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Mathematical structures via e-open sets

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Abstract. Considering the e-kernel defined by Özkoç and Ayhan [18] in a topological space, a new type of generalized closed set is studied through this article. The aim of this paper is to introduce a new class of sets called $ge\Lambda$-closed sets and $ge\Lambda$-open sets in a topological space and to study their properties and characterizations.

Keywords: e-open set, e-kernel, Λ-set, λ-closed set, λ-derived set.

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1. Introduction and preliminaries

First, in 1986, Maki [15] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel (=saturated set) i.e. to the intersection of all open supersets of A. In 1997, Arenas et al. [2] defined and investigated the notion of λ-closed and λ-open sets by using Λ-sets and closed sets. Then, in 2008, Ekici [9] gave a new type of generalized open sets, called e-open sets. In this study, $ge\Lambda$-closed sets are introduced through these defined concepts.

Throughout this present paper, (X,τ), (Y,σ) and (Z,η) (or simply X, Y and Z) represent topological spaces on which no seperation axioms are assumed. For a subset A of a space X, the closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

A subset A of a topological space X is said to be regular open (regular closed [19]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A point x of X is said to be δ-cluster point [21] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighbourhood U of x. The set of all δ-cluster points of A is called the δ-closure [21] of A and is denoted by $cl_\delta(A)$. If $A = cl_\delta(A)$, then

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A is called δ-closed [21], and the complement of a δ-closed set is called δ-open [21]. The set $\{x|(\exists U \in RO(X))(x \in U \subseteq A)\}$ (equally $\{x|(\exists U \in \tau)(x \in U)(int(cl(U)) \subseteq A)\}$) is called the δ-interior of A and is denoted by $int_\delta(A)$.

A subset A is called semi-open [14] (resp. b-open [1], e-open [9]) if $A \subseteq cl(int(A))$ (resp. $A \subseteq cl(int(A)) \cup int(cl(A))$, $A \subseteq cl(int_\delta(A)) \cup int(cl_\delta(A))$). The complement of a semi-open (resp. b-open, e-open) set is called semi-closed [14] (resp. b-closed [1], e-closed [9]). The intersection of all semi-closed (resp. b-closed, e-closed) sets of X containing A is called the semi-closure [14] (resp. b-closure [1], e-closure [9]) of A and is denoted by $scl(A)$ (resp. $bcl(A)$, $e\text{-}cl(A)$). The union of all semi-open (resp. b-open, e-open) sets of X contained in A is called the semi-interior [14] (resp. b-interior [1], e-interior [9]) of A and is denoted by $sint(A)$ (resp. $bint(A)$, $e\text{-}int(A)$). The family of all open (resp. closed, semi-open, semi-closed, b-open, b-closed, e-open, e-closed) subsets of X is denoted by $O(X)$ (resp. $C(\tau)$, $O^S(X)$, $C^S(\tau)$, $O^B(X)$, $C^B(\tau)$, $O^e(X)$, $C^e(\tau)$).

The power set of X is the set of all possible subsets of X and denoted by $\mathcal{P}(X)$.

Definition 1.1 The kernel [15] (resp. s-kernel [16], γ-kernel [10], e-kernel [18]) of A is denoted by $Ker(A)$ (resp. $Ker_s(A)$, $Ker_\gamma(A)$, $Ker_e(A)$) or A^Λ (resp. A^{Λ_s}, A^{Λ_b}, A^{Λ_e}). The kernels are defined as follows:
(a) $Ker(A) := \cap\{U|(A \subseteq U)(U \in O(X))\}$,
(b) $Ker_s(A) := \cap\{U|(A \subseteq U)(U \in O^S(X))\}$,
(c) $Ker_\gamma(A) := \cap\{U|(A \subseteq U)(U \in O^B(X))\}$,
(d) $Ker_e(A) := \cap\{U|(A \subseteq U)(U \in O^e(X))\}$.

In general $Ker(A)$ (resp. $Ker_s(A)$, $Ker_\gamma(A)$, $Ker_e(A)$) neither an open (resp. semi-open, b-open, e-open) set, nor a closed (resp. semi-closed, b-closed, e-closed) set.

Lemma 1.2 [18] Let X be a topological space and $A \subseteq X$. Then

$$A^{\Lambda_e} = \{x|(\forall E \in O^e(X))(A \subseteq E)(x \in E)\}.$$

Definition 1.3 A subset A of X is called Λ-set [15] (resp. Λ_s-set [16], Λ_b-set [7], Λ_e-set [18]) if $A = Ker(A)$ (resp. $A = Ker_s(A)$, $A = Ker_\gamma(A)$, $A = Ker_e(A)$). The collection of all Λ-sets (resp. Λ_s-sets, Λ_b-sets, Λ_e-set) is denoted by $O^\Lambda(X)$ (resp. $O^{\Lambda_s}(X)$, $O^{\Lambda_b}(X)$, $O^{\Lambda_e}(X)$).

