Journal of
Linear and Topological Algebra
Vol. 08*, No.* 04*,* 2019*,* 237- 250

New iteration process for approximating fixed points in Banach spaces

J. D. Bhutia[a,*], K. Tiwary[b]

[a] *Department of Mathematics, Cooch Behar College, Cooch Behar, West Bengal, Pin Code 736101, India.*

[b] *Department of Mathematics, Raiganj University, Raiganj, Pin Code 733134, India.*

Received 2 April 2019; Revised 20 July 2019; Accepted 14 August 2019.

Communicated by Mohammad Sadegh Asgari

Abstract. The object of this paper is to present a new iteration process. We will show that our process is faster than the known recent iterative schemes. We discuss stability results of our iteration and prove some results in the context of uniformly convex Banach space for Suzuki generalized nonexpansive mappings. We also present a numerical example for proving the rate of convergence of our results. Our results improves many known results of the existing literature.

© 2019 IAUCTB. All rights reserved.

Keywords: Iteration process, Suzuki generalized nonexpansive mappings, uniformly convex Banach space, fixed point.

2010 AMS Subject Classification: 47H09, 47H10.

1. Introduction and preliminaries

If $T\colon X \to X$ and p be a point in X such that $Tp = p$, then p is said to be a fixed point of T. It was in the year 1922 when Banach introduced the ground breaking result known as Banach contraction principle, which guarantees the existence of a fixed point. This completely initiated a new dimension of research in the field of nonlinear analysis. This result was done on a complete metric space. From then on, many researchers have considered different spaces and have taken different contraction condition to prove the existence of a fixed point. However, finding the value of the fixed point is not that easy. So to solve this problem, we need an iterative processes that can give us the fixed point.

*Corresponding author.
E-mail address: jigmibhutia@gmail.com (J. D. Bhutia); tiwarykalishankar@yahoo.com (K. Tiwary).

Print ISSN: 2252-0201
Online ISSN: 2345-5934
© 2019 IAUCTB. All rights reserved.
http://jlta.iauctb.ac.ir

It all began by the result given in Mann [11], Ishikawa [9], Agarwal et al. [2], Noor [12], Abbas [1], Ćirić et al. [4], Dukić et al. [5], Vatan Two-step [10], K-iteration process [8], M* iteration process [17], K^* iteration process [18], M-iteration process [19] and so on. All the above process depends on choosing some initial point on the space that would generate a sequence obeying the iterative schemes and converges to the fixed point.

Recently, a new type of iteration has been obtained by Hussian et al. [8] known as K-iteration process. They have considered the contraction condition and (C) condition also known as generalized nonexpansive mapping for approximation of fixed points. Also, Ullah et al. [21] have used this iteration process in CAT(0) space.

A Banach space X is called uniformly convex if for each $\epsilon \in (0,2]$ there exists $\delta > 0$ such that for $x, y \in X$ with $\|x\| \leq 1$ and $\|y\| \leq 1$, $|x-y\| > \epsilon$ implies $\|\frac{x+y}{2}\| \leq \delta$.

Let C be a nonempty subset of a Banach space X. A mapping $T\colon C \to C$ is said to be a contraction if there exist $L \in (0,1)$ such that for all $x, y \in C$, $\|Tx - Ty\| \leq L\|x-y\|$. If $L = 1$ then T is called nonexpansive mapping and quasi nonexpansive if for all $x \in C$ and $p \in F(T)$, we have $\|Tx - p\| \leq \|x - p\|$.

In 2008, Suzuki [14] introduced the concept of generalized nonexpansive mapping. A mapping $T\colon X \to X$ is said to be generalized nonexpansive mappings if for all $x, y \in X$, $\frac{1}{2}\|x - Tx\| \leq \|x-y\|$ implies $\|Tx - Ty\| \leq \|x-y\|$.

Definition 1.1 [3] Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedure sequences that converge to the same fixed point p and $\|u_n - p\| \leq a_n$ and $\|v_n - p\| \leq b_n$, for all $n \geq 0$. If the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ converge to a and b, respectively, and $\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0$, then we say that $\{u_n\}_{n=0}^{\infty}$ converge faster than $\{v_n\}_{n=0}^{\infty}$ to p.

Definition 1.2 [7] Let $\{t_n\}_{n=0}^{\infty}$ be an arbitrary sequence in C. Then, an iteration procedure $x_{n+1} = f(T, x_n)$ converging to a fixed point p is said to be $T-$stable or stable with respect to T, if for $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$, $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \epsilon = 0$ if and only if $\lim_{n\to\infty} t_n = p$.

Lemma 1.3 [22] Let $\{u_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the relation $u_{n+1} \leq (1 - w_n)\, u_n + w_n$, where $w_n \in (0,1)$ for all $n \in N$, $\sum_{n=0}^{\infty} w_n = \infty$ and $\frac{w_n}{w_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} u_n = 0$.

Lemma 1.4 [13] Suppose that X is a uniformly convex Banach space and $\{t_n\}$ be any real sequence such that $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences of X such that $\lim_{n\to\infty} sup\,\|x_n\| \leq r$, $\lim_{n\to\infty} sup\,\|y_n\| \leq r$ and $\lim_{n\to\infty} sup\,\|t_n x_n + (1 - t_n y_n)\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proposition 1.5 [14] Let C be a nonempty subset of a Banach space X and $T\colon C \to C$ be any mapping. Then

(1) If T is nonexpansive, then T is Suzuki generalized nonexpansive mapping.
(2) T is Suzuki generalized nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.

