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Invariant elements in the dual Steenrod algebra

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Abstract. In this paper, we investigate the invariant elements of the dual mod p Steenrod subalgebra $\mathcal{A}_p^*$ under the conjugation map χ and give bounds on the dimensions of $(\chi - 1)(\mathcal{A}_p^*)_d$, where $(\mathcal{A}_p^*)_d$ is the dimension of $\mathcal{A}_p^*$ in degree d.

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1. Introduction

The mod p Steenrod algebra ($\mathcal{A}$), is the algebra of the stable cohomology operations on mod p cohomology, where p is a prime number. It has a Hopf structure and its dual is a commutative Hopf algebra which is isomorphic to the polynomial algebra in generators ξ_k of degree $2^k - 1$ ($k \geqslant 0$) for $p = 2$ and isomorphic to the tensor product of the polynomial algebra in generators ξ_k of degree $2p^k - 2$ ($k \geqslant 1$) and the exterior algebra in generators τ_k of degree $2p^k - 1$ ($k \geqslant 0$) for $p > 2$. The Steenrod algebra and its dual have unique anti-automorphism map χ over themselves. The map χ is also called a conjugation.

Researchers in this area obtained some formulas to determine the image of the certain monomial of the Steenrod algebra to get an information about the image of all monomials under χ. Milnor [7] has given a formula of a conjugation of a Steenrod square (or Steenrod powers) in terms of Milnor bases in a certain degree. Davis [5] computes the conjugation of $\mathcal{P}^{p^{n-1}+\ldots+p+1}$ by Milnor's formula. Straffin [12] has got a formula of the images of Sq^{2^k}'s

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for $k \geqslant 0$, the algebraically generators of the Steenrod algebra, under χ. Silverman [10] extends the formula of Davis and computes the conjugation of $\mathrm{Sq}^{2^j(2^{i+1}-1)} \ldots \mathrm{Sq}^{(2^{i+1}-1)}$. Karaca and I.Y. Karaca [6] have generalized the results of Silverman for the odd prime case. For $p = 2$, let $\mathcal{A}$ denote the mod 2 Steenrod algebra. Crossley and Whitehouse [4] determine the subspace, $\mathcal{A}^{\chi}$, of elements invariant under χ and give bounds on the dimension of this subspace for each degree.

This paper is organized as follows. In section 2, we give the construction of the Steenrod algebra and basic facts related to our study. In section 3, we generalize the results of [4] to odd prime case. Let $\mathcal{A}_p$ denote the subalgebra of the mod p Steenrod algebra generated by the Steenrod powers $\mathcal{P}^k$ for $k \geqslant 0$ and $\mathcal{A}_p^*$ denote its dual. If $(\mathcal{A}_p^*)_d$ is the d degree part of $\mathcal{A}_p^*$ and D_d is the dimension of $(\mathcal{A}_p^*)_d$ then we find bounds on the dimensions of $(\chi-1)(\mathcal{A}_p^*)_d$ and we obtain

$$D_{d-1}/2 \leqslant dim((\chi-1)(\mathcal{A}_p^*)_d) \leqslant D_d. \tag{1}$$

2. Preliminaries

Let p be a prime number. Then the Steenrod operation is a natural transformation

$$\begin{aligned} \mathrm{Sq}^i &: H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2), \quad \text{if } \; p=2 \\ \mathcal{P}^i &: H^n(X;\mathbb{Z}_p) \to H^{n+2i(p-1)}(X;\mathbb{Z}_p) \quad \text{if } \; p>2 \end{aligned}$$

that has certain conditions such as instability. The degree of the element Sq^i is i and the degree of $\mathcal{P}^i$ is $2i(p-1)$. The Steenrod squares Sq^k (or $\mathcal{P}^k$) generate all stable operations in the cohomology theory [9]. In [1, 2], Adem showed that all relations in the Steenrod algebra are generated by the set of Adem relations: For $p=2$, we have