Definition 1.4 A subset A of X is called:
(a) λ-closed set [2] if $A = B \cap C$, where B is a Λ-set and C is a closed set.
(b) g-closed set [13] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is open.
(c) λ-open [2] (resp. g-open [13]) set if $X \setminus A$ is λ-closed (resp. g-closed).
(d) $g\Lambda$-closed set [6] if $cl_\lambda(A) \subseteq U$, whenever $A \subseteq U$ and U is open.
(e) g^*-closed set [12] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is g-open.
(f) Λg-closed set [6] if $cl_\lambda(A) \subseteq U$, whenever $A \subseteq U$ and U is λ-open.
(g) sg-closed set [3] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is semi-open.
(h) gs-closed set [3] if $scl(A) \subseteq U$, whenever $A \subseteq U$ and U is g-open.
(i) $gs\Lambda$-closed set [16] if $cl_\lambda(A) \subseteq U$, whenever $A \subseteq U$ and U is semi-open.
(j) $gb\Lambda$-closed set [17] if $cl_\lambda(A) \subseteq U$, whenever $A \subseteq U$ and U is b-open.
(k) w-closed set [20] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is semi-open.

The collection of all λ-closed (resp. λ-open, g-closed, g-open, g^*-closed, gs-closed, sg-closed, $gs\Lambda$-closed, $gb\Lambda$-closed, $g\Lambda$-closed, Λg-closed, w-closed) subsets of X is denoted by $O^\lambda(X)$ (resp. $C^\lambda(\tau)$, $O^g(X)$, $C^g(\tau)$, $C^{g^*}(\tau)$, $C^{gs}(\tau)$, $C^{sg}(\tau)$, $C^{gs\Lambda}(\tau)$, $C^{gb\Lambda}(\tau)$, $C^{g\Lambda}(\tau)$, $C^{\Lambda g}(\tau)$, $C^\omega(\tau)$).

Lemma 1.5 [2] For a subset A of a space X, the following statements are equivalent:
(1) A is λ-closed;

(2) $A = F \cap cl(A)$, where F is a Λ-set;
(3) $A = Ker(A) \cap cl(A)$.

Definition 1.6 [5] Let X be a topological space and $A \subseteq X$. A point $x \in X$ is called λ-cluster (resp. λ-interior) point of A if for every (resp. there exists a) λ-open set U of X containing x, $A \cap U \neq \emptyset$ (resp. such that $U \subseteq A$). The collection of all λ-cluster (resp. λ-interior) points of A is called the λ-closure (resp. λ-interior) of A and denoted by $cl_\lambda(A)$ (resp. $int_\lambda(A)$).

Lemma 1.7 [4, 5] Let A and B be subsets of a topological space X. Then the following properties hold:
(1) A is λ-closed if and only if $cl_\lambda(A) = A$.
(2) $cl_\lambda(A) \in C^\lambda(\tau)$ and $cl_\lambda(cl_\lambda(A)) = cl_\lambda(A)$.
(3) $cl_\lambda(A) = \cap\{F|(F \in C^\lambda(\tau))(A \subseteq F)\}$.
(4) $A \subseteq cl_\lambda(A) \subseteq cl(A)$.
(5) If $A \subseteq B$, then $cl_\lambda(A) \subseteq cl_\lambda(B)$.
(6) $cl_\lambda(X \setminus A) = X \setminus int_\lambda(A)$.

Definition 1.8 [5] Let X be a topological space and $A \subseteq X$. A point $x \in X$ is called λ-limit point of A if for each λ-open set U containing x, $U \cap (A \setminus \{x\}) \neq \emptyset$. The collection of all λ-limit points of A is called a λ-derived set of A and denoted by $D_\lambda(A)$.

Lemma 1.9 [5] Let A be subset of a topological space X. Then the following properties hold:
(1) $D_\lambda(A) \subseteq D(A)$ where $D(A)$is the derived set of A.
(2) If $A \subseteq X$, then $cl_\lambda(A) = A \cup D_\lambda(A)$.

2. The role of e-open sets as a kernel

Definition 2.1 Let X be a topological space. A subset A of X is called $ge\Lambda$-closed if $cl_\lambda(A) \subseteq U$, whenever $A \subseteq U$ and U is e-open in X.

The collection of all $ge\Lambda$-closed sets of X is denoted by $C^{ge\Lambda}(\tau)$.

Theorem 2.2 Let X be a topological space and $A \subseteq X$. Then, for $A \in C^\lambda(\tau)$, $A \in C^{ge\Lambda}(\tau)$.

