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Non-additive Lie centralizer of infinite strictly upper triangular matrices

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Abstract. Let $\mathcal{F}$ be an field of zero characteristic and $N_\infty(\mathcal{F})$ be the algebra of infinite strictly upper triangular matrices with entries in $\mathcal{F}$, and $f : N_\infty(\mathcal{F}) \to N_\infty(\mathcal{F})$ be a non-additive Lie centralizer of $N_\infty(\mathcal{F})$; that is, a map satisfying that $f([X, Y]) = [f(X), Y]$ for all $X, Y \in N_\infty(\mathcal{F})$. We prove that $f(X) = \lambda X$, where $\lambda \in \mathcal{F}$.

Keywords: Lie centralizer, infinite strictly upper triangular matrices, commuting map.

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1. Introduction and preliminaries

Consider a ring R. An additive mapping $T : R \to R$ is called a left (respectively right) centralizer if $T(ab) = T(a)b$ (respectively $T(ab) = aT(b)$) for all $a, b \in R$. The mapping T is called a centralizer if it is a left and a right centralizer. The characterization of centralizers on algebras or rings has been a widely discussed subject in various areas of mathematics.

In [11], Zalar proved the following interesting result: if R is a 2-torsion free semiprime ring and T is an additive mapping such that $T(a^2) = T(a)a$ (or $T(a^2) = aT(a)$), then T is a centralizer. Vukman [10] considered additive maps satisfying similar condition, namely $2T(a^2) = T(a)a + aT(a)$ for any $a \in R$, and showed that if R is a 2-torsion free semiprime ring, then T is also a centralizer. Since then centralizers have been intensively

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investigated by many mathematicians, let us name only [2–5, 7] and references included in these works.

An additive map $f : R \to R$, where R is a ring, is called a Lie centralizer of R if $f([x, y]) = [f(x), y]$ for all $x, y \in R$, where $[x, y]$ is the Lie product of x and y.

Recently, Ghomanjani and Bahmani [8] dealt with the structure of Lie centralizers of trivial extension algebras, whereas Fošner and Jing [6] studied Lie centralizers of triangular rings.

Comes from the inspiration of this paper articles [1, 4, 6] in which the authors deal with triangular algebras and rings and various maps connected to commutativity. In this note we will consider non-additive Lie centralizers on strictly infinite upper triangular matrices over an field of zero characteristic.

Let us recall one basic fact. Let $\mathcal{F}$ be an field of zero characteristic. Also, let $N_\infty(\mathcal{F})$, $D_\infty(\mathcal{F})$ and $T_\infty(\mathcal{F})$ denote the algebra of strictly infinite $\mathbb{N}\times\mathbb{N}$ upper triangular matrices over $\mathcal{F}$, the algebra of all infinite $\mathbb{N}\times\mathbb{N}$ diagonal matrices over $\mathcal{F}$ and the algebra of all infinite $\mathbb{N}\times\mathbb{N}$ upper triangular matrices over $\mathcal{F}$, respectively.

Throughout this article, J will represent the matrix $J = \sum_{i=1}^{\infty} E_{i,i+1}$ and $I_\infty = \sum_{i=1}^{\infty} E_{i,i}$. By $C_{N_\infty(\mathcal{F})}(X)$, we will denote the centralizer of the element X in the ring $N_\infty(\mathcal{F})$ and $f : N_\infty(\mathcal{F}) \to N_\infty(\mathcal{F})$ will denote a non-additive map satisfying : $f([X, Y]) = [f(X), Y]$ for all $X, Y \in N_\infty(\mathcal{F})$. We will say that f is a non-additive Lie centralizer of $N_\infty(\mathcal{F})$. Notice that it is easy to check that the $N_\infty(\mathcal{F})$ has a trivial center $\mathcal{Z}(N_\infty(\mathcal{F}))$.

