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Solving fractional evolution problem in Colombeau algebra by mean generalized fixed point

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Abstract. The present paper is devoted to the existence and uniqueness result of the fractional evolution equation $D_c^q u(t) = g(t, u(t)) = Au(t) + f(t)$ for the real $q \in (0, 1)$ with the initial value $u(0) = u_0 \in \tilde{\mathbb{R}}$, where $\tilde{\mathbb{R}}$ is the set of all generalized real numbers and A is an operator defined from $\mathcal{G}$ into itself. Here the Caputo fractional derivative D_c^q is used instead of the usual derivative. The introduction of locally convex spaces is to use their topology in order to define generalized semigroups and generalized fixed points, then to show our requested result.

Keywords: Colombeau algebra, locally convex space, generalized semigroup, generalized fixed point.

2010 AMS Subject Classification: 34-XX.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non integer order. Moreover fractional processes have been increased many developments in the last decade. For instance, they are suitable for describing the long memory properties of many time series. A strong motivation for investigating fractional differential equations comes from physics. Fractional diffusion equations describe anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous

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semiconductors or strongly porous materials; see [5, 9]. Colombeau algebras (usually denoted by the letter $\mathcal{G}$) are differential (quotient) algebras with unit, and were introduced by Colombeau [2, 3]. This algebra plays a crucial role in order to give a sense of multiplication of distributions [4, 9]. As a nonlinear extension of distribution theory to deal with nonlinearities and singularities of data and coefficients in PDE theory [9]. These algebras contain the space of distributions D' as a subspace with an embedding realized through convolution with a suitable mollifier. Elements of these algebras are classes of nets of smooth functions. The reason for introducing fractional derivatives was the possibility of solving nonlinear problems with singularities and derivatives of arbitrary real order. Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach fixed point theorem. There exists a vast literature on the topic field and this is very active field of research at present. Fixed point theorems are very important tools for proving the existence and uniqueness of the solutions to various mathematical models (integral and partial equations, variational inequalities, etc). It can be applied to for example: variational inequalities, optimization, and approximation theory. The fixed point theory has been continually studied by many researchers (see [3]). It is rare to find a paper that presented the fixed point theory in Colombeau algebra. We will rely on the work of Martin in [7] and we will used the topology of locally convex spaces in order to give a sense of the concept of fixed point in the such algebra. In this paper, we investigate the existence and uniqueness of solutions to the following problem.

$$\begin{cases} D_c^q u(t) = g(t, u(t)) = Au(t) + f(t), \ t \in [0, b], \\ u(0) = u_0 \in \tilde{\mathbb{R}} \end{cases} \tag{1}$$

where A is an operator defined from $\mathcal{G}$ into itself and $f : [0, b] \to \widetilde{\mathbb{R}}$ is a continuous function. We give a definition of generalized semigroup for study the integral solution of a such equation.

The present paper is organized as follows: after this introduction, we will recall some concept concerning the Colombeaus algebra and fractional calculus in section 2. The new notion of generalized semigroup and some properties take place in section 3. In section 4, we provided the theorem of fixed point in Colombeau algebra. Finally, the existence-uniqueness result for a fractional differential equation is proven in section 5.

2. Preliminaries

Here we list some notations and formulas to be used later. The elements of Colombeau algebras $\mathcal{G}$ are equivalence classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ϵ. Therefore, for any set X, the family of sequences $(u_\epsilon)_{\epsilon \in (0,1)}$ of elements of a set X will be denoted by $X^{(0,1)}$, such sequences will also be called nets and simply written as u_ϵ. Let $n \in \mathbb{N}^*$, as in [4] we define the set $\mathcal{E}(\mathbb{R}^n) = \left(\mathcal{C}^\infty(\mathbb{R}^n)\right)^{(0,1)}$. The set of moderate functions is given as follows:

$$\mathcal{E}_M(\mathbb{R}^n) = \Big\{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^n) | \, \forall K \subset\subset \mathbb{R}^n \& \forall \alpha \in \mathbb{N}_0^n \, \exists N \in \mathbb{N} \, s.t. \sup_{x \in K} |D^\alpha u_\varepsilon(x)| = \mathcal{O}_{\epsilon \to 0}(\varepsilon^{-N})\Big\}.$$

The ideal of negligible functions is defined by

$$\mathcal{N}(\mathbb{R}^n) = \Big\{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^n) | \, \forall K \subset\subset \mathbb{R}^n \& \forall \alpha \in \mathbb{N}_0^n \& \forall p \in \mathbb{N}, \sup_{x \in K} |D^\alpha u_\varepsilon(x)| = \mathcal{O}_{\epsilon \to 0}(\varepsilon^p)\Big\}.$$

The Colombeau algebra is defined as a factor set $\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n)$. Also, we set $\left|\mathcal{E}_M(\mathbb{R}^n)\right| = \left\{ (|u_\epsilon|)_\epsilon ,\ u_\epsilon \in \mathcal{E}_M(\mathbb{R}^n)\right\}$ and $\left|\mathcal{N}(\mathbb{R}^n)\right| = \left\{ (|u_\epsilon|)_\epsilon ,\ u_\epsilon \in \mathcal{N}(\mathbb{R}^n)\right\}$. Moreover, the set of all generalized real numbers is defined by $\widetilde{\mathbb{R}} = \mathcal{E}\left(\mathbb{R}\right)/N\left(\mathbb{R}\right)$, where

$$\mathcal{E}(\mathbb{R}) := \left\{(x_\varepsilon)_\varepsilon \in (\mathbb{R})^{(0,1)} |\, \exists m \in \mathbb{N}, |x_\varepsilon| = \mathcal{O}_{\epsilon\to 0}(\varepsilon^{-m})\right\},$$
$$N(\mathbb{R}) := \left\{(x_\varepsilon)_\varepsilon \in (\mathbb{R})^{(0,1)} |\, \forall m \in \mathbb{N}, |x_\varepsilon| = \mathcal{O}_{\epsilon\to 0}(\varepsilon^{m})\right\}.$$

Note that $\widetilde{\mathbb{R}}$ is a ring obtained by factoring moderate families of real numbers with respect to negligible families.

Proposition 2.1 The space $\mathcal{E}(\mathbb{R})$ is an algebra and $N(\mathbb{R})$ is an ideal of $\mathcal{E}(\mathbb{R})$.