Proof. Follows from the fact that Lemma 1.7(1). ∎

The following example shows that the reverse implication of Theorem 2.2 does not hold, in general.

***Example* 2.3** Let $X = \{e_1, e_2, e_3, e_4\}$ and $\tau = \{\emptyset, X, \{e_1\}, \{e_2\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}\}$. Then the set $\{e_1, e_3, e_4\}$ is $ge\Lambda$-closed, but it is not λ-closed.

Theorem 2.4 Let X be a topological space and $A \subseteq X$. If $A \in O^e(X) \cap C^{ge\Lambda}(\tau)$, then $A \in C^\lambda(\tau)$.

Proof. Let $A \in O^e(X)$ and $A \in C^{ge\Lambda}(\tau)$.

$$\left.\begin{array}{c} A \in O^e(X) \\ A \in C^{ge\Lambda}(\tau) \end{array}\right\} \Rightarrow cl_\lambda(A) \subseteq A \Rightarrow X \setminus A \subseteq X \setminus cl_\lambda(A) = int_\lambda(X \setminus A)$$
$$\Rightarrow X \setminus A \in O^\lambda(X) \Rightarrow A \in C^\lambda(\tau).$$ ∎

Theorem 2.5 Let X be a topological space and $A \subseteq X$. Then, for $A \in C(\tau)$, $A \in C^{ge\Lambda}(\tau)$.

Proof. It follows from the fact that $C(\tau) \subseteq C^{\lambda}(\tau) \subseteq C^{ge\Lambda}(\tau)$. ∎

Theorem 2.6 Let X be a topological space and $U \subseteq X$. Then, for $U \in O(X)$, $U \in C^{ge\Lambda}(\tau)$.

Proof. It follows from the fact that $O(X) \subseteq C^{\lambda}(\tau) \subseteq C^{ge\Lambda}(\tau)$. ∎

Theorem 2.7 Let X be a topological space and $A \subseteq X$. Then, for $A \in C^{ge\Lambda}(\tau)$, $A \in C^{g\Lambda}(\tau)$.

Proof. It follows from the fact that $O(X) \subseteq O^{e}(X)$. ∎

Corollary 2.8 Let X be a topological space. Then we have the following chains:
(a) $O(X) \subseteq O^{\Lambda}(X) \subseteq C^{\lambda}(\tau)$;
(b) $C(\tau) \subseteq C^{\lambda}(\tau) \subseteq C^{ge\Lambda}(\tau) \subseteq C^{g\Lambda}(\tau)$.

Remark 1 *From Definitions 1.4, 2.1 and Example 2.3 we have the following diagram. However, none of the above implications is reversible as shown in the relevant articles.*

$$
\begin{array}{ccccccc}
 & & \text{closed} & \rightarrow & \lambda\text{-closed} & \rightarrow & ge\Lambda\text{-closed} \\
 & & \downarrow & & & & \\
g^{*}\text{-closed} & \rightarrow & g\text{-closed} & \leftarrow & sg\text{-closed} & & w\text{-closed} \\
 & & \uparrow & & \downarrow & \swarrow & \\
\Lambda g\text{-closed} & \rightarrow & g\Lambda\text{-closed} & \leftarrow & gs\Lambda\text{-closed} & \leftarrow & gb\Lambda\text{-closed}
\end{array}
$$

Question Are the concepts $gb\Lambda$-closeness and $ge\Lambda$-closeness independent of each other?

Definition 2.9 [11] A partition topology is a topology which can be induced on any set X by partitioning X into disjoint subsets P, these subsets form the basis for the topology.

Lemma 2.10 [11, 16] Let X be a topological space. Then
(1) X is a partition space if and only if $O(X) \subseteq C(\tau)$.
(2) For a partition space X, $cl(A) = cl_{\lambda}(A)$, where $A \subseteq X$.

Theorem 2.11 Let X be a partition space. Then $C^{ge\Lambda}(\tau) \subseteq C^{\omega}(\tau)$.

Proof. Let $A \in C^{ge\Lambda}(\tau)$, $A \subseteq X$ and $U \in O^{S}(X)$.

$$
\left.\begin{array}{r}(A \subseteq X)(U \in O^{S}(X)) \Rightarrow (A \subseteq X)(U \in O^{e}(X)) \\ A \in C^{ge\Lambda}(\tau)\end{array}\right\} \left.\begin{array}{c}\Rightarrow cl_{\lambda}(A) \subseteq U \\ X \text{ is partition}\end{array}\right\} \Rightarrow
$$

$$
\overset{\text{Lemma 2.10(b)}}{\Rightarrow} cl(A) \subseteq U \Rightarrow A \in C^{\omega}(\tau).
$$ ∎

Theorem 2.12 Let X be a topological space and $A \subseteq X$. Then for $A \in C^{ge\Lambda}(\tau)$, $A \in C^{gb\Lambda}(\tau)$, $A \in C^{gs\Lambda}(\tau)$ and $A \in C^{g\Lambda}(\tau)$.