The main result in this paper is the following:

Theorem 1.1 Let $\mathcal{F}$ be an field of zero characteristic. If $f : N_\infty(\mathcal{F}) \to N_\infty(\mathcal{F})$ is a non-additive Lie centralizer then there exists $\lambda \in \mathcal{F}$ such that $f(X) = \lambda X$ for all $X \in N_\infty(\mathcal{F})$.

Notice that the converse is trivially true: every map $f(X) = \lambda X$ is a (non-additive) Lie centralizer.

2. Proof of the main result

Let's start with some properties of Lie centralizers.

Lemma 2.1 [6] Let f be a non-additive Lie centralizer of $N_\infty(\mathcal{F})$. Then

(1) $f(0) = 0$,
(2) For every $X, Y \in N_\infty(\mathcal{F})$ we have $f([X, Y]) = [X, f(Y)]$,
(3) f is a commuting map, i.e. $f(X)X = Xf(X)$ for all $X \in N_\infty(\mathcal{F})$.

Proof. (1) It suffices to notice that $f(0) = f([0, 0]) = [f(0), 0] = 0$.
(2) Observe that if $f([X, Y]) = [f(X), Y]$, then we have $f(XY - YX) = f(X)Y - Yf(X)$. Interchanging X and Y in the above identity, we have $f(YX - XY) = f(Y)X - Xf(Y)$. Replacing X with $-X$, we arrive at $f(XY - YX) = Xf(Y) - f(Y)X$ which can be written as $f([X, Y]) = [X, f(Y)]$.
(3) From (1), one also gets $[f(X), X] = f([X, X]) = f(0) = 0$. ∎

Remark 1 *Let f be a non-additive Lie centralizer of $N_\infty(\mathcal{F})$ and $X \in C_{N_\infty(\mathcal{F})}(Y)$. Then $f(X) \in C_{N_\infty(\mathcal{F})}(Y)$. Indeed, if $X \in C_{N_\infty(\mathcal{F})}(Y)$, then $[X, Y] = 0$ and*

$$0 = f(0) = f([X, Y]) = [f(X), Y].$$

Lemma 2.2 Let f be a non-additive Lie centralizer of $N_\infty(\mathcal{F})$. Then

(1) if $A \in T_\infty(\mathcal{F})$, then $[D_0, A] = A$ if and only if $A = \sum\limits_{i=1}^{\infty} a_i E_{i,i+1}$;

(2) $f(\sum\limits_{i=1}^{\infty} a_i E_{i,i+1}) = \sum\limits_{i=1}^{\infty} b_i E_{i,i+1}$;

(3) if $A = \sum\limits_{i=1}^{\infty} a_i E_{i,i+1}$ for some $a_i \in A$, then $[J_\infty, A] = 0$ if and only if $A = aJ_\infty$ for some $a \in \mathcal{F}$;

(4) there exists $\lambda \in \mathcal{F}$ such that $f(J) = \lambda J$.

Proof. Let $D_0 = \sum\limits_{k=1}^{\infty} (-k)\, E_{k,k}$.

(1) Consider $A = \sum_{i \leqslant j} a_{ij} E_{ij} \in T_\infty(\mathcal{F})$. Then $[D_0, A] = A$ if and only if $(p-n)\, a_{np} = a_{np}$ for all $1 \leqslant n < p \in \mathbb{N}$, and consequently $A = \sum\limits_{i=1}^{\infty} a_{i,i+1} E_{i,i+1}$.

(2) Hence, if $A = \sum\limits_{i=1}^{\infty} a_i E_{i,i+1}$, $[D_0, A] = A$. Thus $f([D_0, A]) = [D_0, f(A)] = f(A)$. Thus, $f(A) = \sum\limits_{i=1}^{\infty} b_i E_{i,i+1}$.

(3) As in (1), consider $A = \sum\limits_{i=1}^{\infty} a_i E_{i,i+1}$ for some $a_i \in \mathcal{F}$. Then $[J, A] = 0$ if and only if $A = aJ$ for some $a \in \mathcal{F}$.