In the same, we set $\left|\mathcal{E}(\mathbb{R})\right| = \left\{ (|r_\epsilon|)_\epsilon , r_\epsilon \in \mathcal{E}(\mathbb{R})\right\}$ and $\left|N(\mathbb{R})\right| = \left\{ (|r_\epsilon|)_\epsilon , r_\epsilon \in N(\mathbb{R})\right\}$. A fractional integral is defined by

$$J^\alpha f(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \qquad \alpha > 0.$$

Fractional calculus is a branch of mathematical analysis that studies the several different possibilities of defining real number powers or complex number powers of the differentiation operator D. For example, one may ask the question of meaningfully interpreting $D^{\frac{1}{2}}$. It is known that there are many types of derivatives of non-integral order, but in this time we will work with Caputo approach. The fractional derivative of order $\alpha > 0$ in the Caputo sense is defined by

$$D^\alpha f(t) = \tfrac{1}{\Gamma(m-\alpha)} \int_0^t \tfrac{f^{(m)}(\tau) d\tau}{(t-\tau)^{\alpha+1-m}} \qquad m-1 < \alpha < m.$$

Let (f_ε) is a representative of $F \in \mathcal{G}$. Then

$$D^\alpha f_\varepsilon(t) = \tfrac{1}{\Gamma(1-\alpha)} \int_0^t \tfrac{f'_\varepsilon(\tau) d\tau}{(t-\tau)^{\alpha}} \qquad 0 < \alpha < 1.$$

Now, we have

$$\begin{aligned}
\sup_{t\in[0,T]} \left|D^\alpha f_\varepsilon(t)\right| &\leqslant \frac{1}{\Gamma(1-\alpha)} \sup_{t\in[0,T]} \left|\int_0^t \frac{f'(\tau)d\tau}{(t-\tau)^\alpha}\right| \\
&\leqslant \frac{1}{\Gamma(1-\alpha)} ||f'||_{L^\infty([0,T])} \sup_{t\in[0,T]} \int_0^t \frac{d\tau}{(t-\tau)^\alpha} d\tau \\
&\leqslant \frac{1}{\Gamma(1-\alpha)} \varepsilon^{-N} \frac{T^{1-\alpha}}{1-\alpha} \leqslant C_{\alpha,T} \varepsilon^{-N}.
\end{aligned}$$

In general, for $m-1 < \alpha < m$, we have

$$\begin{aligned}
\sup_{t\in[0,T]} \left|D^\alpha f_\varepsilon(t)\right| &\leqslant \frac{1}{\Gamma(m-\alpha)} \sup_{t\in[0,T]} \int_0^t \frac{|f^{(m)}(\tau)|}{(t-\tau)^{\alpha+1-m}} d\tau \\
&\leqslant \frac{1}{\Gamma(m-\alpha)} ||f^{(m)}||_{L^\infty([0,T])} \sup_{t\in[0,T]} \int_0^t \frac{1}{(t-\tau)^{\alpha+1-m}} d\tau \\
&\leqslant \frac{1}{\Gamma(m-\alpha)} \varepsilon^{-N} \frac{T^{m-\alpha}}{m-\alpha} \leqslant C_{\alpha,T} \varepsilon^{-N},
\end{aligned}$$

where constant $C_{\alpha,T}$ depends on two parameters α and T. In order to prove moderateness for higher derivatives a similar calculation is applied.

Let $G_1, G_2 \in \mathcal{G}(\mathbb{R}^n)$ and $G_{1,\varepsilon}, G_{2,\varepsilon}$ their representatives, respectively. We say that $G_1, G_2 \in \mathcal{G}(\mathbb{R}^n)$ are associated and write $G_1 \approx G_2$ if

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} (G_{1,\varepsilon} - G_{2,\varepsilon})\varphi(x)dx = 0.$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

We will end this preliminaries by the Grönwall's inequality.

Lemma 2.2 Let δ, η and u be three functions defined on an interval $I = (a, b)$ such that η and u are continuous. Moreover, suppose that δ is locally integrable on I.

(1) If η is non-negative and u satisfies the integral inequality

$$u(t) \leqslant \delta(t) + \int_a^t \eta(s)u(s)ds$$

for all $t \in I$, then

$$u(t) \leqslant \delta(t) + \int_a^t \delta(s)\eta(s) \exp\left(\int_s^t \eta(r)dr\right)ds.$$

(2) In addition, if the function δ is non-decreasing, then

$$u(t) \leqslant \delta(t) \exp\left(\int_a^t \eta(s)ds\right), \quad t \in I.$$

3. Generalized semigroups

The notion of a semigroup plays a crucial role in order to study an evolutionary problem. As we have known a lot of research is devoted to the linking relationship between semigroups of an operator and its infinitesimal generator, the famous relationship is given by Theorem 3.1 in [10]. In this section we will benefit the classical case and the method of building the Colombeau algebra for giving a sense of the generalized semigroups. We will start by some properties of locally convex spaces.

3.1 *Locally convex spaces*

In this subsection, we recall the concept of locally convex spaces and the notion of completeness in this type of space.

Definition 3.1 Let X be a vector space with a seminorms family $(p_i)_{i\in I}$. For all $i \in I$, we denote τ_i the topology defined by the seminorm p_i, and τ the topology generated by the classes $\bigcup_{i\in I} \tau_i$. The couple $\big(X, \tau\big)$ is called a locally convex space.

A basis of 0-neighborhood is the set of all "balls" of the seminorms $(p_i)_{i\in I}$ is

$$B(i, r) = \Big\{x \in X | \quad p_i(x) < r\Big\} \quad \forall i \in I \text{ and } r > 0.$$

Also, $(x_n)n \in \mathbb{N}$ is a Cauchy sequence if and only if

$$\forall \varepsilon > 0, \forall i \in I\, \exists n_0 \in \mathbb{N}\, s.t.\ \forall n, p \in \mathbb{N} \text{ if } n \geqslant n_0 \Rightarrow p_i(x_{n+p} - x_n) < \varepsilon.$$

X is sequentially complete if any Cauchy sequence converges to an element e in X.

Definition 3.2 We said that $\mathcal{D}$ is dense in locally convex space X if and only if

$$\forall x \in X\ \exists y \in \mathcal{D}\ st.\ \forall \varepsilon > 0\,\&\,\forall i \in I:\ p_i(x-y) < \varepsilon.$$

3.2 *Generalized semigroups*

As we known the semigroup of operator is defined on Banach spaces, but so far we haven't this concept in algebra of Colombeau. So, in order to define this concept we needed to exploit the previous subsection for manipulating such notion. This subsection is devoted to defining the generalized semigroups and its properties.

Definition 3.3 Let X be a locally convex space with a seminorm family $(p_i)_{i\in I}$. We define

$$\mathcal{E}_M(X) := \Big\{(x_\varepsilon)_\varepsilon \in (X)^{(0,1)} |\, \exists m \in \mathbb{N}, \forall i \in I, p_i(x_\varepsilon) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^{-m})\Big\},$$
$$\mathcal{N}(X) := \Big\{(x_\varepsilon)_\varepsilon \in (X)^{(0,1)} |\, \forall m \in \mathbb{N}, \forall i \in I, \quad p_i(x_\varepsilon) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^{m})\Big\}.$$

Also, we define the Colombeau algebra type by $\tilde{X} = \mathcal{E}_M(X)/\mathcal{N}^s(X)$.