Proof. It follows from the fact that $O(X) \subseteq O^{S}(X) \subseteq O^{B}(X) \subseteq O^{e}(X)$. ∎

3. Applications of *e*-open sets as a kernel

Theorem 3.1 Let X be a topological space and $A \subseteq X$. If $A \in C^{ge\Lambda}(\tau)$, then $F \nsubseteq cl_{\lambda}(A) \setminus A$ where $\emptyset \neq F \in C(\tau)$.

Proof. Let $A \in C^{ge\Lambda}(\tau)$. Suppose that $F \subseteq cl_\lambda(A) \setminus A$ where $\emptyset \neq F \in C(\tau)$.

$$\left.\begin{array}{r}(\emptyset \neq F \in C(\tau))(F \subseteq cl_\lambda(A) \setminus A) \Rightarrow (X \setminus F \in O(X) \subseteq eO(X))(A \subseteq X \setminus F) \\ A \in C^{ge\Lambda}(\tau)\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow cl_\lambda(A) \subseteq X \setminus F \Rightarrow F \subseteq X \setminus cl_\lambda(A) \\ F \subseteq cl_\lambda(A) \setminus A \Rightarrow F \subseteq cl_\lambda(A)\end{array}\right\} \Rightarrow F \subseteq (X \setminus cl_\lambda(A)) \cap cl_\lambda(A) \Rightarrow F = \emptyset.$$

This is a contradiction. Hence, $cl_\lambda(A) \setminus A$ does not contain any non-empty closed set. ■

Theorem 3.2 Let X be a topological space and $A \subseteq X$. If $A \in C^{ge\Lambda}(\tau)$, then $T \nsubseteq cl_\lambda(A) \setminus A$ where $\emptyset \neq T \in C^B(\tau)$.

Proof. It is similar to the proof of Theorem 3.1. ■

Theorem 3.3 Let X be a topological space and $A \subseteq X$. If $A \in C^{ge\Lambda}(\tau)$, then $T \nsubseteq cl_\lambda(A) \setminus A$ where $\emptyset \neq T \in C^e(\tau)$.

Proof. It is similar to the proof of Theorem 3.1. ■

Theorem 3.4 Let X be a topological space. Then for each $x \in X$, either $\{x\} \in C^e(\tau)$ or $X \setminus \{x\} \in C^{ge\Lambda}(\tau)$.

Proof. Let $\{x\} \notin C^e(\tau)$.

$$\left.\begin{array}{r}\{x\} \notin C^e(\tau) \Rightarrow X \setminus \{x\} \in O^e(X) \\ X \setminus \{x\} \subseteq X \in O^e(X)\end{array}\right\} \Rightarrow cl_\lambda(X \setminus \{x\}) \subseteq X.$$

Hence, $X \setminus \{x\} \in C^{ge\Lambda}(\tau)$. Thus, $\{x\} \in C^e(\tau)$ or $X \setminus \{x\} \in C^{ge\Lambda}(\tau)$. ■

Recall that a topological space X is called a Hausdorff (or T_2) space iff for every pair of distinct points x and y, there exist two open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. In this context, a topological space X is called a T_1 space iff every singleton set is closed in X. It is obvious that every Hausdorff space is T_1 space.

Theorem 3.5 Let X be a topological space in which each one-point set is closed. Then $C^\lambda(\tau) = C^{ge\Lambda}(\tau)$.

Proof. Let $x \in X$, $\{x\} \in C(\tau)$. Suppose that $A \in C^{ge\Lambda}(\tau)$ and $A \notin C^\lambda(\tau)$.

$$\left.\begin{array}{r}(A \in C^{ge\Lambda}(\tau))(A \notin C^\lambda(\tau)) \Rightarrow cl_\lambda(A) \setminus A \neq \emptyset \Rightarrow \exists x \in cl_\lambda(A) \setminus A \\ \{x\} \in C(\tau)\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\emptyset \neq \{x\} \in C(\tau))(\{x\} \subseteq cl_\lambda(A) \setminus A)$$

This is a contradiction because of Theorem 3.1. Hence $A \in C^\lambda(\tau)$. Therefore, $C^{ge\Lambda}(\tau) \subseteq C^\lambda(\tau)$. Moreover $C^\lambda(\tau) \subseteq C^{ge\Lambda}(\tau)$. Thus, the result follows. ■

Corollary 3.6 Let X be a Hausdorff space. Then $C^\lambda(\tau) = C^{ge\Lambda}(\tau)$.

Definition 3.7 [13] A topological space X is called a $T_{\frac{1}{2}}$-space if every generalized closed subset of X is closed.