Also, the author in [14] proved the following lemma (lemma 7, [14]):

Lemma 1.6 [14] Let C be a nonempty subset of a Banach space X and $T\colon C \to C$ be Suzuki generalized nonexpansive mapping. Then, for all $x, y \in X$, we have

$$\|Tx - Ty\| \leq 3\|Tx - x\| + \|x - y\|.$$

Let C be a nonempty closed convex subset of a Banach space X, and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set $r(x, \{x_n\}) = \lim sup_{n\to\infty}\|x_n - x\|$. The asymptotic radius of $\{x_n\}$ relative to C is given by $r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$ and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is known that, in a uniformly convex Banach space, $A(C, \{x_n\})$ consists of exactly one point.

Throughout this section, we have $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$. In 2014, the authors in [6] introduced the concept of Picard S iteration process as follows:

$$\begin{cases} u_0 \in C, \\ w_n = (1 - \beta_n)u_n + \beta_n T u_n, \\ v_n = (1 - \alpha_n)Tu_n + \alpha_n T w_n, \\ u_{n+1} = Tv_n. \end{cases}$$

Here it was shown that the Picard S iteration process is faster than all other iterations like Picard, Mann, Ishikawa, Noor, SP, CR, S, S^*, Abbas, Normal-S and Two-step Mann iteration process.

In 2015, the authors in [10] introduced new two-step iteration process even faster than the Picard S iteration process as follows:

$$\begin{cases} u_0 \in C, \\ v_n = T[(1 - \beta_n)u_n + \beta_n T u_n], \\ u_{n+1} = T[(1 - \alpha_n)v_n + \alpha_n T v_n]. \end{cases}$$

Then, in the year 2016, the authors in [16] introduced new iteration process known as "Thakur new iteration process" as follows:

$$\begin{cases} u_0 \in C, \\ w_n = (1 - \beta_n)u_n + \beta_n T u_n, \\ v_n = T[(1 - \alpha_n)u_n + \alpha_n T w_n], \\ u_{n+1} = Tv_n. \end{cases}$$

This iteration proved to be faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration process for some class of mappings. Again, in 2016, a new iteration was introduced by the authors in [15] in the following way:

$$\begin{cases} x_0 \in C, \\ z_n = (1 - \beta_n)x_n + \beta_n T x_n], \\ y_n = (1 - \alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = (1 - \gamma_n)Tz_n + \gamma_n T y_n. \end{cases}$$

The authors here proved that their iteration is faster than that of Picard, Mann, Ishikawa, Noor, Agarwal et al., Abbas et. al., iteration process for the contractive mappings in the sense of Berinde [3]. In 2017, the authors in [17] introduced the following iteration process known as M^* iteration process.

$$\begin{cases} x_0 \in C, \\ z_n = (1-\ \beta_n)x_n +\ \beta_n T x_n], \\ y_n = T[(1-\ \alpha_n)x_n +\ \alpha_n T z_n], \\ x_{n+1} = T y_n. \end{cases}$$

Recently, in 2018, the authors in [8] introduced the new iteration process called "K iteration process" and proved some weak and strong convergence theorems for fixed point of Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces.

$$\begin{cases} x_0 \in C, \\ z_n = (1-\ \beta_n)x_n +\ \beta_n T x_n], \\ y_n = T[(1-\ \alpha_n)T x_n +\ \alpha_n T z_n], \\ x_{n+1} = T y_n. \end{cases}$$

K iteration process is faster than the iterations i.e., Picard S iteration, new two step iteration process, Thakur new iteration process. The authors proved some weak and strong convergence results considering Suzuki generalized nonexpansive mappings in the setting of Uniformly convex Banach spaces.

In 2018, the authors in [18] introduced K^* iteration process and proved some weak and strong convergence results considering Suzuki generalized nonexpansive mappings in the setting of Uniformly convex Banach spaces. They have shown that the K^* iteration process is faster than Picard S iteration [6] and S iteration.

$$\begin{cases} x_0 \in C, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T[(1-\alpha_n)z_n + \alpha_n T z_n], \\ x_{n+1} = T y_n. \end{cases}$$

In the same year, the authors in [19] introduced M iteration process as follows:

$$\begin{cases} x_0 \in C, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n. \end{cases}$$

2. Main result

We now introduce a new iteration process call it "J iteration process" as follows:

$$\begin{cases} x_0 \in C, \\ z_n = T[(1-\beta_n)x_n + \beta_n T x_n], \\ y_n = T[(1-\alpha_n)z_n + \alpha_n T z_n], \\ x_{n+1} = T y_n. \end{cases}$$

Here we prove that J iteration process is faster than that of the recent iteration process such as K iteration process, K^* iteration process, M^* iteration process, M iteration process. Here we prove that the sequence of iteration obtained from J iteration process converges to the fixed point of T. The proof is on the lines similar to that of [8] and is the same proof in [20]. Here we use the condition $\sum_{n=0}^{\infty}\beta_n = \infty$ instead of $\sum_{n=0}^{\infty}\beta_n\alpha_n = \infty$.

Theorem 2.1 Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by J iteration process with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\beta_n = \infty$ *or* $(\sum_{n=0}^{\infty}\alpha_n = \infty)$. Then $\{x_n\}_{n=0}^{\infty}$ converge strongly to a unique fixed point of T.