Indeed, $f(J) = \sum\limits_{i=1}^{\infty} a_i E_{i,i+1}$ by (1). Thus, $0 = f(0) = f([J, J]) = [J, f(J)]$. Hence, there exists $\lambda \in \mathcal{F}$ such that $f(J) = \lambda J$. ■

Lemma 2.3 [9] Suppose that $\mathcal{F}$ is an arbitrary field. If $G, H \in UT_\infty(\mathcal{F})$ are such that $g_{i,i+1} = h_{i,i+1} \neq 0$ for all $1 \leqslant i \leqslant n-1$, then G and H are conjugated in $UT_\infty(\mathcal{F})$.

Here $UT_\infty(\mathcal{F})$ is the multiplicative group of infinite $\mathbb{N} \times \mathbb{N}$ upper triangular matrices with only 1's in the main diagonal. From the lemma above we obtain the following corollary.

Corollary 2.4 Let $\mathcal{F}$ be a field. For every $A = \sum_{i \leqslant j} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$, there exists $B \in T_\infty(\mathcal{F})$ such that $B^{-1}AB = J$.

Proof. Let A be a matrix in $N_\infty(\mathcal{F})$ of the mentioned form. Then $I_\infty + A$ is a unitriangular matrix, let's notice first that there exists $B_1 \in D_\infty(\mathcal{F})$ such that $(B_1^{-1}AB_1)_{i,i+1} = 1$ for all $i \in \mathbb{N}$. We can construct $B_1 \in D_\infty(\mathcal{F})$ recursively by:

$$(B_1)_{11} = 1, \qquad (B_1)_{i+1,i+1} = (B_1)_{ii} \cdot (A_{i,i+1})^{-1} \quad \text{for } i \geqslant 1.$$

Consider matrix $B_1^{-1}AB$ and $I_n + B_1^{-1}AB \in UT_\infty(\mathcal{F})$. The unitriangular matrices $I_\infty + J$ and $I_\infty + B_1^{-1}AB$ fulfill the condition in Lemma 2.3. Hence, there exists $B_2 \in UT_\infty(\mathcal{F})$ such that $I_\infty + J = B_2^{-1}(I_\infty + B_1^{-1}AB_1)B_2$. Then, $J = B_2^{-1}(B_1^{-1}AB_1)B_2$. Takin $B = B_1B_2 \in T_\infty(\mathcal{F})$, we obtain $J = B^{-1}AB$ as wanted. ■

Lemma 2.5 Let $A = \sum_{i<j} a_{ij} E_{ij}$, be a matrix in $N_\infty(\mathcal{F})$ with $a_{i,i+1} \neq 0$ for every $i \geqslant 1$. Then there exists $\lambda_A \in \mathcal{F}$ such that $f(A) = \lambda_A A$.

Proof. If $A = \sum_{i<j} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$, there exists $T \in T_\infty(\mathcal{F})$ such that $TAT^{-1} = J$, by the previous corollary. Define $h : N_\infty(\mathcal{F}) \to N_\infty(\mathcal{F})$ by $h(X) =$

$Tf(T^{-1}XT)T^{-1}$. Then h is a non-additive Lie centralizer. Indeed,

$$\begin{aligned} h([A,B]) &= Tf(T^{-1}[A,B]T)T^{-1} \\ &= Tf(T^{-1}(AB-BA)T)T^{-1} \\ &= Tf(T^{-1}ATT^{-1}BT - T^{-1}BTT^{-1}AT)T^{-1} \\ &= Tf([T^{-1}AT, T^{-1}BT])T^{-1} \\ &= T[f(T^{-1}AT), T^{-1}BT]T^{-1} \\ &= T\left(f(T^{-1}AT)T^{-1}BT - T^{-1}BTf(T^{-1}AT)\right)T^{-1} \\ &= Tf(T^{-1}AT)T^{-1}B - BTf(T^{-1}AT)T^{-1} \\ &= [Tf(T^{-1}AT)T^{-1}, B] \\ &= [h(A), B] \end{aligned}$$

for all $A, B \in N_\infty(\mathcal{F})$. Hence, $h(J) = \lambda_A J$ by Lemma 2.2. Then

$$Tf(A)T^{-1} = Tf(T^{-1}(TAT^{-1})T)T^{-1} = h(J) = \lambda_A J = \lambda_A TAT^{-1}.$$

