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Hyers–Ulam–Rassias stability of impulsive Volterra integral equation via a fixed point approach

R. Shah[a,*], A. Zada[a]

[a]*Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan.*

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Abstract. In this paper, we establish the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability of impulsive Volterra integral equation by using a fixed point method.

Keywords: Ulam–Hyers stability, impulsive Volterra integral equation, fixed point method, Lipschitz condition.

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1. Introduction and preliminaries

We say that a functional equation is stable, if for every approximate solution, there exists an exact solution near it. In 1940, Ulam posed the following problem concerning the stability of functional equations [21]: we are given a group G and a metric group G' with metric $d(.,.)$. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if $f : G \to G'$ satisfies

$$d(f(uv), f(u)f(v)) < \delta \tag{1}$$

for all $u, v \in G$, then a homomorphism $h : G \to G'$ exists with $d(f(u), h(u)) < \epsilon$ for all $u \in G$? The problem for the case of approximately additive mappings was solved by Hyers [12] when G and G' are Banach spaces. Thereafter, this type of stability is called the Ulam–Hyers stability. In 1978, Rassias [18] proved the existence of unique

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*Corresponding author.
E-mail address: safeer_ rahim@yahoo.com (R. Shah); zadababo@yahoo.com (A.Zada).

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linear mappings near approximate additive mappings that provide generalization of the Hyers result. By using the notion of Cadariu and Radu [6], Jung [13] applied the fixed point method to the investigation of Volterra integral equation. They verified that if a continuous function $v : I \to \mathbb{C}$ satisfies the Volterra integral equation of second kind such that

$$\left| v(t) - \int_c^t f(\xi, v(\xi))d\xi \right| \leqslant \phi(t)$$

for all $t \in I$, then there exists a unique continuous function $v_0 : I \to \mathbb{C}$ and a constant K such that

$$v_0(t) = \int_c^t f(\xi, v_0(\xi))d\xi \quad \text{and} \quad | \, v(t), v_0(t) \, | \leqslant K\phi(t) \ \text{ for all } \ t \in I.$$

The theory of nonlinear impulsive differential and integral equations and inclusions have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics (see [3]). In paper [11], Guo established some existence theorems of external solutions for nonlinear impulsive Volterra equations on a finite interval with a finite number of moments of impulse effect in Banach spaces, and offered some applications to initial value problems for the first order impulsive differential equations in Banach spaces. Seeing that many problems in applied mathematics lead to the study of systems of differential or integral equations, the existence of solutions for system of nonlinear impulsive Volterra integral equations on the infinite interval $\mathbb{R}^+$ with an infinite number of moments of impulse effect in Banach spaces is studied.

In 2009, Benchohra et al. [5] discussed the existence and uniqueness of solutions to impulsive fractional differential equations. Also they extend the idea of impulsive fractional differential equations in Banach spaces, see [4]. Moreover, Balachandran et al. [2] proved some new results in Banach spaces for fractional impulsive integro-differential equations. In this paper, we presented the stability results in the sense of Hyers and Ulam of impulsive Volterra integral equation by applying fixed point approach. For detailed study of Ulam-type stability with different approaches, we recommend some papers such as [1, 7, 9, 10, 14–17, 19, 20, 22–30].

The main purpose of this paper is to examine the stability of impulsive Volterra integral equation of second kind given by

$$v(t) = \int_c^t f(\alpha, v(\alpha))d\alpha + \sum_{c<t_k<t} I_k(v(t_k^-)), \tag{2}$$

where f is a continuous function and c is a fixed real number, $I_k : \mathbb{C} \to \mathbb{C}$, $k = 1, 2, \dots, m$ and $v(t_k^-)$ represents the left limit of $v(t)$ at $t = t_k$.