Proof. Since T is a contraction mapping in a Banach space, T has a unique fixed point in C. Let us suppose that p is a fixed point of T. From J iteration process, we get

$$\begin{aligned}
\|z_n - p\| &= \|T[(1-\beta_n)x_n + \beta_n Tx_n] - Tp\| \\
&\leqslant k\|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\
&\leqslant k\|(1-\beta_n)(x_n - p) + \beta_n(Tx_n - p)\| \\
&\leqslant k\|(1-\beta_n)(x_n - p) + \beta_n(Tx_n - p)\| \\
&\leqslant k(1-\beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \\
&\leqslant k\{(1-\beta_n)\|x_n - p\| + k\beta_n\|x_n - p\|\} \\
&\leqslant k\{1-\beta_n(1-k)\}\|x_n - p\|.
\end{aligned}$$

Now,

$$\begin{aligned}
\|y_n - p\| &= \|T[(1-\alpha_n)z_n + \alpha_n Tz_n] - Tp\| \\
&\leqslant k\|(1-\alpha_n)z_n + \alpha_n Tz_n - p\| \\
&\leqslant k\|(1-\alpha_n)z_n + \alpha_n Tz_n - p\| \\
&\leqslant k\|(1-\alpha_n)(z_n - p) + \alpha_n(Tz_n - p)\| \\
&\leqslant k\|(1-\alpha_n)(z_n - p) + k\alpha_n(z_n - p)\| \\
&\leqslant k(1-\alpha_n)\|z_n - p)\| + k^2\alpha_n\|(z_n - p)\| \\
&\leqslant k(1-\alpha_n + k\alpha_n)\|(z_n - p)\| \leq k\|z_n - p\| \\
&\leqslant k^2\{1-\beta_n(1-k)\}\|x_n - p\|.
\end{aligned}$$

Then,

$$\|x_{n+1} - p\| \leqslant \|Ty_n - p\| \leqslant k^3\{1-\beta_n(1-k)\}\|x_n - p\|.$$

By repeating the above process, we get

$$\begin{aligned}
\|x_n - p\| &\leqslant k^3\{1-\beta_{n-1}(1-k)\}\|x_{n-1} - p\|, \\
\|x_{n-1} - p\| &\leqslant k^3\{1-\beta_{n-2}(1-k)\}\|x_{n-2} - p\|, \\
&\vdots \\
\|x_1 - p\| &\leqslant k^3\{1-\beta_0(1-k)\}\|x_0 - p\|.
\end{aligned}$$

Therefore, we obtain $\|x_{n+1} - p\| \leqslant k^{3(n+1)}\|x_0 - p\| \prod_{i=0}^{n}\{1 - \beta_i(1-k)\}$. Now, $k < 1$ so $(1-k) > 0$ and $\beta_n \leqslant 1$ for all $n \in N$. Therefore, we get $1 - \beta_n(1-k) < 1$ for all $n \in \mathbb{N}$. Again, we know that $1 - x \leq e^{-x}$, for all $x \in [0,1]$. So we have

$$\|x_{n+1} - p\| \leq k^{3(n+1)}\|x_0 - p\| e^{-(1-k)\sum_{i=0}^{n}\beta_i}.$$

Taking the limits $n \to \infty$ both sides, we get $\lim_{n\to\infty} \|x_n - p\| = 0$. ■

Remark 1 *In the above theorem, replace the condition $\sum_{n=0}^{\infty} \beta_n = \infty$ by $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then we take $\|z_n - p\| \leqslant k\|x_n - p\|$ and we get $\|y_n - p\| \leqslant k^2\{1 - \alpha_n(1-k)\}\|x_n - p\|$. Thus,*

$$\|x_{n+1} - p\| \leqslant k^{3(n+1)}\|x_0 - p\| \prod_{i=0}^{n}\{1 - \alpha_i(1-k)\}.$$

Therefore, we get the desired result.

Theorem 2.2 Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping with a fixed point p. For a given $x_0 = u_0$, let $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be an iterative sequence generated by J iteration process and K^*-iteration process [18] respectively, with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in [0, 1] satisfying $\beta \leqslant \beta_n < 1$ and $\alpha \leqslant \alpha_n < 1$, for some $\alpha, \beta > 0$ and for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$.

Proof. From inequality (10) of Theorem 3.2 in [18], we have

$$\|u_{n+1} - p\| \leqslant k^{2(n+1)}\|u_0 - p\| \prod_{i=0}^{n}\{1 - \alpha_i(1-k)\}.$$

Since $\alpha \leqslant \alpha_n$ for all $n \in \mathbb{N}$, we obtain $\|u_{n+1} - p\| \leqslant k^{2(n+1)}\|u_0 - p\|\{1 - \alpha\,(1-k)\}^{n+1}$. Let $a_n = k^{2(n+1)}\|u_0 - p\|\{1 - \alpha\,(1-k)\}^{n+1}$. Now, from Remark 1, we get

$$\|x_{n+1} - p\| \leqslant k^{3(n+1)}\|x_0 - p\| \prod_{i=0}^{n}\{1 - \alpha_i(1-k)\}.$$

Again $\alpha \leqslant \alpha_n$ for all $n \in \mathbb{N}$ gives

$$\|x_{n+1} - p\| \leqslant k^{3(n+1)}\|x_0 - p\|\{1 - \alpha(1-k)\}^{n+1}.$$

Let $b_n = k^{3(n+1)}\|x_0 - p\|\{1 - \alpha(1-k)\}^{n+1}$. Then,

$$\frac{b_n}{a_n} = \frac{k^{3(n+1)}\|x_0 - p\|\{1 - \alpha(1-k)\}^{n+1}}{k^{2(n+1)}\|u_0 - p\|\{1 - \alpha\,(1-k)\}^{n+1}} = \frac{b_n}{a_n} = k^{n+1}.$$

Thus, we get $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$. Hence, the result follows. ■

Here, we prove that the J iteration process is faster than that of the K iteration process [8].