Proposition 3.8 [2] For a topological space X, the followings are equivalent:
(1) X is a $T_{\frac{1}{2}}$-space;
(2) Every subset of X is λ-closed.

Theorem 3.9 Let X be a $T_{\frac{1}{2}}$-space. Then, for each subset A of X, $A \in C^{ge\Lambda}(\tau)$.

Proof. The proof immediately follows from Proposition 3.8 and Theorem 2.2. ■

Definition 3.10 [2] A topological space X is called a $T_{\frac{1}{4}}$-space if for every finite subset F of X and every $y \notin F$,there exists a set A_y containing F and disjoint from $\{y\}$ such that A_y is either open or closed.

Proposition 3.11 [2] For a topological space X, the followings are equivalent:
(1) X is a $T_{\frac{1}{4}}$-space;
(2) Every finite subset of X is λ-closed.

Theorem 3.12 Let X be a $T_{\frac{1}{4}}$-space. Then, for any finite subset A of X, $A \in C^{ge\Lambda}(\tau)$.

Proof. The proof immediately follows from Proposition 3.11 and Theorem 2.2. ■

Definition 3.13 [8] A topological space X is said to be a door space if every subset of X is either open or closed.

Theorem 3.14 Let X be a door space. Then $C^{ge\Lambda}(\tau) = \mathcal{P}(X)$.

Proof. The proof is obvious from Theorems 2.5 and 2.6. ■

Theorem 3.15 Let X be a partition space. Then $C^{ge\Lambda}(\tau) = C^{g}(\tau)$.

Proof. Let $A \in C^{ge\Lambda}(\tau)$, $A \subseteq U$ and $U \in O(X)$.

$$\left.\begin{array}{r}(A \subseteq U)(U \in O(X)) \\ A \in C^{ge\Lambda}(\tau)\end{array}\right\} \left.\begin{array}{l}\Rightarrow cl_{\lambda}(A) \subseteq U \\ X \text{ is partition}\end{array}\right\} \overset{\text{Lemma 2.10(b)}}{\Rightarrow} cl(A) \subseteq U \Rightarrow A \in C^{g}(\tau).$$

Moreover, $C^{g}(\tau) \subseteq C^{ge\Lambda}(\tau)$. Thus, $C^{ge\Lambda}(\tau) = C^{g}(\tau)$. ■

Theorem 3.16 Let X be a topological space and A be a $ge\Lambda$-closed subset of X. Then $A \in C^{\lambda}(\tau)$ if and only if $cl_{\lambda}(A) \setminus A \in C(\tau)$.

Proof. *Necessity. Let* $A \in C^{\lambda}(\tau)$.
$A \in C^{\lambda}(\tau) \Rightarrow cl_{\lambda}(A) = A \Rightarrow cl_{\lambda}(A) \setminus A = \emptyset \in C(\tau)$.
Sufficiency. Let $A \in C^{ge\Lambda}(\tau)$ *and* $cl_{\lambda}(A) \setminus A \in C(\tau)$.

$$\left.\begin{array}{r}A \in C^{ge\Lambda}(\tau) \overset{\text{Theorem 3.1}}{\Rightarrow} (\emptyset \neq F \in C(\tau))(F \nsubseteq cl_{\lambda}(A) \setminus A) \\ cl_{\lambda}(A) \setminus A \in C(\tau)\end{array}\right\} \Rightarrow cl_{\lambda}(A) \setminus A = \emptyset$$

$\Rightarrow cl_{\lambda}(A) = A \Rightarrow A \in C^{\lambda}(\tau)$. ■

Theorem 3.17 Let X be a topological space. If $C^{ge\Lambda}(\tau) \subseteq C^{\lambda}(\tau)$, then for each $x \in X$, either $\{x\} \in C^{e}(\tau)$ or $\{x\} \in O^{\lambda}(X)$.

Proof. Let $C^{ge\Lambda}(\tau) \subseteq C^{\lambda}(\tau)$ and $\{x\} \notin C^{e}(\tau)$.
$\{x\} \notin C^{e}(\tau) \Rightarrow (X \setminus \{x\} \notin O^{e}(X))(X \setminus \{x\} \subseteq X \in O^{e}(X)) \Rightarrow cl_{\lambda}(X \setminus \{x\}) \subseteq X$

$$\left.\begin{array}{r}\Rightarrow X \setminus \{x\} \in C^{ge\Lambda}(\tau) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow X \setminus \{x\} \in C^{\lambda}(\tau) \Rightarrow \{x\} \in O^{\lambda}(X).$$

Therefore, $\{x\}$ is either e-closed or λ-open. ■

Theorem 3.18 [2] Let X be a topological space and $\{A_i | i \in \Lambda\}$ be an arbitrary collection of λ-closed sets. Then $\bigcap_{i} A_i \in C^{\lambda}(\tau)$.