Definition 1.1 If for each function $v(t)$ satisfying

$$\left| v(t) - \int_c^t f(\alpha, v(\alpha))d\alpha - \sum_{c<t_k<t} I_k(v(t_k^-)) \right| \leqslant \varphi(t),$$

where $\varphi(t) \geqslant 0$ for all $t \in I$, there exists a solution $v_0(t)$ of the impulsive Volterra integral equation (2) and a constant $K > 0$ with $|v(t) - v_0(t)| \leqslant K\varphi(t)$ for all $t \in I$, where K is independent of $v(t)$ and $v_0(t)$, then we say that the implusive Volterra integral equation (2) has the Hyers–Ulam–Rassias stability. If $\varphi(t)$ is a constant function in the above inequalities, we say that the implusive Volterra integral equation (2) has the Hyers–Ulam stability.

For a nonempty set Y, we introduce the definition of generalized metric on Y as follows.

Definition 1.2 A mapping $d : Y \times Y \to [0,\infty]$ is called a generalized metric on set Y iff d holds:
(a) $d(u,v) = 0$ if and only if $u = v$;
(b) $d(u,v) = d(v,u)$ for all $u, v \in Y$;
(c) $d(u,w) \leqslant d(u,v) + d(v,w)$ for all $u, v, w \in Y$.

We now introduce one of the fundamental results of fixed point theory. For the proof, we refer to [8]. This theorem will play an important role in proving our main theorems.

Theorem 1.3 Let (Y,d) be a generalized complete metric space. Assume that $\Theta : Y \to Y$ is a strictly contractive operator with $L < 1$, where L is a Lipschitz constant. If there exists a nonnegative integer k such that $d(\Theta^{k+1}u, \Theta^k u) < \infty$ for some $u \in Y$, the the following are true:
(a) The sequence $\Theta^n u$ converges to a fixed point u^* of Θ;
(b) u^* is the unique fixed point of Θ in $Y^* = \Big\{v \in Y \mid d(\Theta^k u, v) < \infty\Big\}$;
(c) If $v \in Y^*$, then $d(v,u^*) \leqslant \frac{1}{1-L}\, d(\Theta v, v)$.

In this paper, using the fixed point Theorem 1.3, we study the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability of the impulsive Volterra integral equation (2).

2. Hyers–Ulam–Rassias stability

In this section, we are going to prove the Hyers–Ulam–Rassias stability of impulsive Volterra integral equation (2).

Theorem 2.1 Suppose $I = [a_1, b_1]$ be given for fixed real numbers a_1, b_1 with $a_1 < b_1$ and K, L_1, L_2 are positive constants with $0 < KL_1 + L_2 < 1$. Let $f : I \times \mathbb{C} \to \mathbb{C}$ is continuous function which satisfy the Lipschitz condition

$$\Big|f(t,u_1) - f(t,v_1)\Big| \leqslant L_1\Big|u_1 - v_1\Big| \tag{3}$$

for any $t \in I$ and $u_1, v_1 \in \mathbb{C}$.
Moreover, $I_k : \mathbb{C} \to \mathbb{C}$ and there exists a constant L_2 such that

$$\Big|I_k(u_1) - I_k(v_1)\Big| \leqslant L_2\Big|u_1 - v_1\Big| \tag{4}$$

for all $u_1, v_1 \in \mathbb{C}$.
Let $v : I \to \mathbb{C}$ be a continuous function such that

$$\Big|v(t) - \int_c^t f(\alpha, v(\alpha))d\alpha - \sum_{c<t_k<t} I_k(v(t_k^-))\Big| \leqslant \varphi(t) \tag{5}$$

for all $t \in I$ and for some $c \in I$, where $I_k : \mathbb{C} \to \mathbb{C}$, $k = 1, 2, ..., m$ and $v(t_k^-)$ represents the left limit of $v(t)$ at $t = t_k$, where $\varphi : I \to (0, \infty)$ is a continuous function with

$$\Big| \int_c^t \varphi(\zeta) d\zeta \Big| \leqslant K\varphi(t) \tag{6}$$

for all $t \in I$, then there exists a unique continuous function $v_0 : I \to \mathbb{C}$ such that

$$v_0(t) = \int_c^t f(\alpha, v_0(\alpha)) d\alpha + \sum_{c<t_k<t} I_k(v_0(t_k^-)), \tag{7}$$

and

$$|v(t) - v_0(t)| \leqslant \frac{1}{1 - (KL_1 + L_2)} \varphi(t) \tag{8}$$

for all $t \in I$.