Theorem 2.3 Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping with a fixed point p. For a given $x_0 = u_0$, let

$\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be an iterative sequence generated by J iteration process and K-iteration process [8] respectively, with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in [0, 1] satisfying $\beta \leqslant \beta_n < 1$ and $\alpha \leqslant \alpha_n < 1$ for some $\alpha, \beta > 0$ and for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$.

Proof. From Theorem 2.1, we have $\|x_{n+1} - p\| \leqslant k^{3(n+1)}\|x_0 - p\| \prod_{i=0}^{n}\{1 - \beta_i(1-k)\}$. Since $\alpha \leqslant \alpha_n$ and $\beta \leqslant \beta_n$ for all $n \in \mathbb{N}$. We get

$$\begin{aligned}\|x_{n+1} - p\| &\leqslant k^{3(n+1)}\|x_0 - p\| \prod_{i=0}^{n}\{1 - \beta(1-k)\}\\ &= k^{3(n+1)}\|x_0 - p\|\{1 - \beta(1-k)\}^{n+1}.\end{aligned}$$

Now, from Theorem 3.1 of [8], we get $\|u_{n+1} - p\| \leqslant k^{3(n+1)}\|u_0 - p\| \prod_{i=0}^{n}\{1 - \beta_i\alpha_i(1-k)\}$. Using $\alpha \leqslant \alpha_n$ and $\beta \leqslant \beta_n$ for all $n \in \mathbb{N}$, we get

$$\begin{aligned}\|u_{n+1} - p\| &\leqslant k^{3(n+1)}\|u_0 - p\| \prod_{i=0}^{n}\{1 - \beta\alpha(1-k)\}\\ &= k^{3(n+1)}\|u_0 - p\|\{1 - \beta\alpha(1-k)\}^{n+1}.\end{aligned}$$

Define

$$\begin{aligned}\|a_n\| &= k^{3(n+1)}\|x_0 - p\|\{1 - \beta(1-k)\}^{n+1},\\ \|b_n\| &= k^{3(n+1)}\|u_0 - p\|\{1 - \beta\alpha(1-k)\}^{n+1}.\end{aligned}$$

Then, we have

$$\frac{a_n}{b_n} = \frac{k^{3(n+1)}\|x_0 - p\|\{1 - \beta(1-k)\}^{n+1}}{k^{3(n+1)}\|u_0 - p\|\{1 - \beta\alpha(1-k)\}^{n+1}} = \frac{\|x_0 - p\|\{1 - \beta(1-k)\}^{n+1}}{\|u_0 - p\|\{1 - \beta\alpha(1-k)\}^{n+1}}.$$

Taking limit as $n \to \infty$, we obtain $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. ■

Here we prove that the J iteration process is faster than that of the M iteration process as described in [19].

Theorem 2.4 Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping with a fixed point p. For a given $x_0 = u_0$, let $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be an iterative sequence generated by J iteration process and M iteration process as in [19] respectively, with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in [0, 1] satisfying $\beta \leq \beta_n < 1$ and $\alpha \leq \alpha_n < 1$, for some $\alpha, \beta > 0$ and for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$.

Proof. From Theorem 2.1, we have $\|x_{n+1} - p\| \leqslant k^{3(n+1)}\|x_0 - p\| \prod_{i=0}^{n}\{1 - \beta_i(1-k)\}$. Since $\beta \leqslant \beta_n$ for all $n \in \mathbb{N}$. We get

$$\|x_{n+1} - p\| \leqslant k^{3(n+1)}\|x_0 - p\| \prod_{i=0}^{n}\{1 - \beta(1-k)\} \leqslant k^{3(n+1)}\|x_0 - p\|\{1 - \beta(1-k)\}^{n+1}.$$

Now, for M iteration process, we have

$$\begin{cases} u_0 \in C, \\ w_n = (1-\beta_n)u_n + \beta_n T u_n, \\ v_n = T w_n, \\ u_{n+1} = T v_n. \end{cases}$$

Then

$$\begin{aligned} \|w_n - p\| &= \|(1-\beta_n)u_n + \beta_n T u_n - p\| \\ &\leqslant \|(1-\beta_n)(u_n - p) + \beta_n (T u_n - p)\| \\ &\leqslant (1-\beta_n)\|u_n - p\| + \beta_n \|T u_n - p\| \\ &\leqslant (1-\beta_n)\|u_n - p\| + k\beta_n \|u_n - p\| \\ &\leqslant (1-\beta_n(1-k))\|u_n - p\|. \end{aligned}$$

Now,

$$\|v_n - p\| \leqslant \|T w_n - p\| \leqslant k\|w_n - p\| \leqslant k(1-\beta_n(1-k))\|u_n - p\|.$$

Therefore, we get

$$\|u_{n+1} - p\| \leqslant \|T v_n - p\| \leqslant k\|v_n - p\| \leqslant k^2(1-\beta_n(1-k))\|u_n - p\|.$$

By repeating the above process, we have

$$\begin{aligned} \|u_n - p\| &\leqslant k^2\{1-\beta_{n-1}(1-k)\}\|u_{n-1} - p\|, \\ \|u_{n-1} - p\| &\leqslant k^2\{1-\beta_{n-2}(1-k)\}\|u_{n-2} - p\|, \\ &\vdots \\ \|u_1 - p\| &\leqslant k^2\{1-\beta_0(1-k)\}\|u_0 - p\|. \end{aligned}$$