Multiplying the left and right sides by T^{-1} and T respectively yields $f(A) = \lambda_A A$. ■

Now, we wish to extend Lemma 2.5 to all elements of $N_\infty(\mathcal{F})$. In order to do it, let's introduce the set that we will denote by $\mathcal{S} = \{B = (b_{ij}) \in N_\infty(\mathcal{F}) : \ b_{i,i+1} \neq 0 \ \forall i \geqslant 1\}$. This set has an important property that is established below.

Lemma 2.6 Let $\mathcal{F}$ be a field. Every element of $N_\infty(\mathcal{F})$ can be written as a sum of at most two elements of $\mathcal{S}$.

Proof. If $a_{i,i+1} \neq 0$ for all $i \geqslant 1$, then A belongs in $\mathcal{S}$, so there is nothing to prove. If A is not in $\mathcal{S}$, then we can define B_1 and B_2 as follows:

$$(B_1)_{ij} = \begin{cases} a_{i,i+1} - b_i & \text{if } j = i+1 \\ a_{ij} & \text{if } j > i+1, \end{cases} \qquad , \qquad (B_2)_{ij} = \begin{cases} b_i & \text{if } j = i+1 \\ 0 & \text{otherwise}, \end{cases}$$

where b_i is a nonzero different elements of $\mathcal{F}$ from $a_{i,i+1}$. It is easy to see that B_1 and B_2 are in $\mathcal{S}$, and $A = B_1 + B_2$, so we wanted. ■

Lemma 2.7 Let $\mathcal{F}$ be a field. Then $f(A+B) = f(A) + f(B)$ for arbitrary elements $A, B \in N_\infty(\mathcal{F})$.

Proof. For any A, B, X of $N_\infty(\mathcal{F})$, we have

$$\begin{aligned} [f(A+B), X] &= f([A+B, X]) \\ &= [A+B, f(X)] \\ &= [A, f(X)] + [B, f(X)] \\ &= [f(A), X] + [f(B), X] \\ &= [f(A) + f(B), X], \end{aligned}$$

which implies that $f(A+B)-f(A)-f(B) \in \mathcal{Z}(N_\infty(\mathcal{F}))$. Thus, $f(A+B) = f(A)+f(B)$. ■

Proof of Theorem 1.1: Let $A, B \in \mathcal{S}$ be two non-commuting elements. By lemma 2.5, $f(A) = \lambda_A A$, $f(B) = \lambda_B B$, $\lambda_A, \lambda_B \in \mathcal{F}$. Since f is non-additive Lie centralizer, we have

$$\begin{aligned} f([A,B]) &= [f(A), B] = \lambda_A[A,B] \\ &= [A, f(B)] = \lambda_B[A,B] \end{aligned}$$

Then, $[A,B] \neq 0$ implies that $\lambda_A = \lambda_B$. If $A, B \in \mathcal{S}$ commute, then we take $C \in \mathcal{S}$ that does not commute neither with A nor with B. As we have just seen, $\lambda_A = \lambda_C$ and $\lambda_B = \lambda_C$. Given $X \in N_\infty(\mathcal{F})$. We know by lemma 2.6 that there exist $A, B \in \mathcal{S}$ such that $X = A+B$ (we can assume that $X \notin S$). Then $f(X) = f(A)+f(B)$ by lemma 2.7. Thus, $f(X) - \lambda_A A - \lambda_B B = f(X) - \lambda X$ for $\lambda \in \mathcal{F}$ such that $f(A) = \lambda A$ for all $A \in S$; that is, $f(X) = \lambda X$, and Theorem 1.1 is proved.

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