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A new subclass of harmonic mappings with positive coefficients

A. R. Haghighi[a,*], N. Asghary[b], A. Sedghi[c]

[a] *Department of Mathematics, Technical and Vocational, University (TVU), Tehran, Iran.*
[b,c] *Department of Mathematics, Islamic Azad University, Central Tehran Branch, Tehran, Iran.*

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Abstract. Complex-valued harmonic functions that are univalent and sense-preserving in the open unit disk U can be written as form $f = h + \bar{g}$, where h and g are analytic in U. In this paper, we introduce the class $S^1_H(\beta)$, where $1 < \beta \leqslant 2$, and consisting of harmonic univalent function $f = h + \bar{g}$, where h and g are in the form $h(z) = z + \sum\limits_{n=2}^{\infty} |a_n| z^n$ and $g(z) = \sum\limits_{n=2}^{\infty} |b_n| \bar{z}^n$ for which

$$\mathrm{Re}\left\{z^2(h''(z) + g''(z)) + 2z(h'(z) + g'(z)) - (h(z) + g(z)) - (z-1)\right\} < \beta.$$

It is shown that the members of this class are convex and starlike. We obtain distortions bounds extreme point for functions belonging to this class, and we also show this class is closed under convolution and convex combinations.

Keywords: Convex combinations, extreme points, harmonic starlike functions, harmonic univalent functions.

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1. Introduction

Let U be a simply connected domain and $f = u + iv$ be a harmonic map on U (the continuous complex-valued function $f = u + iv$ is called harmonic if both u and v are

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*Corresponding author.
E-mail address: ah.haghighi@gmail.com (A. R. Haghighi); nasimsana@yahoo.com (N. Asghary); ali_sedghi1389@yahoo.com (A. Sedghi).

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real harmonic in U). Also, in any simply connected domain, one can write $f = h + \bar{g}$, where h and g are analytic in D and h is called the analytic part and g is called the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ (see [1, 4]).

Silverman in [6] has been defined the class S_H^0, where functions belonging in this class are the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) \sum_{n=2}^{\infty} b_n z^n.$$

They proved that the coefficient conditions

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leqslant 1 \text{ and } \sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leqslant 1$$

are sufficient conditions for function $f = h + \bar{g}$ to be harmonic starlike convex function, respectively.

Consider by S_H^1 the subclass of S_H^0 consisting of functions of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} |a_n| z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} |b_n| z^n \, |b_1| < 1. \tag{1}$$

Let $1 < \beta \leqslant 2$. By $R_H(\beta)$, we mean the subclass of S_H^1 satisfying the condition

$$\operatorname{Re}\left\{h'(z) + g'(z)\right\} < \beta. \tag{2}$$

This class was studied by Dixit and Porwal [2, 3]. Consider $S_H^1(\beta)$ for $1 < \beta \leqslant 2$ and denote the class S_H^1 satisfying the condition

$$h(z) = z + \sum_{n=2}^{\infty} |a_n| z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} |b_n| \bar{z}^n \tag{3}$$

for which

$$\operatorname{Re}\left\{z^2(h''(z) + g''(z)) + 2z(h'(z) + g'(z)) - (h(z) + g(z)) - (z - 1)\right\} < \beta. \tag{4}$$

We will show that if $f \in S_H^1(\beta)$, then f is univalent and sense preserving in U (see Remark *1*). In this paper, we generalize the obtained results in [2] for the class of $S_H^1(\beta)$.

2. Main Results

Theorem 2.1 A function of the form (3) is in $S_H^1(\beta)$ if and only if

$$\sum_{n=2}^{\infty} \left(n^2 + n - 1\right)(|a_n| + |b_n|) \leqslant \beta - 1, \quad z \in U. \tag{5}$$

Proof. Let $\sum\limits_{n=2}^{\infty}\left(n^2+n-1\right)(|a_n|+|b_n|)\leqslant\beta-1$. It is sufficient to show that

$$\left|\frac{z^2(h''(z)+g''(z))+2z(h'(z)+g'(z))-(h(z)+g(z))-(z-1)-1}{z^2(h''(z)+g''(z))+2z(h'(z)+g'(z))-(h(z)+g(z))-(z-1)-(2\beta-1)}\right|<1.$$