Theorem 3.19 Let X be a topological space and $A, F \subseteq X$. Then, for $A \in O^{e}(X) \cap C^{ge\Lambda}(\tau)$ and $F \in C^{\lambda}(\tau), A \cap F \in C^{ge\Lambda}(\tau)$.

Proof. Let $A \in O^{e}(X) \cap C^{ge\Lambda}(\tau)$ and $F \in C^{\lambda}(\tau)$.

$$\left.\begin{array}{r}A \in O^{e}(X) \cap C^{ge\Lambda}(\tau) \\ \text{Theorem 2.4}\end{array}\right\} \left.\begin{array}{r}\Rightarrow A \in C^{\lambda}(\tau) \\ F \in C^{\lambda}(\tau)\end{array}\right\} \overset{\text{Theorem 3.18}}{\Rightarrow} A \cap F \in C^{\lambda}(\tau)$$

$\overset{\text{Theorem 2.2}}{\Rightarrow} A \cap F \in C^{ge\Lambda}(\tau)$. ■

Theorem 3.20 Let X be a topological space and $A \subseteq X$. Then, for $A \in C^{ge\Lambda}(\tau)$, $e\text{-}cl(\{x\}) \cap A \neq \emptyset$ for every $x \in cl_\lambda(A)$.

Proof. Let $A \in C^{ge\Lambda}(\tau)$. Suppose that $e\text{-}cl(\{x\}) \cap A = \emptyset$ for some $x \in cl_\lambda(A)$.

$$\left.\begin{array}{r}(\exists x \in cl_\lambda(A))(e\text{-}cl(\{x\}) \cap A = \emptyset) \Rightarrow (A \subseteq X \setminus e\text{-}cl(\{x\}))(X \setminus e\text{-}cl(\{x\}) \in O^e(X)) \\ A \in C^{ge\Lambda}(\tau)\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow cl_\lambda(A) \subseteq X \setminus e\text{-}cl(\{x\}) \\ x \in cl_\lambda(A)\end{array}\right\} \Rightarrow x \notin e\text{-}cl(\{x\}).$$

This is a contradiction. Hence, $e\text{-}cl(\{x\}) \cap A \neq \emptyset$ for every $x \in cl_\lambda(A)$. ■

Theorem 3.21 For a topological space X, the following statements are equivalent:
(1) $O^e(X) \subseteq C^\lambda(\tau)$;
(2) $\mathcal{P}(X) \subseteq C^{ge\Lambda}(\tau)$.

Proof. (1) $\Rightarrow$ (2) : Let $A \subseteq X$ and $A \subseteq U$, where $U \in O^e(X)$.

$$\left.\begin{array}{r}(A \subseteq X)(A \subseteq U) \overset{\text{Lemma 1.7(5)}}{\Rightarrow} cl_\lambda(A) \subseteq cl_\lambda(U) \\ U \in O^e(X) \overset{\text{Hypothesis}}{\Rightarrow} U \in C^\lambda(\tau) \overset{\text{Lemma 1.7(1)}}{\Rightarrow} U = cl_\lambda(U)\end{array}\right\} \Rightarrow cl_\lambda(A) \subseteq U$$

$\Rightarrow A \in C^{ge\Lambda}(\tau)$.

(2) $\Rightarrow$ (1) : Let $A \in O^e(X)$.

$$\left.\begin{array}{r}A \in O^e(X) \Rightarrow A \in \mathcal{P}(X) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow A \in C^{ge\Lambda}(\tau) \overset{\text{Theorem 2.2}}{\Rightarrow} A \in C^\lambda(\tau).$$ ■

Theorem 3.22 Let X be a topological space. Let $A, B \in C^{ge\Lambda}(\tau)$ with $D(A) \subseteq D_\lambda(A)$ and $D(B) \subseteq D_\lambda(B)$. Then $A \cup B \in C^{ge\Lambda}(\tau)$.

Proof. Let $A \cup B \subseteq U$ where $U \in O^e(X)$ and let $D(A) \subseteq D_\lambda(A)$ and $D(B) \subseteq D_\lambda(B)$.

$$\left.\begin{array}{r}\left.\begin{array}{r}(D(A) \subseteq D_\lambda(A))(D(B) \subseteq D_\lambda(B)) \\ \text{Lemma 1.9(1)}\end{array}\right\} \Rightarrow (D(A) = D_\lambda(A))(D(B) = D_\lambda(B)) \\ cl(A) = A \cup D(A)\end{array}\right\} \Rightarrow$$

$$\overset{\text{Lemma 1.9(2)}}{\Rightarrow} (cl(A) = A \cup D_\lambda(A) = cl_\lambda(A))(cl(B) = B \cup D_\lambda(B) = cl_\lambda(B)) \ldots (*)$$

$$\left.\begin{array}{r}(A \cup B \subseteq U)(U \in O^e(X)) \Rightarrow (A \subseteq U)(B \subseteq U)(U \in O^e(X)) \\ A, B \in C^{ge\Lambda}(\tau)\end{array}\right\} \Rightarrow$$

$$\Rightarrow cl_\lambda(A \cup B) \overset{\text{Lemma 1.7(4)}}{\subseteq} cl(A \cup B) = cl(A) \cup cl(B) \overset{(*)}{=} cl_\lambda(A) \cup cl_\lambda(B) \subseteq U.$$ ■

The following theorem is a characterization of $ge\Lambda$-closed sets.