First, we are looking if it is possible to define a map $A : \tilde{X} \longrightarrow \tilde{X}$ by means of a given family $(A_\varepsilon)_{\varepsilon\in(0,1)}$ of maps $A_\varepsilon : X \longrightarrow X$ where A_ε is a linear and continuous operator. The general requirement is given in the following lemma

Lemma 3.4 Let $(A_\varepsilon)_{\varepsilon\in(0,1)}$ be a given family of maps $A_\varepsilon : X \longrightarrow X$. For each $(x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$ and $(y_\varepsilon)_\varepsilon \in \mathcal{N}(X)$, suppose that

(1) $\Big(A_\varepsilon x_\varepsilon\Big)_\varepsilon \in \mathcal{E}_M(X)$,

(2) $\Big(A_\varepsilon(x_\varepsilon + y_\varepsilon)\Big)_\varepsilon - \Big(A_\varepsilon x_\varepsilon\Big)_\varepsilon \in \mathcal{N}(X)$.

Then

$$A : \begin{cases} \tilde{X} \longrightarrow \tilde{X} \\ x = \Big[x_\varepsilon\Big] \longmapsto Ax = \Big[A_\varepsilon x_\varepsilon\Big], \end{cases}$$

is well defined.

Proof. From the first property we see that the class $\Big[(A_\varepsilon x_\varepsilon)_\varepsilon\Big] \in \tilde{X}$. Let $x_\varepsilon + y_\varepsilon$ be another representative of $x = [x_\varepsilon]$. From the second property, we have $\Big(A_\varepsilon(x_\varepsilon + y_\varepsilon)\Big)_\varepsilon - \Big(A_\varepsilon x_\varepsilon\Big)_\varepsilon \in \mathcal{N}(X)$ and $\Big[\big(A_\varepsilon(x_\varepsilon + y_\varepsilon)\big)_\varepsilon\Big] = \Big[\big(A_\varepsilon x_\varepsilon)\big)_\varepsilon\Big] \in \tilde{X}$. Thus, A is well defined. ■

Now, we will give the definition of generalized semigroups on the Colombeau's algebra.

Definition 3.5 We define $\mathcal{SE}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$ as the space of all nets $\big(S_\varepsilon\big)_\varepsilon$ of strongly continuous mappings $S_\varepsilon : \mathbb{R}_+ \longrightarrow \mathcal{L}_c(X)$, $\varepsilon \in (0,1)$, with the property that, for every $T > 0$, there is $a \in \mathbb{R}$ such that

$$\sup_{t\in[0,T)} p_i\left(S_\varepsilon(t)\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^a) \tag{2}$$

for all $i \in I$ and $\mathcal{SN}(\mathbb{R}_+ : \mathcal{L}_c(X))$ is the space of nets $(N_\varepsilon)_\varepsilon$ of strongly continuous mappings $N_\varepsilon : \mathbb{R}_+ \longrightarrow \mathcal{L}_c(X),\ \varepsilon \in (0,1)$ with the properties:

For every $b \in \mathbb{R}$ and $T > 0$

$$\sup_{t\in[0,T)} p_i\left(N_\varepsilon(t)\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^b). \tag{3}$$

There exist $t_0 > 0$ and $a \in \mathbb{R}$ such that

$$\sup_{t<t_0} p_i\left(\frac{N_\varepsilon(t)}{t}\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^a) \tag{4}$$

for all $i \in I$.
There exists a net $\big(H_\varepsilon\big)_\varepsilon$ in $\mathcal{L}_c(X)$ and $\varepsilon_0 \in (0,1)$ such that

$$\lim_{t\longrightarrow 0} \frac{N_\varepsilon(t)}{t} = H_\varepsilon x, \;\; x \in X. \tag{5}$$

For all $b > 0$ and $i \in I$,

$$p_i\left(H_\varepsilon\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^b). \tag{6}$$

Remark 1 *Let us remark that, because of* (4), *it is enough that* (5) *holds for all $x \in D$ where D is a dense subspace of X.*

The following proposition show that the previous notion is in type Colombeau's algebra. Namely this concept take place in our context.

Proposition 3.6 $\mathcal{SE}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$ is algebra with respect to composition and $\mathcal{SN}\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$ is an ideal of $\mathcal{SE}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$.

Proof. Let $\big(S_\varepsilon(t)\big)_\varepsilon \in \mathcal{SE}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$ and $\big(N_\varepsilon(t)\big)_\varepsilon \in \mathcal{SN}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$. We will prove only the second assertion, i.e., that

$$\big(S_\varepsilon(t)N_\varepsilon(t)\big)_\varepsilon, \big(N_\varepsilon(t)S_\varepsilon(t)\big)_\varepsilon \in \mathcal{SN}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big),$$

where $S_\varepsilon(t)N_\varepsilon(t)$ denotes the composition. Let $\varepsilon \in (0,1)$. By the properties (2) and (5) of the Definition 3.5, for some $a \in \mathbb{R}$ and every $b \in \mathbb{R}$, we have

$$p_i\left(S_\varepsilon(t)N_\varepsilon(t)\right) \leqslant p_i\left(S_\varepsilon(t)\right) p_i\left(N_\varepsilon(t)\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^{a+b})$$

for all $i \in I$. The same holds for $p_i\left(N_\varepsilon(t)S_\varepsilon(t)\right)$ for all $i \in I$. Further, the properties (2) and (5) of the definition yield

$$\sup_{t<t_0} p_i\left(\frac{S_\varepsilon(t)N_\varepsilon(t)}{t}\right) \leqslant \sup_{t<t_0} p_i\left(S_\varepsilon(t)\right) \sup_{t<t_0} p_i\left(N_\varepsilon(t)\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^a)$$

for some $t_0 > 0$, $a \in \mathbb{R}$ and for all i. Also,

$$\sup_{t<t_0} p_i\left(\frac{S_\varepsilon(t)N_\varepsilon(t)}{t}\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^a)$$

for some $t_0 > 0$, $a \in \mathbb{R}$ and for all i. Now, let $\varepsilon \in (0,1)$ be fixed. For all $i \in I$, we have

$$p_i\left(\frac{S_\varepsilon(t)N_\varepsilon(t)}{t}x - S_\varepsilon(0)H_\varepsilon x\right) = p_i\left(S_\varepsilon(t)\frac{N_\varepsilon(t)}{t}x - S_\varepsilon(t)H_\varepsilon x + S_\varepsilon(t)H_\varepsilon x - S_\varepsilon(0)H_\varepsilon x\right)$$
$$\leqslant p_i\left(S_\varepsilon(t)\right)\left(\frac{N_\varepsilon(t)}{t}x - S_\varepsilon(t)H_\varepsilon x\right) + p_i\left(S_\varepsilon(t)H_\varepsilon x - S_\varepsilon(0)H_\varepsilon x\right).$$