We have

$$\left|\frac{\sum\limits_{n=2}^{\infty}(n^2+n-1)(|a_n|+|b_n|)|z|^{n-1}}{2(\beta-1)-\sum\limits_{n=2}^{\infty}(n^2+n-1)(|a_n|+|b_n|)|z|^{n-1}}\right|\leqslant \frac{\left|\sum\limits_{n=2}^{\infty}(n^2+n-1)(|a_n|+|b_n|)\right|}{\left|2(\beta-1)-\sum\limits_{n=2}^{\infty}(n^2+n-1)(|a_n|+|b_n|)\right|},$$

which is bounded from above by 1 (by hypothesis), and the sufficient part is proved.
Conversely, suppose that

$$\operatorname{Re}\left\{z^2(h''(z)+g''(z))+2z(h'(z)+g'(z))-(h(z)+g(z))-(z-1)\right\}<\beta.$$

We have

$$\operatorname{Re}\left\{\sum_{n=2}^{\infty}\left(n^2+n-1\right)|a_n|z^n+\sum_{n=2}^{\infty}(n^2+n-1)|b_n|\bar{z}^n\right\}<\beta.$$

The above condition must hold for all values of z with $|z|=r<1$. Upon choosing the values of z to be real and $z\longrightarrow 1^-$, we obtain

$$\sum_{n=2}^{\infty}\left(n^2+n-1\right)(|a_n|+|b_n|)\leqslant\beta-1,$$

which gives the necessary part. ■

Remark 1 *Let* $\sum\limits_{n=2}^{\infty}\left(n^2+n-1\right)(|a_n|+|b_n|)\leqslant\beta-1$ *and* $|z_1|<|z_2|<1$, *we have*

$$\begin{aligned}
|f(z_1)-f(z_2)| &\geqslant |h(z_1)-h(z_2)|-|g(z_1)-g(z_2)|\\
&\geqslant |z_1-z_2|\left(1-\sum_{n=2}^{\infty}n|a_n||z_2|^{n-1}-\sum_{n=2}^{\infty}n|b_n||z_2|^{n-1}\right)\\
&\geqslant |z_1-z_2|\left(1-|z_2|\left(\sum_{n=2}^{\infty}(n^2+n-1)(|a_n|+|b_n|)\right)\right)\\
&\geqslant |z_1-z_2|(1-|z_2|)(\beta-1)>0.
\end{aligned}$$

Hence, f is univalent in U. Also, f is sense preserving in U, because

$$|h'(z)| = |1+\sum_{n=2}^{\infty} n|a_n|z^n| \geqslant 1-\sum_{n=2}^{\infty} n|a_n||z^n| > 1-\sum_{n=2}^{\infty}(n^2+n-1)|a_n|$$
$$\geqslant \sum_{n=2}^{\infty}(n^2+n-1)|b_n|$$
$$\geqslant \sum_{n=2}^{\infty} n|b_n||\bar{z}^n-1| \geqslant |g'(z)|.$$

Thus, if $f \in S_H^1(\beta)$, then f will be univalent and sense preserving in U.

Next, we determine bounds for the class $S_H^1(\beta)$.

Theorem 2.2 If $f \in S_H^1(\beta)$, then

$$|f(z)| \leqslant r+\frac{1}{5}(\beta-1)r^2, \quad |z|=r<1,$$
$$|f(z)| \geqslant r-\frac{1}{5}(\beta-1)r^2, \quad |z|=r<1.$$

The bounds are sharp for the functions $f(z)=z+\frac{1}{5}(\beta-1)z^2$ and $f(z)=z+\frac{1}{5}(\beta-1)\bar{z}^2$.