Theorem 3.23 Let X be a topological space and $A \subseteq X$. Then $A \in C^{ge\Lambda}(\tau)$ if and only if $cl_\lambda(A) \subseteq Ker_e(A)$.

Proof. *Necessity.* Let $A \in C^{ge\Lambda}(\tau)$. Suppose that $x \in cl_\lambda(A)$ but $x \notin Ker_e(A)$.

$$\left.\begin{array}{r}x \notin Ker_e(A) \overset{\text{Lemma 1.2}}{\Rightarrow} (\exists E \in O^e(X))(A \subseteq E)(x \notin E) \\ A \in C^{ge\Lambda}(\tau)\end{array}\right\} \Rightarrow cl_\lambda(A) \subseteq E.$$

This is a contradiction. Hence, $cl_\lambda(A) \subseteq Ker_e(A)$.

Sufficiency. Let $A \subseteq X$, $cl_\lambda(A) \subseteq Ker_e(A)$ and $A \subseteq U$ where $U \in O^e(X)$.

$$\left.\begin{array}{r}(A \subseteq U)(U \in O^e(X)) \Rightarrow Ker_e(A) \subseteq U \\ cl_\lambda(A) \subseteq Ker_e(A)\end{array}\right\} \Rightarrow cl_\lambda(A) \subseteq U.$$ ■

Theorem 3.24 Let X be a topological space. Let $A, B \subseteq X$ such that $A \in C^{ge\Lambda}(\tau)$ and $A \subseteq B \subseteq cl_\lambda(A)$. Then $B \in C^{ge\Lambda}(\tau)$.

Proof. Let $B \subseteq U$ where $U \in O^e(X)$ and let $A \in C^{ge\Lambda}(\tau)$ and $A \subseteq B \subseteq cl_\lambda(A)$.

$$\left.\begin{array}{r}(B \subseteq U)(U \in O^e(X))(A \subseteq B \subseteq cl_\lambda(A)) \Rightarrow (A \subseteq U)(U \in O^e(X)) \\ A \in C^{ge\Lambda}(\tau)\end{array}\right\} \Rightarrow$$

$$\Rightarrow cl_\lambda(A) \subseteq U \ldots (*)$$

$$A \subseteq B \subseteq cl_\lambda(A) \overset{\text{Lemma 1.7}}{\Rightarrow} cl_\lambda(B) \subseteq cl_\lambda\left(cl_\lambda(A)\right) = cl_\lambda(A) \overset{(*)}{\subseteq} U.$$ ■

4. Complement of $ge\Lambda$-closed sets

Definition 4.1 Let X be a topological space. A is called $ge\Lambda$-open if $X \setminus A$ is $ge\Lambda$-closed set. Equivalently, a subset A of X is said to be $ge\Lambda$-open if $F \subseteq int_\lambda(A)$, whenever $F \subseteq A$ and $F \in C^e(\tau)$.

The collection of all $ge\Lambda$-open sets of X is denoted by $O^{ge\Lambda}(X)$.

Theorem 4.2 Let X be a topological space and $A \subseteq X$. Then $A \in O^{ge\Lambda}(X)$ if and only if $F \subseteq int_\lambda(A)$ whenever $F \subseteq A$ and $F \in C^e(\tau)$.

Proof. *Necessity.* Let $A \in O^{ge\Lambda}(X)$ and let $F \subseteq A$ and $F \in C^e(\tau)$.

$$\left.\begin{array}{r}(F \subseteq A)(F \in C^e(\tau)) \Rightarrow (X \setminus A \subseteq X \setminus F)(X \setminus F \in O^e(X)) \\ A \in O^{ge\Lambda}(X) \Rightarrow X \setminus A \in C^{ge\Lambda}(\tau)\end{array}\right\} \Rightarrow$$

$$\Rightarrow cl_\lambda(X \setminus A) \subseteq X \setminus F$$

$$\Rightarrow F \subseteq X \setminus cl_\lambda(X \setminus A) = int_\lambda(A).$$

Sufficiency. Let $F \subseteq int_\lambda(A)$ where $F \subseteq A$ and $F \in C^e(\tau)$.

$$\left.\begin{array}{r}(F \subseteq A)(F \in C^e(\tau)) \Rightarrow (X \setminus A \subseteq X \setminus F)(X \setminus F \in O^e(X)) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\Rightarrow cl_\lambda(X \setminus A) = X \setminus int_\lambda(A) \subseteq X \setminus F \Rightarrow X \setminus A \in C^{ge\Lambda}(\tau) \Rightarrow A \in O^{ge\Lambda}(X).$$ ■

Theorem 4.3 Let X be a topological space. Then $O^\lambda(X) \subseteq O^{ge\Lambda}(X)$.