Therefore, we get $\|u_{n+1} - p\| \leqslant k^{2(n+1)}\|u_0 - p\| \prod_{i=0}^{n}\{1-\beta_i(1-k)\}$. Now, since $\beta \leqslant \beta_n$ for all $n \in \mathbb{N}$, we have $\|u_{n+1} - p\| \leqslant k^{2(n+1)}\|u_0 - p\|\{1-\beta(1-k)\}^{n+1}$. Define,

$$\begin{aligned} \|a_n\| &= k^{3(n+1)}\|x_0 - p\|\{1-\beta(1-k)\}^{n+1}, \\ \|b_n\| &= k^{2(n+1)}\|u_0 - p\|\{1-\beta(1-k)\}^{n+1}. \end{aligned}$$

Then we get

$$\Psi_n = \frac{a_n}{b_n} = \frac{k^{3(n+1)}\|x_0 - p\|\{1-\beta(1-k)\}^{n+1}}{k^{2(n+1)}\|u_0 - p\|\{1-\beta(1-k)\}^{n+1}} = k^{(n+1)}.$$

Now, $k < 1$. So we get $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \Psi_n = 0$. Thus, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$. ■

Here we consider the rate of convergence of M^* iteration process under contraction mapping and compare it with the J iteration process.

Theorem 2.5 Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping with a fixed point p. For a given $x_0 = u_0$, let $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be an iterative sequence generated by J iteration process and M^* iteration process as in [17] respectively, with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0, 1]$ satisfying $\beta \leqslant \beta_n < 1$ and $\alpha \leqslant \alpha_n < 1$, for some $\alpha, \beta > 0$ and for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$.

Proof. Since T is a contraction mapping in a Banach space, T has a unique fixed point in C. Let us suppose that p is a fixed point of T. From J iteration process, we get

$$\begin{aligned}
\|z_n - p\| &= \|T[(1-\beta_n)x_n + \beta_n T x_n] - Tp\| \\
&\leqslant k\|(1-\beta_n)x_n + \beta_n T x_n - p\| \\
&\leqslant k\|(1-\beta_n)(x_n - p) + \beta_n (T x_n - p)\| \\
&\leqslant k\|(1-\beta_n)(x_n - p) + \beta_n (T x_n - p)\| \\
&\leqslant k(1-\beta_n)\|x_n - p\| + \beta_n \|T x_n - p\| \\
&\leqslant k\{(1-\beta_n)\|x_n - p\| + k\beta_n \|x_n - p\|\} \\
&\leqslant k\{1 - \beta_n(1-k)\}\|x_n - p\|.
\end{aligned}$$

Now,

$$\begin{aligned}
\|y_n - p\| &= \|T[(1-\alpha_n)z_n + \alpha_n T z_n] - Tp\| \\
&\leqslant k\|(1-\alpha_n)z_n + \alpha_n T z_n - p\| \\
&\leqslant k\|(1-\alpha_n)z_n + \alpha_n T z_n - p\| \\
&\leqslant k\|(1-\alpha_n)(z_n - p) + \alpha_n (T z_n - p)\| \\
&\leqslant k\|(1-\alpha_n)(z_n - p) + k\alpha_n (z_n - p)\| \\
&\leqslant k(1-\alpha_n)\|z_n - p)\| + k^2\alpha_n \|(z_n - p)\| \\
&\leqslant k\{1 - \alpha_n(1-k)\}\|(z_n - p)\| \\
&\leqslant k^2\{1 - \alpha_n(1-k)\}\{1 - \beta_n(1-k)\}\|x_n - p\|.
\end{aligned}$$

Then, $\|x_{n+1} - p\| \leqslant \|Ty_n - p\| \leq k^3\{1 - \alpha_n(1-k)\}\{1 - \beta_n(1-k)\}\|x_n - p\|$. By repeating the above process, we get

$$\begin{aligned}
\|x_n - p\| &\leqslant k^3\{1 - \alpha_{n-1}(1-k)\}\{1 - \beta_{n-1}(1-k)\}\|x_{n-1} - p\|, \\
\|x_{n-1} - p\| &\leqslant k^3\{1 - \alpha_{n-2}(1-k)\}\{1 - \beta_{n-2}(1-k)\}\|x_{n-2} - p\|, \\
&\vdots \\
\|x_1 - p\| &\leqslant k^3\{1 - \alpha_0(1-k)\}\{1 - \beta_0(1-k)\}\|x_0 - p\|.
\end{aligned}$$

Therefore, we get

$$\|x_{n+1} - p\| \leqslant k^{3(n+1)}\|x_0 - p\| \prod_{i=0}^{n}\{1 - \alpha_i(1-k)\}\{1 - \beta_i(1-k)\}.$$

Since $\alpha \leqslant \alpha_n$ and $\beta \leqslant \beta_n$ for all $n \in \mathbb{N}$, we get

$$\|x_{n+1} - p\| \leqslant k^{3(n+1)}[\{1 - \alpha(1-k)\}\{1 - \beta(1-k)\}]^{n+1}\|x_0 - p\|,$$

which implies that

$$\|x_{n+1} - p\| \leqslant k^{3(n+1)}[1 - (1-k)\{\alpha(1 - \beta k) + \beta(1+\alpha)\}]^{n+1}\|x_0 - p\|.$$