By the (2) and (4) of the Definition 3.5 as well as by the continuity of $t \longrightarrow S_\varepsilon(t)(H_\varepsilon x)$ at zero, it follows that the last expression tend to zero as $t \mapsto 0$. Similarly, we have

$$p_i\left(\frac{N_\varepsilon(t)S_\varepsilon(t)}{t}x - H_\varepsilon S_\varepsilon(0)x\right) = \left(\frac{N_\varepsilon(t)}{t}S_\varepsilon(t)x - \frac{N_\varepsilon(t)}{t}S_\varepsilon(0)x + \frac{N_\varepsilon(t)}{t}S_\varepsilon(0)x - H_\varepsilon S_\varepsilon(0)x\right)$$
$$\leqslant p_i\left(\frac{N_\varepsilon(t)}{t}\right)p_i\left(S_\varepsilon(t)x - S_\varepsilon(0)x\right) + p_i\left(\frac{N_\varepsilon(t)}{t}(S_\varepsilon(0)x) - H_\varepsilon(S_\varepsilon(0)x)\right).$$

Assumptions (2), (4) and (5) imply that the last expression tends to zero as $t \mapsto 0$. Thus, (5) is proved in both cases. ■

Now, we define Colombeau type algebra as the factor algebra

$$\mathcal{SG}\big(\mathbb{R}_+ : \mathcal{L}(X)\big) = \mathcal{SE}_M\big(\mathbb{R}_+ : \mathcal{L}(X)\big)/\mathcal{SN}\big(\mathbb{R}_+ : \mathcal{L}(X)\big).$$

Elements of $\mathcal{SG}\big(\mathbb{R}_+ : \mathcal{L}(X)\big)$ will be denoted by $S = [S_\varepsilon]$, where $(S_\varepsilon)_\varepsilon$ is a representative of the above class.

Definition 3.7 $S \in \mathcal{SG}\big(\mathbb{R}_+ : \mathcal{L}(X)\big)$ is a called a Colombeau C_0-Semigroup if it has a representative $(S_\varepsilon)_\varepsilon$ such that for some $\varepsilon_0 > 0$, S_ε is a C_0-semigroup for every $\varepsilon < \varepsilon_0$.

In the sequel, we will use only representatives $\big(S_\varepsilon\big)_\varepsilon$ of a Colombeau C_0-semigroup S which are C_0-semigroups for ε small enough.

Proposition 3.8 Let $\big(S_\varepsilon\big)_\varepsilon$ and $\big(\tilde{S}_\varepsilon\big)_\varepsilon$ be representatives of a Colombeau C_0-semigroup S with the infinitesimal generators A_ε for $\varepsilon < \varepsilon_0$ and $\tilde{A}_\varepsilon$ for $\varepsilon < \tilde{\varepsilon}_0$, respectively, where ε_0 and $\tilde{\varepsilon}_0$ correspond (in the sense of Definition 3.7) to $\big(S_\varepsilon\big)_\varepsilon$ and $\big(\tilde{S}_\varepsilon\big)_\varepsilon$, respectively. Then, $D\big(A_\varepsilon\big) = D\big(\tilde{A}_\varepsilon\big)$ for every $\varepsilon < \bar{\varepsilon} = \min\big\{\varepsilon_0, \tilde{\varepsilon}_0\big\}$ and $A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to an element of $\mathcal{L}(X)$, denoted again by $A_\varepsilon - \tilde{A}_\varepsilon$. Moreover, for every $a \in \mathbb{R}$,

$$p_i\left(A_\varepsilon - \tilde{A}_\varepsilon\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^a), \ \forall i. \tag{7}$$

Proof. Denote $N_\varepsilon(S_\varepsilon - \tilde{S}_\varepsilon)_\varepsilon \in \mathcal{SN}(\mathbb{R}_+, \mathcal{L}(X))$. Let $\varepsilon < \bar{\varepsilon}_0$ be fixed and $x \in X$. we have

$$\tfrac{S_\varepsilon(t)x - x}{t} - \tfrac{\tilde{S}_\varepsilon(t)x - x}{t} = \tfrac{N_\varepsilon(t)}{t}x.$$

This implies by letting $t \mapsto 0$, that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$. Now, for $x \in D(A_\epsilon)$, we have

$$\left(A_\varepsilon - \tilde{A}_\varepsilon\right)x = \lim_{t\longrightarrow 0}\frac{S_\varepsilon(t)x - x}{t} - \lim_{t\longrightarrow 0}\frac{\tilde{S}_\varepsilon(t)x - x}{t} = \lim_{t\longrightarrow 0}\frac{N_\varepsilon(t)}{t}x = H_\varepsilon x. \tag{8}$$

Since $D(A_\varepsilon)$ is dense in X, properties (4), (5) and (7) imply that for every $a \in \mathbb{R}$,

$$p_i\left(A_\varepsilon - \tilde{A}_\varepsilon\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^a).$$

■

Now, we define the infinitesimal generator of a Colombeau C_0-semigroup S. Let $\mathcal{A}$ be the set of pairs $((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon)$, where A_ε is a closed linear operator on X with the dense domain $D(A_\varepsilon) \subset X$ for all $\varepsilon \in (0,1)$. We introduce an equivalence relation in A

$$\Big(\big(A_\varepsilon\big)_\varepsilon, \big(D(A_\varepsilon)\big)_\varepsilon\Big) \sim \Big(\big(\tilde{A}_\varepsilon\big)_\varepsilon, \big(D(\tilde{A}_\varepsilon)\big)_\varepsilon\Big).$$

If there exist $\varepsilon_0 \in (0,1)$ such that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \varepsilon_0$, and for every $a \in \mathbb{R}$ there exist $C > 0$ and $\varepsilon_a \leqslant \varepsilon_0$ such that, for $x \in D(A_\varepsilon)$, $p_i\left(A_\varepsilon - \tilde{A}_\varepsilon)x\right) \leqslant C\varepsilon^a p_i(x)$, $\forall i\ x \in D(A_\varepsilon)$, $\varepsilon \leqslant \varepsilon_a$. Since A_ε has a dense domain in X, $R_\varepsilon := A_\varepsilon - \tilde{A}_\varepsilon$ ca, be extended to be an operator in $\mathcal{L}_c(X)$ satisfying $p_i\left((A_\varepsilon - \tilde{A}_\varepsilon)x\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^a)$, for every $a \in \mathbb{R}$. such an operator R_ε is called the zero operator.

We denote by A the corresponding element of the quotient space $\mathcal{A}/\sim$. Due to Proposition 3.8, the following definition makes sense.

Definition 3.9 $A \in \mathcal{A}/\sim$ is the infinitesimal generator of a Colombeau C_0-semigroup S if there exists a representative $(A_\varepsilon)_\varepsilon$ of A such that A_ε is the infinitesimal generator of S_ε, for ε small enough.

By Pazy [10], we have the following proposition.