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$$

for all $i, j > 0$ such that $i < 2j$. For the odd prime number p,

$$\mathcal{P}^i\mathcal{P}^j = \sum_{k=0}^{[\frac{i}{p}]} (-1)^{i+k} \binom{(p-1)(j-k)-1}{i-pk} \mathcal{P}^{i+j-k}\mathcal{P}^k,$$

where $i < pj$ and

$$\begin{aligned} \mathcal{P}^i\beta\mathcal{P}^j = &\sum_{k=0}^{[\frac{i}{p}]} (-1)^{i+k} \binom{(p-1)(j-k)}{i-pk} \beta\mathcal{P}^{i+j-k}\mathcal{P}^k \\ &- \sum_{k=0}^{[\frac{i-1}{p}]} (-1)^{i+k-1} \binom{(p-1)(j-k)-1}{i-pk-1} \mathcal{P}^{i+j-k}\beta\mathcal{P}^k \end{aligned}$$

such that $i \leqslant pj$ for all $i, j > 0$, where β is the Bockstein homomorphism

$$\beta : H^n(X;\mathbb{Z}_p) \to H^{n+1}(X;\mathbb{Z}_p).$$

Then the mod 2 Steenrod algebra is the graded associative algebra over $\mathbb{Z}_2$ generated by the elements Sq^i of degree i for $i \geqslant 0$ subject to the Adem relations. Likewise the construction of the mod 2 Steenrod algebra, for an odd prime number p, the mod p Steenrod algebra is the graded associative algebra over $\mathbb{Z}_p$ generated by the elements $\mathcal{P}^i$ of degree $2i(p-1)$ for $i > 0$, and β of degree 1 such that $\beta^2 = 0$ subject to the Adem relations. For details, we refer to references [3, 8, 11, 13].

The Adem relations lead us to have a minimal algebraic generating set for the Steenrod algebra. For $p = 2$, the mod 2 Steenrod algebra is generated as an algebra by Sq^0 and Sq^{2^k} for $k \geqslant 0$. For $p > 2$, the mod p Steenrod algebra is generated as an algebra by β and the operations $\mathcal{P}^{p^k}$ for $k \geqslant 0$ [9].

Milnor [7] has shown that there is a unique algebra map $\psi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that for all $i \geqslant 0$

$$\begin{aligned} \psi(\mathrm{Sq}^i) &= \sum_{j+k=i} \mathrm{Sq}^j \otimes \mathrm{Sq}^k, \quad \text{if } p = 2 \\ \psi(\mathcal{P}^i) &= \sum_{j+k=i} \mathcal{P}^j \otimes \mathcal{P}^k \quad \text{and} \\ \psi(\beta) &= \beta \otimes 1 + 1 \otimes \beta, \quad \text{if } p > 2, \end{aligned}$$

which makes it into a Hopf algebra and incorporates a conjugation map $\chi : \mathcal{A} \to \mathcal{A}$. Conjugation map χ is an anti-automorphism of the Steenrod algebra and $\chi^2 = 1$. The conjugation in $\mathcal{A}$ is determined in Steenrod operations by

$$\begin{aligned} \chi(\mathrm{Sq}^k) &= \sum_{1 \leqslant i \leqslant k} \mathrm{Sq}^i \chi(\mathrm{Sq}^{k-i}) \quad \text{if } \ p = 2, \\ \chi(\mathcal{P}^k) &= \sum_{1 \leqslant i \leqslant k} \mathcal{P}^i \chi(\mathcal{P}^{k-i}) \quad \text{if } \ p > 2. \end{aligned}$$

Milnor [7] has shown that the dual of the mod 2 Steenrod algebra is a polynomial algebra on generators ξ_i, $i \geqslant 0$, of degree $2^i - 1$ and for an odd prime number p, the dual of the mod p Steenrod algebra is the tensor product of a polynomial algebra on generators ξ_i, $i \geqslant 1$, of degree $2(p^i - 1)$ and an exterior algebra on generators τ_i, $i \geqslant 0$, of degree $2p^i - 1$. The co-product ψ on the dual Steenrod algebra $\mathcal{A}^*$ is given by the formulas [7]

$$\begin{aligned} \psi(\xi_k) &= \sum_{0 \leqslant i \leqslant k} \xi_{k-i}^{2^i} \otimes \xi_i, \quad \text{if } p = 2, \\ \psi(\xi_k) &= \sum_{0 \leqslant i \leqslant k} \xi_{k-i}^{p^i} \otimes \xi_i \quad \text{and} \\ \psi(\tau_k) &= \tau_k \otimes 1 + \sum_{0 \leqslant i \leqslant k} \xi_{k-i}^{p^i} \otimes \tau_i \qquad \text{if } p > 2. \end{aligned}$$

The conjugation formula on the generators of the dual Steenrod algebra is

$$\chi(\xi_n) = \sum_{\alpha \in \mathrm{Part(n)}} (-1)^{\ell(\alpha)} \prod_{i=1}^{\ell(\alpha)} \xi_{\alpha(i)}^{p^{\sigma(i)}}, \tag{2}$$

where $\alpha = (\alpha(1), \alpha(2), \ldots \alpha(\ell(\alpha)))$ is an ordered partition α of the integer n with length $\ell(\alpha)$, Part(n) is the set of all ordered partitions of n and $\sigma(i)$ is the partial sum $\sigma(i) = \sum_{j=1}^{i-1} \alpha(j)$.