Proof. It is obvious from Theorem 2.2. ■

Theorem 4.4 Let X be a topological space. Then $O(X) \subseteq O^{ge\Lambda}(X)$.

Proof. Follows from the fact that $C(\tau) \subseteq C^\lambda(\tau)$ and Theorem 2.2. ■

Theorem 4.5 Let X be a topological space. Then $C(\tau) \subseteq O^{ge\Lambda}(X)$.

Proof. Follows from the fact that $O(X) \subseteq C^\lambda(\tau) \subseteq C^{ge\Lambda}(\tau)$. ■

Theorem 4.6 Let X be a topological space and $A \subseteq X$. If $A \in C^e(\tau) \cap O^{ge\Lambda}(X)$, then $A \in O^\lambda(X)$.

Proof. It is obvious from Theorem 2.4. ■

Theorem 4.7 Let X be a topological space and $A \in O^{ge\Lambda}(X)$. If $int_\lambda(A) \subseteq B \subseteq A$, then $B \in O^{ge\Lambda}(X)$.

Proof. Let $A \in O^{ge\Lambda}(X)$ and $int_\lambda(A) \subseteq B \subseteq A$.

$$\left.\begin{array}{r}int_\lambda(A) \subseteq B \subseteq A \Rightarrow X \setminus A \subseteq X \setminus B \subseteq X \setminus int_\lambda(A) \\ A \in O^{ge\Lambda}(X) \Rightarrow X \setminus A \in C^{ge\Lambda}(\tau)\end{array}\right\} \overset{\text{Theorem 3.24}}{\Rightarrow}$$

$$\Rightarrow X \setminus B \in C^{ge\Lambda}(\tau) \Rightarrow B \in O^{ge\Lambda}(X).$$ ■

Theorem 4.8 Let X be a topological space. If $A \in O^{ge\Lambda}(X)$ and $G \in O^e(X)$ with $int_\lambda(A) \cup (X \setminus A) \subseteq G$, then $G = X$.

Proof. Let $A \in O^{ge\Lambda}(X)$ and $G \in O^e(X)$ with $int_\lambda(A) \cup (X \setminus A) \subseteq G$.

$$\left.\begin{array}{r} int_\lambda(A)\cup(X\setminus A)\subseteq G\Rightarrow X\setminus G\subseteq(X\setminus int_\lambda(A))\cap A=cl_\lambda(X\setminus A)\setminus X\setminus A \\ (A\in O^{ge\Lambda}(X)\Rightarrow X\setminus A\in C^{ge\Lambda}(\tau))(G\in O^e(X)\Rightarrow X\setminus G\in C^e(\tau)) \end{array}\right\}\Rightarrow$$

$$\overset{\text{Theorem 3.3}}{\Rightarrow} X\setminus G=\emptyset\Rightarrow X=G. \qquad \blacksquare$$

Theorem 4.9 Let X be a topological space and $A\subseteq X$. If $A\in C^{ge\Lambda}(\tau)$, then $cl_\lambda(A)\backslash A\in O^{ge\Lambda}(X)$.

Proof. Let $A\in C^{ge\Lambda}(\tau)$ and let $F\subseteq cl_\lambda(A)\setminus A$ where $F\in C^e(\tau)$.

$$\left.\begin{array}{r} (F\subseteq cl_\lambda(A)\setminus A)(F\in C^e(\tau)) \\ A\in C^{ge\Lambda}(\tau) \end{array}\right\}\overset{\text{Theorem 3.3}}{\Rightarrow} F=\emptyset\subseteq int_\lambda(cl_\lambda(A)\setminus A). \qquad \blacksquare$$

Theorem 4.10 Let X be a door space. Then $\mathcal{P}(X)\subseteq O^{ge\Lambda}(X)$.

Proof. Let A be a subset of a door space X.

$$\left.\begin{array}{r} A\subseteq X \\ X \text{ is door} \end{array}\right\}\Rightarrow(A\in O(X))\vee(A\in C(\tau))\overset{\text{Theorem 4.4 or 4.5}}{\Rightarrow} A\in O^{ge\Lambda}(X). \qquad \blacksquare$$

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