Proposition 3.10 Let S be a Colombeau C_0-semigroup with the infinitesimal generator A. Then there exists $\varepsilon_0 \in (0,1)$ such that

- Mapping $t \longmapsto S_\varepsilon(t)x : \mathbb{R}_+ \longrightarrow X$ is continuous for every $x \in X$ and $\varepsilon < \varepsilon_0$;
- For $\varepsilon < \varepsilon_0$ and $x \in X$,

$$\lim_{h\longrightarrow 0} \int_t^{t+h} S_\varepsilon(s)xds = S_\varepsilon(t)x;$$

- For $\varepsilon < \varepsilon_0$ and $x \in X$,

$$\int_0^t S_\varepsilon(s)xds \quad \in D(A_\varepsilon);$$

- For all $x \in D(A_\varepsilon)$ and $t \geqslant 0$, $S_\varepsilon(t)x \in D(A_\varepsilon)$ and

$$\frac{d}{dt}S_\varepsilon(t)x = A_\varepsilon S_\varepsilon(t)x = S_\varepsilon(t)A_\varepsilon x, \quad \varepsilon < \varepsilon_0; \tag{9}$$

- Let $\big(S_\varepsilon\big)_\varepsilon$ and $\big(\tilde{S}_\varepsilon\big)_\varepsilon$ be representative of Colombeau C_0-semigroup S, with infinitesimal generators A_ε and $\tilde{A}_\varepsilon$, $\varepsilon < \varepsilon_0$, respectively. Then, for all $a \in \mathbb{R}$ $t \geqslant 0$ and for all i,

$$p_i\left(\frac{d}{dt}S_\varepsilon(t) - \tilde{A}_\varepsilon S_\varepsilon(t)\right) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^a). \tag{10}$$

- For every $x \in D\big(A_\varepsilon\big)$ and every $t, s \geqslant 0$,

$$S_\varepsilon(t)x - S_\varepsilon(s)x = \int_s^t S_\varepsilon(\tau)A_\varepsilon xd\tau = \int_s^t A_\varepsilon S_\varepsilon(\tau)xd\tau.$$

Now, we will discuss a condition given the equality between two generalized semigroups.

Theorem 3.11 Let S and $\tilde{S}$ be Colombeau C_0-semigroups with infinitesimal generators A and $\tilde{A}$, respectively. If $A=\tilde{A}$, then $S=\tilde{S}$.

Proof. Let ε be small enough and $x\in D\big(A_\varepsilon\big)=D\big(\tilde{A}_\varepsilon\big)$. Proposition 3.10 implies that for $t\geqslant 0$, the mapping $s\longmapsto \tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x$ is differentiable $t\geqslant s\geqslant 0$ and

$$\tfrac{d}{ds}\big(\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x\big)=-\tilde{A}_\varepsilon\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x+\tilde{S}_\varepsilon(t-s)A_\varepsilon S_\varepsilon(s)x,\quad t\geqslant s\geqslant 0.$$

The assumption $A=\tilde{A}$ implies that $A_\varepsilon=\tilde{A}_\varepsilon+R_\varepsilon$, where R_ε is a zero operator. Since $\tilde{A}_\varepsilon$ commutes with $\tilde{S}_\varepsilon$, for every $x\in D\big(A_\varepsilon\big)$ and $t\geqslant s\geqslant 0$,

$$\tfrac{d}{ds}\big(\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x\big)=S_\varepsilon(t-s)R_\varepsilon S_\varepsilon(s)x,$$

which implies that

$$\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x-\tilde{S}_\varepsilon(t)x=\int_0^s\tilde{S}_\varepsilon(t-u)R_\varepsilon S_\varepsilon(u)xdu. \tag{11}$$

Let $s=t$ in (11). Then, for $t\geqslant 0$ and $x\in D(A_\varepsilon)$, we obtain

$$S_\varepsilon(t)x-\tilde{S}_\varepsilon(t)x=\int_0^s\tilde{S}_\varepsilon(t-u)R_\varepsilon S_\varepsilon(u)xdu. \tag{12}$$

Since $D(A_\varepsilon)$ is dense in X, uniform boundedness of S and $\tilde{S}$ on $[0,t]$ implies that (9) holds for every $y\in X$. Let us prove that $(N_\varepsilon)_\varepsilon=(S_\varepsilon-\tilde{S}_\varepsilon)_\varepsilon\in\mathcal{SN}(\mathbb{R}_+:\mathcal{L}_c(X))$. Proposition 3.10 and (10) imply that for some $C>0$, $a,\tilde{a}\in\mathbb{R}$, $x\in X$ and for all i,

$$\sup_{t\in[0,T)}p_i\left(N_\varepsilon(t)x\right)\leqslant\sup_{t\in[0,T)}\int_0^t p_i\left(\tilde{S}_\varepsilon(t-u)\right)p_i\left(R_\varepsilon\right)\left(S_\varepsilon(u)\right)p_i\left(x\right)du\leqslant T\,C\,\varepsilon^{a+\tilde{a}}p_i\left(R_\varepsilon x\right).$$

Since $p_i\left(R_\varepsilon\right)=\mathcal{O}_{\epsilon\to 0}(\varepsilon^b)$ for all $b\in\mathbb{R}$, $\big(N_\varepsilon(t)\big)_\varepsilon$ satisfies (3) in Definition 3.7. Condition (3) follows from the boundedness of $(\tilde{S}_\varepsilon)_\varepsilon$,$(S_\varepsilon)_\varepsilon$ on bounded domain $[0,t)$, the properties of $(R_\varepsilon)_\varepsilon$ and the following expression:

$$p_i\left(\frac{N_\varepsilon(t)}{t}\right)=p_i\left(\frac{1}{t}\int_0^t\tilde{S}_\varepsilon(t-u)R_\varepsilon S_\varepsilon(u)xdu\right)\leqslant p_i\left(\tilde{S}_\varepsilon(t)\right)p_i\left(R_\varepsilon\right)\left(S_\varepsilon\right)\leqslant const$$

for some $t_0>0$, $x\in X$, $t\leqslant t_0$ and for all i. Also, for all $x\in D\big(A_\varepsilon\big)$, we have

$$\lim_{t\longrightarrow 0}\frac{N_\varepsilon(t)}{t}=\lim_{t\longrightarrow 0}\frac{\tilde{S}_\varepsilon(t)x-x}{t}-\lim_{t\longrightarrow 0}\frac{S_\varepsilon(t)x-x}{t}=R_\varepsilon x.$$

Since it is enough that (5) holds for a dense subset of X see Remark 1, this completes the proof. ■

4. Generalized fixed points

In this section we will presented the notion of fixed point in Colombeau algebra.

4.1 *Contractions in locally convex and complete spaces*

This subsection is devoted to discuss the contraction map in locally convex spaces, which led us to define the contraction map in type Colombeau's algebra. Through this section X is also a locally convex space.

Definition 4.1 A map $A_\varepsilon : X \longrightarrow X$ is called a contraction if there exists $k_i < 1$ for all $i \in I$ such that for all $(x_\varepsilon, y_\varepsilon) \in X \times X$, $p_i(A_\varepsilon x_\varepsilon - A_\varepsilon y_\varepsilon) \leqslant k_i \, p_i(x_\varepsilon - y_\varepsilon)$.

We have the following result.

Theorem 4.2 Any contraction $A_\varepsilon : X \longrightarrow X$ has a fixed point. If X is Hausdorff, this fixed point is unique.