Let $a_n = k^{3(n+1)}[1 - (1-k)\{\alpha(1 - \beta k) + \beta(1+\alpha)\}]^{n+1}\|x_0 - p\|$. From M^* iteration process, we have

$$\begin{cases} u_0 \in C, \\ w_n = (1 - \beta_n)u_n + \beta_n T u_n], \\ v_n = T[(1 - \alpha_n)u_n + \alpha_n T w_n], \\ u_{n+1} = T v_n. \end{cases}$$

and

$$\begin{aligned} \|w_n - p\| &= \|(1-\beta_n)u_n + \beta_n T u_n - Tp\| \\ &\leqslant \|(1-\beta_n)(u_n - p) + \beta_n(Tu_n - p)\| \\ &\leqslant (1-\beta_n)\|u_n - p\| + \beta_n\|Tu_n - p\| \\ &\leqslant (1-\beta_n)\|u_n - p\| + k\beta_n\|u_n - p\| \\ &\leqslant \{1 - \beta_n(1-k)\}\|u_n - p\|. \end{aligned}$$

Now,

$$\begin{aligned} \|v_n - p\| &= \|T[(1-\alpha_n)u_n + \alpha_n T w_n] - Tp\| \\ &\leqslant k\|(1-\alpha_n)u_n + \alpha_n T w_n - p\| \\ &\leqslant k\|(1-\alpha_n)(u_n - p) + \alpha_n(Tw_n - p)\| \\ &\leqslant k(1-\alpha_n)\|u_n - p\| + k\alpha_n\|w_n - p\| \\ &\leqslant k(1-\alpha_n)\|u_n - p)\| + k\alpha_n\{1 - \beta_n(1-k)\}\|(u_n - p)\| \\ &\leqslant k[1 - \alpha_n + k\alpha_n - k\alpha_n\beta_n(1-k)\}]\|(u_n - p)\| \\ &\leqslant k[1 - (1-k)\alpha_n - k\alpha_n\beta_n(1-k)\}]\|(u_n - p)\| \\ &\leqslant k[1 - \alpha_n(1-k)(1-k\beta_n)\}]\|(u_n - p)\|. \end{aligned}$$

Then, $\|u_{n+1} - p\| \leqslant \|Ty_n - p\| \leqslant k^2[1 - \alpha_n(1-k)(1-k\beta_n)]\|(u_n - p)\|$. By repeating the above process, we get

$$\begin{aligned} \|u_n - p\| &\leqslant k^2[1 - \alpha_{n-1}(1-k)(1-k\beta_{n-1})]\|(u_{n-1} - p)\|, \\ \|u_{n-1} - p\| &\leqslant k^2[1 - \alpha_{n-2}(1-k)(1-k\beta_{n-2})]\|(u_{n-2} - p)\|, \\ &\vdots \\ \|u_1 - p\| &\leqslant k^2[1 - \alpha_0(1-k)(1-k\beta_0)]\|(u_0 - p)\|. \end{aligned}$$

Therefore, we have

$$\|u_{n+1}-p\| \leqslant k^{2(n+1)}\|u_0-p\|\prod_{i=0}^{n}[1-\alpha_i(1-k)(1-k\beta_i)].$$

Since $\alpha \leqslant \alpha_n$ and $\beta \leqslant \beta_n$ for all $n \in \mathbb{N}$, we have

$$\|u_{n+1}-p\| \leqslant k^{2(n+1)}[1-\alpha(1-k)(1-k\beta)]^{n+1}\|u_0-p\|.$$

Let $b_n = k^{2(n+1)}[1-\alpha(1-k)(1-k\beta)]^{n+1}\|u_0-p\|$. Now,

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{k^{3(n+1)}\,[1-(1-k)\{\alpha(1-\beta k)+\beta(1+\alpha)\}]^{n+1}\,\|x_0-p\|}{k^{2(n+1)}\,[1-\alpha(1-k)(1-k\beta)]^{n+1}\,\|u_0-p\|}\\ &= k^{n+1}\;\frac{\|x_0-p\|\,[1-(1-k)\,\{\alpha\,(1-\beta k)+\beta\,(1+\alpha)\}]^{n+1}}{\|u_0-p\|\,[1-\alpha\,(1-k)\,(1-k\,\beta)]^{n+1}}.\end{aligned}$$

Again, since $\frac{1-(1-k)\{\alpha(1-\beta k)+\beta(1+\alpha)\}}{1-\alpha(1-k)(1-k\beta)} < 1$ and by taking limits as $n \to \infty$, we get $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$. Thus, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to p faster than $\{u_n\}_{n=0}^{\infty}$. ■

We now present an example by taking $T(x) = (x+2)^{\frac{1}{2}}$ a contraction mapping and $\alpha_n = \beta_n = \gamma_n = \frac{1}{4}$ for all $n \in \mathbb{N}$.

It.	J iteration	K^* iteration	K iteration	M^* iteration	M iteration
0	4	4	4	4	4
1	2.018287635607934	2.022430709480216	2.025983430909863	2.089209472207758	2.090225974648651
2	2.000188396786911	2.000284309634019	2.000386162189640	2.004438767284909	2.004553853178992
4	2.000000020044780	2.000003609326324	2.000005750773716	2.000222089966109	2.000231178486197
5	2.000000000206761	2.000000045821514	2.000000085643808	2.000011115179405	2.000011739347189
6	2.000000000002133	2.000000000581718	2.000000001275457	2.000000556301295	2.000000596138247
7	2.000000000000022	2.000000000007385	2.000000000018995	2.000000027842227	2.000000030272644
8	2	2.000000000000094	2.000000000000283	2.000000001393471	2.000000001537283
9	2	2.000000000000001	2.000000000000004	2.000000000069742	2.000000000078065
10	2	2	2	2.000000000003491	2.000000000003964
11	2	2	2	2.000000000000175	2.000000000000201
12	2	2	2	2.000000000000009	2.000000000000010
13	2	2	2	2.000000000000000	2.000000000000000
14	2	2	2	2	2
15	2	2	2	2	2
16	2	2	2	2	2
17	2	2	2	2	2
18	2	2	2	2	2
19	2	2	2	2	2
20	2	2	2	2	2
21	2	2	2	2	2
22	2	2	2	2	2
23	2	2	2	2	2

Thus it is clear from the table that J iteration process converges faster than that of the above iterative schemes.