3. Conjugation Invariants

Let $\mathbb{F}_p$ be a field with p elements.

Proposition 3.1 Let A be a Hopf algebra. Let $A^\chi = \mathrm{Ker}(\chi - 1)$ be the invariant elements of A under the antiautomorphism $\chi : A \to A$. If A is a (co)commutative Hopf algebra over $\mathbb{F}_p$, then $\mathrm{Im}(\chi + 1) = \mathrm{Ker}(\chi - 1)$ and $\mathrm{Im}(\chi - 1) = \mathrm{Ker}(\chi + 1)$.

Proof. Let $a \in \mathrm{Ker}(\chi - 1)$. This means that $\chi(a) = a$. If we take an element $b \in A$ as $b = \frac{1}{2}a$, then $(\chi+1)(b) = a$ and hence $a \in \mathrm{Im}(\chi+1)$. On the other hand, let $a \in \mathrm{Im}(\chi+1)$ so that there is an element b in A such that $(\chi + 1)(b) = a$. Then, we obtain

$$(\chi - 1)(a) = (\chi - 1)(\chi + 1)(b) = (\chi^2 - 1)(b) = 0$$

and, hence $a \in \mathrm{Ker}(\chi - 1)$.

Let $a \in \mathrm{Ker}(\chi + 1)$. This means that $\chi(a) = -a$. If we take an element $b \in \mathcal{A}$ as $b = -\frac{1}{2}a$, then $(\chi - 1)(b) = a$ and hence $a \in \mathrm{Im}(\chi - 1)$. Let $a \in \mathrm{Im}(\chi - 1)$ so that there is an element b in $\mathcal{A}$ such that $(\chi - 1)(b) = a$. Then, we have

$$(\chi + 1)(a) = (\chi + 1)(\chi - 1)(b) = (\chi^2 - 1)(b) = 0$$

and hence, $a \in \mathrm{Ker}(\chi + 1)$. ■

Then we have the corrollary follows from the Proposition 3.1.

Corollary 3.2 $\mathrm{Im}(\chi - 1)$ is a subalgebra of A.

Let $\mathcal{A}_p$ be the mod p Steenrod algebra reduced by the Steenrod powers $\mathcal{P}^k$. We now consider the dual mod p Steenrod algebra $\mathcal{A}_p^*$. Let $(\mathcal{A}_p^*)_d$ be the part of $\mathcal{A}_p^*$ in degree d and use the notation $(r_1, r_2, \ldots, r_n)$ to denote the monomial $\xi_1^{r_1}\xi_2^{r_2}\ldots\xi_k^{r_k}$. We order the monomials of a given degree in $\mathcal{A}_p^*$ by the right lexicographic orderings. For instance,

$$(1, 2, 0, 4) < (3, 1, 1, 5) < (0, 0, 0, 0, 1).$$

Proposition 3.3 The matrix of the conjugation map $\chi : (\mathcal{A}_p^*)_d \to (\mathcal{A}_p^*)_d$ in each degree is lower triangular with respect to right lexicographic ordering of the monomial bases.

Proof. By (2), we have $\chi(\xi_n) = -\xi_n + P_n$, where P_n is a polynomial in $\xi_1, \ldots, \xi_{n-1}$ so that P_n is strictly lower than ξ_n. Hence for any monomial M in $\xi_1, \ldots, \xi_n$, we have $\chi(M) = \pm M + Q$, where Q is strictly lower than M. It follows that the matrix is lower triangular. ■

Definition 3.4 [4] Let $\xi_1^{r_1}\xi_2^{r_2}\ldots\xi_k^{r_n}$ be the monomial in $(\mathcal{A}_p^*)_d$. If $r_n = 1$, then the monomial is called uniterminal.

Proposition 3.5 In $(\mathcal{A}_p^*)_d$, the image of unitermials under $\chi - 1$ is linearly independent.