Proof. Start from $x_{0\varepsilon} \in X$ and define $x_{(n+1)\varepsilon} = A_\varepsilon(x_{n\varepsilon})$ by induction. It is easy to verify that $x_{n\varepsilon}$ is a Cauchy sequence in the complete space X and converges to some $x_\varepsilon \in X$. The contraction property of the map A_ε implies obviously its continuity. Then, passing to the limit in $x_{(n+1)\varepsilon} = A_\varepsilon(x_{n\varepsilon})$, we obtain that x_ε is a fixed point of X. If X is Hausdorff, then there exists $V \in V(0)$ such that $z_\varepsilon \notin V$ for all $z_\varepsilon \neq 0$. Hence, there exists i (depending on z_ε) such that $pi(z_\varepsilon) > 0$. If x_ε and y_ε are two different fixed points of X, then there exists j (depending on $x_\varepsilon - y_\varepsilon$) such that

$$0 < p_j\big(x_\varepsilon - y_\varepsilon\big) = p_j\big(A_\varepsilon(x_\varepsilon) - A_\varepsilon(y_\varepsilon)\big) \leqslant k_j \, p_j\big(x_\varepsilon - y_\varepsilon\big) < pi\big(x_\varepsilon - y_\varepsilon\big).$$

■

4.2 *Contraction operator in $\tilde{X}$*

We will give a notion of contraction map in type Colombeau algebra.

Definition 4.3 A map $A : \tilde{X} \to \tilde{X}$ is called a contraction if only if
a) for each $(x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$, $(A_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$.
b) each A_ε is a contraction in $\big(X, \tau_\varepsilon\big)$ endowed with the family $Q_\varepsilon = (q_{\varepsilon,i})_{i \in I}$ and the corresponding contraction constants are denoted by $l_{\varepsilon,i} < 1$,
c) for each $i \in I$ and $\varepsilon \in (0,1]$ there exists $a_{\varepsilon,i}, b_{\varepsilon,i} > 0$ such that $a_{\varepsilon,i}\, p_i \leqslant q_{\varepsilon,i} \leqslant b_{\varepsilon,i}\, p_i$.
d) for each $i \in I$ and $\varepsilon \in (0,1]$, $(\frac{b_{\varepsilon,i}}{a_{\varepsilon,i}})_\varepsilon, (\frac{1}{1-l_{\varepsilon,i}})_\varepsilon \in |\mathcal{E}_M(\mathbb{R})|$.

The essential result given in this theorem.

Theorem 4.4 Any contraction $A : \tilde{X} \longrightarrow \tilde{X}$ has a fixed point in $\tilde{X}$.

Proof. Consider condition **(a)** which is (1) in Lemma 3.4. Let $(i_\varepsilon)_\varepsilon \in \mathcal{N}(X)$ and $(x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$. Then we have

$$p_i(A_\varepsilon(x_\varepsilon + i_\varepsilon) - A_\varepsilon x_\varepsilon) = p_i(A_\varepsilon(x_\varepsilon + i_\varepsilon - x_\varepsilon)) = p_i(A_\varepsilon i_\varepsilon) \leqslant C \;\; p_i(i_\varepsilon).$$

Thus, $(A_\varepsilon(x_\varepsilon+i_\varepsilon)-A_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{N}(X)$ and the condition (2) in Lemma 3.4 is verified. Hence, A is well defined. From Theorem 4.2, we know that each A_ε has a fixed point z_ε obtained from limit of the Cauchy sequence $z_{n\varepsilon}$ defined by induction by $z_{(n+1)\varepsilon} = A_\varepsilon(z_{n\varepsilon})$. Starting from $z_0 = [z_{0\varepsilon}] \in \tilde{X}$, we deduce that $z_1 = [A_\varepsilon(z_{0\varepsilon})] \in \tilde{X}$ and $z_1 - z_0 \in \tilde{X}$. That is to say $p_i(z_{1\varepsilon} - z_{0\varepsilon})_\varepsilon \in |\mathcal{E}_M(\mathbb{R})|$. By induction we can compute

$$q_{\varepsilon,i}(z_{n+p,\varepsilon} - z_{n,\varepsilon}) \leqslant \frac{l^n_{\varepsilon,i}}{1-l_{\varepsilon,i}} q_{\varepsilon,i}(z_{1,\varepsilon} - z_{0,\varepsilon})$$

for all $n, p \in \mathbb{N}$. Then

$$q_{\varepsilon,i}(z_{p,\varepsilon} - z_{0,\varepsilon}) \leqslant \frac{1}{1-l_{\varepsilon,i}} q_{\varepsilon,i}(z_{1,\varepsilon} - z_{0,\varepsilon}).$$

Take the limit z_ε of $z_{p\varepsilon}$ in (X, τ_ε) when $p \longrightarrow +\infty$, we get

$$q_{\varepsilon,i}(z_\varepsilon - z_{0,\varepsilon}) \leqslant \frac{1}{1-l_{\varepsilon,i}} q_{\varepsilon,i}(z_{1,\varepsilon} - z_{0,\varepsilon}).$$

Now, $q_{\varepsilon,i}(z_\varepsilon) \leqslant q_{\varepsilon,i}(z_\varepsilon - z_{0,\varepsilon}) + q_{\varepsilon,i}(z_{0,\varepsilon})$, which

$$p_i(z_\varepsilon) \leqslant \frac{1}{a_{\varepsilon,i}} q_{\varepsilon,i}(z_\varepsilon) \leqslant \frac{b_{\varepsilon,i}}{a_{\varepsilon,i}} [\frac{1}{1-l_{\varepsilon,i}} (p_i(z_\varepsilon - z_{0,\varepsilon}) + p_i(z_{0,\varepsilon}))].$$

Then, from the hypotheses $(p_i(z_\varepsilon))_\varepsilon \in |\mathcal{E}_M(\mathbb{R})|$, that is to say $(z_\varepsilon))_\varepsilon \in \mathcal{E}_M(X)$. If $z = [z_\varepsilon]$, then $Az = [A_\varepsilon z_\varepsilon] = [z_\varepsilon] = z$; that is, z is a fixed point of A. ■

5. Existence-uniqueness result

We consider the existence and uniqueness result for a fractional differential equation given by

$$\begin{cases} D_c^q u(t) = g(t, u(t)) = Au(t) + f(t), \ t \in [0, b], \\ u(0) = u_0 \in \tilde{\mathbb{R}} \end{cases} \tag{13}$$

where D_c^q is the Caputo derivative of order $0 < q < 1$, $u_0 \in \tilde{R}$, $g \in \mathcal{C}\big(J \times \tilde{\mathbb{R}}; \tilde{\mathbb{R}}\big)$, $J = [0, b]$ and $u \in \mathcal{C}\big(J; \tilde{\mathbb{R}}\big)$, $I^q u \in D(A)$, $f : J \longrightarrow X$ is continuous. In the forthcoming analysis, we need the following hypothesis:

H1 : the linear operator $A_\varepsilon : D(A_\varepsilon) \subset X \longrightarrow X$ (X Banach space) satisfies the Hille-yosida condition, that is, there exist two constant $\omega \in \mathbb{R}$ and M_1 such that $]w, +\infty[\subset \rho(A_\varepsilon)$ and

$$\| (\lambda I - A_\varepsilon)^{-k} \|_{\mathcal{L}(X)} \leqslant \frac{M_1}{(\lambda - w)^k}$$

for all $\lambda > \omega$ and $k \geqslant 1$.