Theorem 2.6 Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by J iteration process, with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \beta_n = \infty$ for all $n \in \mathbb{N}$. Then the J iterative process is $T-$ stable.

Proof. Let $\{x_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence in C. Also, let the sequence generated by J iterative process be $x_{n+1} = f(T, x_n)$ converging to unique fixed point p (follows from Theorem 2.1) and $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$. We will prove that $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = p$. Let $\lim_{n\to\infty} \epsilon_n = 0$. Then, we have

$$\|t_{n+1} - p\| \leqslant \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - p\| = \epsilon_n + \|t_{n+1} - p\|.$$

From Theorem 2.1, we get $\leqslant \epsilon_n + k^3\{1 - \beta_n(1-k)\}\|t_n - p\|$. Since $0 < k < 1$ and $0 \leqslant \beta_n \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \epsilon_n = 0$, then from the above inequality and using Lemma 1.3, we get $\lim_{n\to\infty} \|t_n - p\| = 0$. Hence, $\lim_{n\to\infty} t_n = p$. Conversely, let $\lim_{n\to\infty} t_n = p$. Then, we have

$$\begin{aligned} \epsilon_n = \|t_{n+1} - f(T, t_n)\| &\leqslant \|t_{n+1} - p\| + \|f(T, t_n) - p\| \\ &\leqslant \|t_{n+1} - p\| + k^3\{1 - \beta_n(1-k)\}\|t_n - p\|. \end{aligned}$$

Thus, we have $\lim\limits_{n\to\infty} \epsilon_n = 0$. Hence, the J iteration process is stable with respect to T. ■

3. Convergence for Suzuki generalized nonexpansive mappings under J iteration process

Lemma 3.1 Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \phi$. For arbitrary chosen $x_0 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by J iteration process with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \beta_n = \infty$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$.

Proof. Suppose $p \in F(T)$ and since C is convex $(1-\beta_n)x_n + \beta_n T x_n \in C$ for all $n \in \mathbb{N}$. Now, T is a Suzuki generalized nonexpansive mapping

$$\tfrac{1}{2}\|p - Tp\| = 0 \leqslant \|p - ((1-\beta_n)x_n + \beta_n T x_n)\|,$$

which implies that

$$\|Tp - T[(1-\beta_n)x_n + \beta_n T x_n]\| \leqslant \|p - ((1-\beta_n)x_n + \beta_n T x_n)\|.$$

Now, we have

$$\begin{aligned} \|z_n - p\| &= \|T[(1-\beta_n)x_n + \beta_n T x_n] - Tp\| \\ &\leqslant \|(1-\beta_n)x_n + \beta_n T x_n - p\| \\ &\leqslant \|(1-\beta_n)(x_n - p) + \beta_n(T x_n - p)\| \\ &\leqslant \|(1-\beta_n)(x_n - p) + \beta_n(T x_n - p)\| \\ &\leqslant (1-\beta_n)\|x_n - p\| + \beta_n\|T x_n - p\| \\ &\leqslant \{(1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\|\} \\ &\leqslant \|x_n - p\| \end{aligned}$$

and

$$\begin{aligned}\|y_n - p\| &= \|T[(1-\alpha_n)z_n + \alpha_n T z_n] - Tp\| \\ &\leqslant \|(1-\alpha_n)z_n + \alpha_n T z_n - p\| \\ &\leqslant \|(1-\alpha_n)z_n + \alpha_n T z_n - p\| \\ &\leqslant \|(1-\alpha_n)(z_n - p) + \alpha_n(T z_n - p)\| \\ &\leqslant \|(1-\alpha_n)(z_n - p) + \alpha_n(z_n - p)\| \\ &\leqslant (1-\alpha_n)\|z_n - p)\| + \alpha_n\|(z_n - p)\| \\ &\leqslant \|z_n - p\| \leqslant \|x_n - p\|.\end{aligned}$$

Then $\|x_{n+1} - p\| \leq \|Ty_n - p\| \leq \|x_n - p\|$, which implies that $\{\|x_n - p\|\}$ is bounded and non-increasing sequence for all $p \in F(T)$. Thus, $\lim_{n\to\infty}\|x_n - p\|$ exists. ■

Theorem 3.2 Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by J iteration process with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[a, b]$ for some a and b satisfying $0 < a \leq b < 1$. Then $F(T) \neq \phi$ if and only if $\{x_n\}_{n=0}^{\infty}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