Proof. Let $f \in S_H^1(\beta)$. Taking the absolute value of f, we have

$$|f(z)| = \left|z+\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=2}^{\infty}|b_n|\bar{z}^n\right| \leqslant |z|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n$$
$$\leqslant r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \leqslant r+\frac{1}{5}\sum_{n=2}^{\infty}(n^2+n-1)(|a_n|+|b_n|)r^2$$
$$\leqslant r+\frac{1}{5}(\beta-1)r^2$$

and

$$|f(z)| \geqslant |z|-\left|\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=2}^{\infty}|b_n|\bar{z}^n\right| \geqslant |z|-\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n$$
$$\geqslant r-\frac{1}{5}\sum_{n=2}^{\infty}(n^2+n-1)(|a_n|+|b_n|)r^2 \geqslant r-\frac{1}{5}(\beta-1)r^2.$$

The functions $z+\frac{1}{5}(\beta-1)z^2$ and $z+\frac{1}{5}(\beta-1)\bar{z}^2$ show that the given bounds are sharp. ■

Corollary 2.3 If $f \in S_H^1(\beta)$, then we have

$$\left\{w:|w|<\frac{1}{5}(6-\beta)\right\} \subset f(U). \tag{6}$$

Now, we determine the extreme point of the closed convex hulls of $S_H^1(\beta)$, denoted by clco $S_H^1(\beta)$.

Theorem 2.4 $f \in clcoS_H^1(\beta)$ if and only if

$$f(z) = \sum_{n=1}^{\infty}(\lambda_n h_n + \gamma_n g_n), \tag{7}$$

in which we have $h_1(z) = z$, $h_n(z) = z + \frac{\beta-1}{n^2+n-1} z^n$ and $g_n = z + \frac{\beta-1}{n^2+n-1}\bar{z}^n$ for $n = 1, 2, ...,$ and also, $\gamma_1 = 0$, $\gamma_n, \lambda_n \geqslant 0$ and $\lambda_1 = 1 - \sum\limits_{n=2}^{\infty}(\lambda_n + \gamma_n) \geqslant 0$. In particular, the extreme points of $S_H^1(\beta)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. For functions f of the form (7), write

$$f(z) = \sum_{n=1}^{\infty}(\lambda_n h_n + \gamma_n g_n) = z + \sum_{n=2}^{\infty}\left(\frac{\beta-1}{n^2+n-1}\right)\lambda_n z^n + \sum_{n=2}^{\infty}\left(\frac{\beta-1}{n^2+n-1}\right)\gamma_n \bar{z}^n.$$

Then, we have

$$\begin{aligned} 1 \geqslant 1 - \lambda_1 &= \sum_{n=2}^{\infty}(\lambda_n + \gamma_n) \\ &= \sum_{n=2}^{\infty}\left(\frac{n^2+n-1}{\beta-1}\right)\left(\frac{\beta-1}{n^2+n-1}\right)\lambda_n + \sum_{n=2}^{\infty}\left(\frac{n^2+n-1}{\beta-1}\right)\left(\frac{\beta-1}{n^2+n-1}\right)\gamma_n, \end{aligned}$$

which implies that $f \in clcoS_H^1(\beta)$. Conversely, suppose that $f \in clcoS_H^1(\beta)$. Set

$$\lambda_n = \frac{n^2+n-1}{\beta-1}|a_n| \text{ and } \gamma_n = \frac{n^2+n-1}{\beta-1}|b_n| \text{ for } n = 2, 3, ...$$

Then note that by Theorem 2.1, $0 \leqslant \lambda_n \leqslant 1$ and $0 \leqslant \gamma_n \leqslant 1$. According to definition, we have $\lambda_1 = 1 - \sum_{n=2}^{\infty}(\lambda + \gamma_n)$ and $\gamma_1 = 0$ and also, by Theorem 2.1, $\lambda_1 \geqslant 0$. Consequently, we obtain $\sum_{n=1}^{\infty}(\lambda_n h_n + \gamma_n g_n)$ as required. ■

Theorem 2.5 If $f \in S_H^1(\beta)$, then f is a harmonic starlike convex function.