H2 : $Q_\varepsilon(t)$ is continuous in the uniform operator topology for $t > 0$ and $\{Q_\varepsilon(t)\}_{t \geqslant 0}$ is uniformly bounded, that is, there exists $M_2 > 1$ such that $\sup\limits_{t \geqslant 0} |Q_\varepsilon(t)| < M_2$.

We need the following definition before we proceed further.

Definition 5.1 Let $g \in \tilde{\mathbb{R}}$, We tell that g is globally Lipschitz if for all $t \in J$ and $\varepsilon \in]0, 1]$ there exists $k_\varepsilon(t) > 0$ such that for all $(y, z) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}}$, we have

$$| g_\varepsilon(t, y_\varepsilon) - g_\varepsilon(t, z_\varepsilon) | \leqslant k_\varepsilon(t) \mid y\varepsilon - z_\varepsilon \mid,$$

where $\sup\limits_{t \in J} k_\varepsilon(t) = M_{T,\varepsilon} < +\infty$

Now, we will presented the existence and uniqueness result of our problem.

Theorem 5.2 Assume that the hypotheses H_1 and H_2 hold and g satisfied a global Lipschitz, then (13) admit unique solution.

Proof. For $u_0 \in \tilde{\mathbb{R}}$, $g \in \mathcal{C}(J \times \tilde{\mathbb{R}}; \tilde{\mathbb{R}})$, $u \in C(J; X)$ and $I^q u \in D(A)$, the problem reduces to finding a fixed point of the map $\phi : \tilde{\mathbb{R}} \longrightarrow \tilde{\mathbb{R}}$ such that for all t$\in J$,$\phi(x)(t) = u_0 + AI^q u(t) + I^q f(t)$. In order to prove the result, we will check the assumptions a, b, c and d of Definition 4.3 and apply Theorem 4.4.

a) We pose $\phi_\varepsilon(u)(t) = u_{\varepsilon 0} + A_\varepsilon I^q u_\varepsilon(t) + I^q f_\varepsilon(t)$ for all $t \in J$. It is clear that ϕ_ε is defined from $\mathcal{C}^\infty(J, X)$ into $\mathcal{C}^\infty(J, X)$. The topology τ is given by the family of norms $(p_T)_{T \in J}$

such that $p_T(u_\varepsilon) = \sup_{t\in[0,T]} \left|u_\varepsilon(t)\right|$ for all $u_\varepsilon \in C^\infty(J,X)$. Now, let $(u_\varepsilon)_\varepsilon \in \mathcal{E}^s_M(J)$. Since

$$\phi_\varepsilon(u_\varepsilon)(t) = u_{\varepsilon 0} + A_\varepsilon I^q u_\varepsilon(t) + I^q f_\varepsilon(t)$$

and g is Lipschitz, then $\left|g_\varepsilon(2u_\varepsilon(t)) - g_\varepsilon(u_\varepsilon(t))\right| \leqslant k_\varepsilon(t)\left|u_\varepsilon(t)\right|$ and $\left|A_\varepsilon(u_\varepsilon(t))\right| \leqslant k_\varepsilon(t)\left|u_\varepsilon(t)\right|$. So $(A_\varepsilon u_\varepsilon(t))_\varepsilon \in \mathcal{E}^s_M(J)$ and we have

$$I^q u_\varepsilon(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} u_\varepsilon(s)ds,$$

which implies that

$$\left|I^q u_\varepsilon(t)\right| \leqslant \frac{b^q}{\Gamma(q)}\left|u_\varepsilon(t)\right|.$$

Hence, $I^q u_\varepsilon(t) \in \mathcal{E}^s_M(J)$. So,

$$\left|\phi_\varepsilon(u_\varepsilon)(t)\right| \leqslant \left|u_{\varepsilon 0}\right| + \left|A_\varepsilon I^q u_\varepsilon(t)\right| + \left|I^q f_\varepsilon(t)\right|.$$

Thus, $p_T(\phi_\varepsilon(u_\varepsilon)) \in |\mathcal{E}^r_M|$ and $(\phi_\varepsilon(u_\varepsilon))_\varepsilon \in \mathcal{E}^s_M(J)$.
b) First, we have to write (1) in term of representatives

$$\begin{cases} D^q_c u_\varepsilon(t) = g_\varepsilon(t, u_\varepsilon(t)) = A_\varepsilon\, u_\varepsilon(t) + f_\varepsilon(t), \\ u_\varepsilon(0) = u_{0\varepsilon} \in \mathbb{R} \end{cases} \tag{14}$$

Here the topology τ_ε is given by the family of norms $(q_{T,\varepsilon})_{T\in\mathbb{R}^+}$ such that for all $u_\varepsilon \in \mathcal{C}^\infty(J,X)$, we have

$$q_{T,\varepsilon}(u_\varepsilon) = \sup_{t\in[0,T]} \left|u_\varepsilon(t)\right| \exp\left(-t\frac{b^{q-1}}{\Gamma(q)} M_{T,\varepsilon}\right)$$

For all $u_\epsilon, v_\epsilon \in \mathcal{C}^\infty(J,X)$, we have

$$\phi_\varepsilon(u_\varepsilon)(t) - \phi_\varepsilon(v_\varepsilon)(t) = I^q(g_\varepsilon(t,u_\varepsilon(t)) - g_\varepsilon(t,v_\varepsilon(t))) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(g_\varepsilon(s,u_\varepsilon(s)) - g_\varepsilon(s,v_\varepsilon(s)))ds,$$

which implies that

$$\left|\phi_\varepsilon(u_\varepsilon)(t) - \phi_\varepsilon(v_\varepsilon)(t)\right| \leqslant \int_0^t \frac{(b)^{q-1}}{\Gamma(q)} M_{T,\varepsilon}\left|u_\varepsilon(s) - v_\varepsilon(s)\right|ds,$$

and

$$e^{-t\frac{b^{q-1}}{\Gamma(q)}} M_{T,\varepsilon}\left|\phi_\varepsilon(u_\varepsilon)(t) - \phi_\varepsilon(v_\varepsilon)(t)\right| \leqslant e^{-t\frac{b^{q-1}}{\Gamma(q)} M_{T,\varepsilon}} \int_0^t \frac{(b)^{q-1}}{\Gamma(q)} M_{T,\varepsilon}\left|u_\varepsilon(s) - v_\varepsilon(s)\right|ds.$$

