dm(x) = \langle f, g\rangle.
\end{aligned}$$

In the special case $p = 2$*, we conclude that* $Q_2 : L^2(K:G:H) \to L^2(K\setminus G/H,\mu)$ *is a unitary mapping, we use this, to get some results for the Hilbert space* $L^2(K\setminus G/H,\mu)$*. For a given Hilbert space, it is often useful to find an orthonormal basis, a Riesz basis or a frame as the generalization of a basis, to get a sequence* $\{g_n\}_{n\in\mathbb{N}}$ *such that any vector* f *can be written as* $f = \sum_{n=1}^{\infty} c_n g_n$ *for some scalers* c_n*,* $n \in \mathbb{N}$ *(See [1]). To construct a frame for the Hilbert space* $L^2(K\setminus G/H,\mu)$*, it is instrumental to assert that the mapping* $Q_2 : L^2(G) \to L^2(K\setminus G/H,\mu)$ *is the orthogonal projection of* $L^2(G)$ *on* $L^2(K\setminus G/H,\mu)$*, by considering* $L^2(K\setminus G/H,\mu)$ *as a closed subspace of* $L^2(G)$*.*
In fact, Theorem 4.1 shows that for all $1 \leqslant p \leqslant +\infty$*, the mapping* $f \mapsto \rho^{\frac{1}{p}}\big(Q_p(f)\circ q\big)$ *is a projection on* $L^p(G)$*. Particularly, one may easily check that*

$$\begin{cases} L^2(G) \to L^2(K:G:H) \subseteq L^2(G) \\ \qquad f \mapsto \rho^{\frac{1}{2}}\big(Q_2(f)\circ q\big), \end{cases}$$

is the orthogonal projection of $L^2(G)$ *on* $L^2(K\setminus G/H)$*. This help us to study the structure of the* L^p*-space constructing on a double coset space via those created on a topological group. For instance,* Q_2 *maps every frame* $L^2(G)$ *onto a frame of* $L^2(K\setminus G/H,\mu)$ *and if* $\{\varphi_n\}_{n\in\mathbb{N}}$ *is a frame for* $L^2(K\setminus G/H,\mu)$*, then* $\{\rho^{\frac{1}{2}}(\varphi\circ q)\}_{n\in\mathbb{N}}$ *is a frame sequence in* $L^2(G)$*.*

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