Proof. Let $f \in S_H^1(\beta)$. Then, by Theorem 2.1, we have

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leqslant \sum_{n=2}^{\infty}(n^2+n-1)(|a_n| + |b_n|) \leqslant 1.$$

Thus, f is starlike. Also,

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leqslant \sum_{n=2}^{\infty}(n^2+n-1)(|a_n| + |b_n|) \leqslant 1.$$

Hence, f is convex. This completes the proof. ■

For our next theorem, we need to define the convolution of two harmonic functions as follows. For harmonic function of the form

$$f(z)=z+\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=2}^{\infty}|b_n|\bar{z}^n \text{ and } F(z)=z+\sum_{n=2}^{\infty}|A_n|z^n+\sum_{n=2}^{\infty}|B_n|\bar{z}^n.$$

We define their convolution

$$(f*F)(z)=f(z)*F(z)=z+\sum_{n=2}^{\infty}|a_nA_n|z^n+\sum_{n=2}^{\infty}|b_nB_n|\bar{z}^n. \tag{8}$$

Theorem 2.6 For $1<\beta\leqslant\alpha\leqslant 2$, if $f\in S_H^1(\beta)$ and $F(z)\in S_H^1(\alpha)$, then $f*F\in S_H^1(\beta)\subseteq S_H^1(\alpha)$.

Proof. At the first, we have

$$f(z)*F(z)=z+\sum_{n=2}^{\infty}|a_nA_n|z^n+\sum_{n=2}^{\infty}|b_nB_n|\bar{z}^n$$

.

We shall show that the coefficients of $f*F$ satisfy the required condition given in Theorem 2.1. For $F(z)\in S_H^1$, note that $|A_n|\leqslant 1$ and $|B_n|\leqslant 1$. Now, for the $f*F$, we have

$$\sum_{n=2}^{\infty}\frac{n^2+n-1}{\beta-1}|a_n||A_n|+\sum_{n=2}^{\infty}\frac{n^2+n-1}{\beta-1}|b_n||B_n|\leqslant\sum_{n=2}^{\infty}\frac{n^2+n-1}{\beta-1}(|a_n|+|b_n|)\leqslant 1.$$

Therefore, $f*F\in S_H^1(\beta)\subseteq S_H^1(\alpha)$. ■

Theorem 2.7 The class $S_H^1(\beta)$ is closed under convex combination.

Proof. Let $f_i(z)\in S_H^1(\beta)$ for $i\in\mathbb{N}$, where $f_i(z)$ is given by

$$f_i(z)=z+\sum_{n=2}^{\infty}|a_{n_i}|z^n+\sum_{n=2}^{\infty}|a_{n_i}|\bar{z}^n.$$

Then, by Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\frac{n^2+n-1}{\beta-1}(|a_{n_i}|+|b_{n_i}|)\leqslant 1.$$

For $\sum\limits_{i=1}^{\infty}t_i=1, 0\leqslant t_i\leqslant 1$, the convex combination of f_i may be written as:

$$\sum_{i=1}^{\infty}t_if_i(z)=z+\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{n_i}|\right)z^n+\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{n_i}|\right)\bar{z}^n.$$

Then, by Theorem 2.1, we have

$$1 = \sum_{i=1}^{\infty} t_i \geqslant \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{n^2+n-1}{\beta-1} (|a_{n_i}| + |b_{n_i}|) \right)$$
$$= \sum_{n=2}^{\infty} \frac{n^2+n-1}{\beta-1} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}| + \sum_{i=1}^{\infty} t_i |b_{n_i}| \right).$$

Therefore, $\sum_{i=1}^{\infty} t_i f_i(z) \in S_H^1(\beta)$. ■

Remember the σ-neighborhood of f from [5] as follows:

$$N_\sigma(f) = \left\{ F : F(z) = z + \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=2}^{\infty} |B_n| \bar{z}^n, \sum_{n=2}^{\infty} n|a_n - A_n| + |b_n - B_n| \leqslant \sigma \right\}.$$

Theorem 2.8 Let $f \in S_H^1(\beta)$ and $\sigma \leqslant 2-\beta$. If $F \in N_\sigma(f)$, then F is harmonic starlike.

Proof. Let

$$F(z) = z + \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=2}^{\infty} |B_n| \bar{z}^n$$

belongs to $N_\sigma(f)$. We have

$$\sum_{n=2}^{\infty} n(|A_n| + |B_n|) \leqslant \sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + \sum_{n=2}^{\infty} n(|a_n| + |b_n|)$$
$$\leqslant \sigma + \beta - 1$$
$$\leqslant 1.$$

Hence, $F(z)$ is a harmonic starlike function. ■

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