Proof. Suppose $F(T) \neq \phi$ and choose $p \in F(T)$. Then, by the Lemma 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists and $\{x_n\}$ is bounded. Let $\lim_{n\to\infty}\|x_n - p\| = r$ for some $r \geq 0$. Then, from Lemma 3.1, we obtain $\lim_{n\to\infty} sup\|z_n - p\| \leq \lim_{n\to\infty} sup\|x_n - p\| = r$. By Proposition 1.5 (ii), we have $\lim_{n\to\infty} sup\|Tx_n - p\| \leq \lim_{n\to\infty} sup\|x_n - p\| = r$. Again, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|Ty_n - Tp\| \leqslant \|y_n - p\| \\ &\leqslant \|T[(1-\alpha_n)z_n + \alpha_n T z_n] - p\| \\ &\leqslant \|(1-\alpha_n)z_n + \alpha_n T z_n - p\| \\ &\leqslant \|(1-\alpha_n)(z_n - p) + \alpha_n(T z_n - p)\| \\ &\leqslant \|(1-\alpha_n)(z_n - p) + \alpha_n(z_n - p)\|,\end{aligned}$$

which implies that $\|x_{n+1} - p\| \leq \|z_n - p\|$. Therefore, we have $r \leqslant \lim_{n\to\infty} inf\|z_n - p\|$; that is,

$$r \leqslant \lim_{n\to\infty} inf\|z_n - p\| \leqslant \lim_{n\to\infty} sup\|z_n - p\| \leqslant r,$$

which implies $r = \lim_{n\to\infty}\|z_n - p\|$. Thus, we get

$$r = \lim_{n\to\infty}\|T[(1-\beta_n)x_n + \beta_n T x_n] - p\| \leqslant \lim_{n\to\infty}\|(1-\beta_n)x_n + \beta_n T x_n - p\|$$

Hence,

$$r \leqslant \lim_{n\to\infty}\|(1-\beta_n)(x_n - p) + \beta_n(Tx_n - p)\| \leqslant r.$$

Thus, $r = \lim_{n\to\infty}\|(1-\beta_n)(x_n - p) + \beta_n(Tx_n - p)\|$. By Lemma 1.4, we get $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Conversely, $\{x_n\}_{n=0}^{\infty}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

Let $p \in A(C, \{x_n\})$. By Lemma 1.6, we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \lim_{n\to\infty} \sup \|x_n - Tp\| \\ &\leqslant \lim_{n\to\infty} \sup \left(3\|Tx_n - x_n\| + \|x_n - p\|\right) \\ &\leqslant \lim_{n\to\infty} \sup \|x_n - p\|. \end{aligned}$$

This implies that $Tp \in A(C, \{x_n\})$. Since X is uniformly convex Banach space, it follows that $A(C, \{x_n\})$ is singleton. Hence, we have $Tp = p$. Thus, $F(T) \neq \phi$. ■

Acknowledgements

The authors of this paper are thankful to the referee for their careful reading and suggestions towards the improvement of this paper.

References

[1] M. Abbas, T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Mat. Vesn. 66 (2) (2014), 223-234.
[2] R. P. Agarwal, D. O'Regan, D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (1) (2007), 61-79.
[3] V. Berinde, Iterative Approximation of Fixed Points, Springer, Berlin, 2007.
[4] Lj. Ćirić, A. Rafiq, S. Radenović, M. Rajović, J. S. Ume, On Mann implicit iterations for strongly accreative and strongly pseudo-contractive mappings, App. Math. Comput. 198 (2008), 128-137.
[5] D. Dukić, Lj. Paunović, S. Radenović, Convergence of iterates with errors of uniformly quasi-Lipschitzian mappings in cone metric spaces, Kragujevac J. Math. 35 (3) (2011), 399-410.
[6] F. Gursoy, V. Karakaya, A Picard S-hybrid type iteration method for solving differential equation with retarded argument, arXiv:1403.2546v2, 2014.
[7] A. M. Harder, Fixed Point Theory and Stability Results for Fixed Point Iteration Procedures, Ph.D Thesis, University of Missouri-Rolla, Missouri, 1987.
[8] N. Hussain, K. Ullah, M. Arshad, Fixed point approximation for Suzuki generalized nonexpansive mappings via new iteration process, J. Nonlinear Convex Anal. 19 (8) (2018), 1383-1393.
[9] S. Ishikawa, Fixed points by a new iteration method, Proc. Am. Math. Soc. 44 (3) (1974), 147-150.
[10] V. Karakaya, N. E. H. Bouzara, K. Dogan, Y. Atalan, On different results for a new two-step iteration method under weak-contraction mapping in Banach spaces, arXiv:1507.00200v1.
[11] W. R. Mann, Mean value methods in iteration, Proc. Am. Math. Soc. 4 (1953), 506-510.
[12] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (1) (2000), 217-229.
[13] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Aust. Math. Soc. 43 (1) (1991), 153-159.
[14] T. Suzuki, Fixed point theorems and convergence theorems for some generalized non-expansive mappings, J. Math. Anal. 340 (2) (2008), 1088-1095.
[15] B. S. Thakur, D. Thakur, M. Postolache, A new iteration scheme for approximating fixed points of nonexpansive mappings, Filomat. 30 (10) (2016), 2711-2720.
[16] B. S. Thakur, D. Thakur, M. Postolache, A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings, Appl. Math. Comp. 275 (2016), 147-155.
[17] K. Ullah, M. Arshad, New iteration process and numerical reckoning fixed point in Banach spaces, U.P.B. Sci. Bull. (Series A). 79 (4) (2017), 113-122.
[18] K. Ullah, M. Arshad, New three-step iteration process and fixed point approximation in Banach spaces, J. Linear. Topological. Algebra. (7) (2) (2018), 87-100.
[19] K. Ullah, M. Arshad, Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process, Filomat. 32 (1) (2018), 187-196.
[20] K. Ullah, M. Arshad, On different results for new three step iteration process in Banach spaces, SpringerPlus. (2016), 2016:1616.
[21] K. Ullah, K. Iqbal, M. Arshad, Some convergence results using K iteration process in CAT(0) spaces, Fixed Point Theory Appl. (2018), 2018:11.
[22] X. Weng, Fixed point iteration for local strictly pseudocontractive mapping, Proc. Am. Math. Soc. 113 (1991), 727-731.