Now,

$$e^{-t\frac{b^{q-1}}{\Gamma(q)} M_{T,\varepsilon}} \int_0^t \frac{(b)^{q-1}}{\Gamma(q)} M_{T,\varepsilon}\left|u_\varepsilon(s) - v_\varepsilon(s)\right|ds = e^{-t\frac{b^{q-1}}{\Gamma(q)} M_{T,\varepsilon}} \int_0^t \frac{b^{q-1}}{\Gamma(q)} M_{T,\varepsilon} e^{-s\frac{b^{q-1}}{\Gamma(q)} M_{T,\varepsilon}} e^{s\frac{b^{q-1}}{\Gamma(q)} M_{T,\varepsilon}}\left|u_\varepsilon(s) - v_\varepsilon(s)\right|ds.$$

implies that

$$e^{-t\frac{b^{q-1}}{\Gamma(q)}M_{T,\varepsilon}}\int_0^t \frac{(b)^{q-1}}{\Gamma(q)}M_{T,\varepsilon}|u_\varepsilon(s)-v_\varepsilon(s)|ds \leqslant e^{-t\frac{b^{q-1}}{\Gamma(q)}M_{T,\varepsilon}}q_{T,\varepsilon}(u_\varepsilon - v_\varepsilon)\int_0^t \frac{b^{q-1}}{\Gamma(q)}M_{T,\varepsilon}e^{-t\frac{b^{q-1}}{\Gamma(q)}M_{T,\varepsilon}}ds.$$

Thus,

$$e^{-t\frac{b^{q-1}}{\Gamma(q)}M_{T,\varepsilon}}\int_0^t \frac{(b)^{q-1}}{\Gamma(q)}M_{T,\varepsilon}\big|u_\varepsilon(s)-v_\varepsilon(s)\big|ds \leqslant q_{T,\varepsilon}(u_\varepsilon - v_\varepsilon)(1-e^{-t\frac{b^{q-1}}{\Gamma(q)}M_{T,\varepsilon}}).$$

As consequence $q_{T,\varepsilon}\big(\phi_\varepsilon(u_\varepsilon)-\phi_\varepsilon(v_\varepsilon)\big) \leqslant q_{T,\varepsilon}(u_\varepsilon - v_\varepsilon)(1-e^{-b\frac{b^{q-1}}{\Gamma(q)}M_{T,\varepsilon}})$. So, ϕ_ε is a contraction in $(\mathcal{C}^\infty(J,\mathbb{R}),\tau_\varepsilon)$.
c) We can write for all $T \in J$ and $u_\varepsilon \in C^\infty(J,X)$,

$$\sup_{t\in[0,T]}\Big\{\big|u_\varepsilon(t)\big|\, e^{-\frac{b^q}{\Gamma(q)}M_{T,\varepsilon}}\Big\} \leqslant \sup_{t\in[0,T]}\Big\{\big|u_\varepsilon(t)\big|\, e^{-t\frac{b^{q-1}}{\Gamma(q)}M_{T,\varepsilon}}\Big\} \leqslant \sup_{t\in[0,T]}\big|u_\varepsilon(t)\big|.$$

Then $e^{-\frac{b^q}{\Gamma(q)}M_{T,\varepsilon}}p_T \leqslant q_{T,\varepsilon} \leqslant p_T$.
d) For all $T \in J$, we have $\big(e^{\frac{b^q}{\Gamma(q)}M_{T,\varepsilon}}\big)_\varepsilon \in \mathcal{E}_M^r$ and

$$\Big(\frac{1}{1-(1-e^{-b\frac{b^{q-1}}{\Gamma(q)}M_{T,\varepsilon}})}\Big)_\varepsilon = (e^{\frac{b^q}{\Gamma(q)}M_{T,\varepsilon}})_\varepsilon \in \mathcal{E}_M^r.$$

Finally, from definition contraction map on generalized function of Colombeau the following mapping

$$\phi : \begin{cases} \tilde{\mathbb{R}} \longrightarrow \tilde{\mathbb{R}}, \\ u(t) = [u_\varepsilon(t)] \longmapsto \phi(u)(t) = [\phi_\varepsilon(u_\varepsilon)(t)] \end{cases}$$

is a contraction. So, from Theorem 4.4, the mapping ϕ has a fixed point. Since z_ε being the unique fixed point of ϕ_ε, we are going to prove that z is the unique fixed point of ϕ and therefore, the unique solution of (1). If $v = [v_\varepsilon]$ is another fixed point of ϕ, then we have $v_\varepsilon = \phi_\varepsilon(v_\varepsilon) + \rho_\varepsilon$ with $\rho_\varepsilon \in \mathcal{N}^s(\mathbb{R})$. Thus, $(p_T(i_\varepsilon))_\varepsilon \in |\mathcal{N}^r|$ and

$$w_\varepsilon(t) - v_\varepsilon(t) = I^q g_\varepsilon(t, w_\varepsilon(t)) - I^q g_\varepsilon(t, v_\varepsilon(t)) - \rho_\varepsilon(t).$$

Then

$$w_\varepsilon(t) - v_\varepsilon(t) = A_\varepsilon I^q(w_\varepsilon(t) - v_\varepsilon(t)) - \rho_\varepsilon(t),$$

and

$$w_\varepsilon(t) - v_\varepsilon(t) = \rho_\varepsilon + A_\varepsilon \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(w_\varepsilon(s) - v_\varepsilon(s))ds,$$

implies that

$$p_i\left(w_\varepsilon(t) - v_\varepsilon(t)\right) \leqslant p_i\left(\rho_\varepsilon\right) + \int_0^t M_{T,\varepsilon}\frac{b^{q-1}}{\Gamma(q)}p_i\left(w_\varepsilon(s) - v_\varepsilon(s)\right)ds$$

for all i. Since $\rho_\epsilon \in \mathcal{N}(\mathbb{R})$, then $p_i(\rho_\epsilon) \leqslant \epsilon^a$ for all $a \in \mathbb{R}$. Hence, for all i,

$$p_i\left(w_\varepsilon(t) - v_\varepsilon(t)\right) \leqslant \epsilon^a + \int_0^t M_{T,\varepsilon} \frac{b^{q-1}}{\Gamma(q)} p_i\left(w_\varepsilon(s) - v_\varepsilon(s)\right) ds.$$

Now, by Lemma 2.2, we get $p_i\big(w_\varepsilon(t) - v_\varepsilon(t)\big) \leqslant \epsilon^a\, e^{M_{T,\varepsilon}\frac{b^{q-1}}{\Gamma(q)}t}$ for all $i \in I$. Also, we have $e^{M_{T,\varepsilon}\frac{b^q}{\Gamma(q)}} \in |\mathcal{E}^r_M|$ and $(p_T(i_\varepsilon))_\varepsilon \in |\mathcal{N}^r|$. Then $(p_T(w_\varepsilon - v_\varepsilon))_\varepsilon \in |\mathcal{N}^r|$. So, $w = v$. ■

6. Conclusion

In this paper, we solved fractional differential evolution problem with initial value is a generalized number. We define an operator from Colombeau's algebra into itself and rely on the topology of locally convex spaces for defining the notion of generalized contraction mapping. From the definition of contraction mapping on $\mathcal{G}$ and generalized semigroups we were able to study the problem proposed at the beginning.

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