Proof. For a uniterminal monomial $(r_1, \dots, r_{n-1}, 1)$, we have

$$\begin{aligned}\chi(\xi_1^{r_1}\xi_2^{r_2}\dots\xi_{n-1}^{r_{n-1}}\xi_n) &= \pm\chi(\xi_1)^{r_1}\chi(\xi_2)^{r_2}\dots\chi(\xi_{n-1})^{r_{n-1}}\chi(\xi_n)\\ &= \pm(-\xi_1)^{r_1}(-\xi_2+\dots.)^{r_2}\dots(-1\xi_{n-1}+\dots)^{r_{n-1}}(\dots+\xi_1\xi_{n-1}^p+\dots)\\ &= \pm(\xi_1^{r_1}\xi_2^{r_2}\dots\xi_{n-1}^{r_{n-1}}\xi_1\xi_{n-1}^p)(\dots)\\ &= \pm(\xi_1^{r_1+1}\xi_2^{r_2}\dots\xi_{n-1}^{r_{n-1}+p})(\dots).\end{aligned}$$

Therefore, the image of the monomial $(r_1, \dots, r_{n-1}, 1)$ under $\chi - 1$ contains the monomial $(r_1 + 1, r_2, \dots, r_{n-1} + p)$ as a summand. Now, we claim that with respect to the right lexicographic ordering, no lower uniterminal monomial than $(r_1, \dots, r_{n-1}, 1)$ has $(r_1 + 1, r_2, \dots, r_{n-1} + p)$ as a summand in its image under $\chi - 1$. Assuming not, let $(s_1, \dots, s_{n'-1}, 1)$ be another monomial lower than $(r_1, \dots, r_{n-1}, 1)$ contains the monomial $(r_1 + 1, r_2, \dots, r_{n-1} + p)$ as a summand in its image and let $n' \leqslant n$. If $n' \leqslant n - 1$, then $(\chi - 1)(s_1, \dots, s_{n'-1}, 1)$ will have no summand which contains ξ_{n-1}-exponent greater than 1. The matrix of a conjugation map is a lower triangular with respect to right lexicographic ordering so that $(\chi - 1)(s_1, \dots, s_{n'-1}, 1)$ doesn't have $(r_1 + 1, r_2, \dots, r_{n-1} + p)$ as a summand. If $n' = n$, then $(\chi - 1)(s_1, \dots, s_{n'-1}, 1)$ contains a summand $(s_1 + 1, \dots, s_{n'-1} + p)$ that is less than $(r_1 + 1, r_2, \dots, r_{n-1} + p)$ so that it cannot contain $(r_1 + 1, r_2, \dots, r_{n-1} + p)$ as a summand.

Let Q be a linear combination of images under $\chi - 1$ of uniterminal monomials. Let P is a linear combination of uniterminal monomials so we may write $Q = (\chi - 1)(P)$. Suppose that $(r_1, \dots, r_{n-1}, 1)$ is the highest monomial which appears in P. Then our claim shows that $(\chi - 1)(P)$ has $(r_1 + 1, r_2, \dots, r_{n-2}, r_{n-1} + 2)$ as a summand and cannot be a zero. This completes the proof. ■

Let U_d denote the number of uniterminal monomials in degree d. It is easy to see that $\dim((\chi - 1)(\mathcal{A}_p^*)_d) \geqslant U_d$. Using Lemma 3.3 in [4], we obtain some results about U_d.

Lemma 3.6 Let D_d denote the dimension of $(\mathcal{A}_p^*)_d$. Then

(i) $U_d \geqslant \frac{D_{d-1}}{2}$,
(ii) $U_d = D_{d-1} - U_{d-1}$.

Proof. We map uniterminal monomials in degree d with another degree d monomials that is not uniterminal by the pairing

$$(r_1, r_2, \dots, r_{n-1}, 1) \longleftrightarrow (r_1 + 1, r_2, \dots, r_{n-1} + p).$$

Note that the monomials left unpaired are characterized by the fact that they begin with zero and not uniterminal. The number of these is $D_d - 2U_d$ which is less than or equal to the total number beginning with zero which is $D_d - D_{d-1}$. This gives us the first claim.

The number of uniterminal monomials starting with zero in degree d is $U_d - U_{d-1}$ so that the number of unpaired monomials is $(D_d - D_{d-1}) - (U_d - U_{d-1})$. Hence, we have

$$D_d - 2U_d = (D_d - D_{d-1}) - (U_d - U_{d-1}),$$

which implies that $-U_d = -D_{d-1} + U_{d-1}$ and this gives us the second claim. ■

Under these circumstances, we give bounds on the dimensions of $(\chi - 1)(\mathcal{A}_p^*)_d$. Finally, we obtain the following result.

Theorem 3.7 Let D_d be the dimension of $(\mathcal{A}_p^*)_d$. Then, we have

$$\frac{D_{d-1}}{2} \leqslant \dim((\chi - 1)(\mathcal{A}_p^*)_d) \leqslant D_d.$$

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