Proof. First, we define a set $Y = \{h_0 : I \to \mathbb{C} | h_0 \text{ is continuous}\}$ and introduced a generalized metric on Y as follows:

$$d(f, h_0) = \inf\{C \in [0, \infty], |f(t) - h_0(t)| \leqslant C\varphi(t), \text{ for all } t \in I\}. \tag{9}$$

We can see easily that (Y, d) is a complete generalized metric space, see [13]. Consider the operator $\Theta : Y \to Y$ defined by

$$(\Theta h_1)(t) = \int_c^t f(\alpha, h_1(\alpha)) d\alpha + \sum_{c<t_k<t} I_k(h_1(t_k^-)) \tag{10}$$

for all $h_1 \in Y$ and $t \in I$. Next, we will check that the operator Θ is strictly contractive on the set Y. Suppose $g_1, h_1 \in Y$ and assume that $C_{g_1h_1} \in [0, \infty]$ be a constant with $d(g_1, h_1) \leqslant C_{g_1h_1}$ for any $g_1, h_1 \in Y$. By (9), we can write

$$|g_1(t) - h_1(t)| \leqslant C_{g_1h_1}\varphi(t) \tag{11}$$

for all $t \in I$. From inequalities (3), (4), (6), (10) and (11) it follows that

$$\begin{aligned}
|(\Theta g_1)(t) - (\Theta h_1)(t)| &= \Big| \int_c^t \Big\{ f(\alpha, g_1(\alpha)) - f(\alpha, h_1(\alpha)) \Big\} d\alpha \Big| \\
&\quad + \Big| \sum_{c<t_k<t} \Big\{ I_k(g_1(t_k^-)) - I_k(h_1(t_k^-)) \Big\} \Big| \\
&\leqslant L_1 C_{g_1h_1} \Big| \int_c^t \varphi(\alpha) d\alpha \Big| + L_2 \sum_{c<t_k<t} \Big| g_1(t_k^-) - h_1(t_k^-)) \Big| \\
&\leqslant (KL_1 + L_2) C_{g_1h_1} \varphi(t)
\end{aligned}$$

for all $t \in I$, i.e., $d(\Theta g_1, \Theta h_1) \leqslant (KL_1 + L_2)C_{g_1h_1}$. Hence, we may conclude that $d(\Theta g_1, \Theta h_1) \leqslant (KL_1 + L_2)d(g_1, h_1)$ for any $g_1, h_1 \in Y$, where $0 < KL_1 + L_2 < 1$.

It follows from (10) that for any arbitrary $h_0 \in Y$, there exists a constant $C \in [0, \infty]$ with

$$|\Theta h_0(t) - h_0(t)| = \Big| \int_c^t g_1(\alpha, h_0(\alpha))d\alpha + \sum_{c<t_k<t} I_k(h_1(t_k^-)) - h_0(t)\Big| \leqslant C\varphi(t)$$

for all $t \in I$. Since g_1, h_0 are bounded and $\min_{t\in I} \varphi(t) > 0$, then equation (9) implies that

$$d(\Theta h_0, h_0) < \infty. \tag{12}$$

So, according to Theorem 1.3 (a), $v_0 : I \to \mathbb{C}$ a continuous function exists in a way that $\Theta^n h_0 \to h_0$ in (Y, d) and $\Theta v_0 = v_0$, i.e., v_0 satisfies (7) for all $t \in I$. Next, we show that $\{g_1 \in Y|d(h_0, g_1) < \infty\} = Y$, where h_0 was selected with the property (12). Let $g_1 \in Y$, since we know that g_1 and h_0 are bounded on closed interval I for and $\min_{t\in I} \varphi(t) > 0$, then a constant exists $C_{g_1} \in [0, \infty]$ such that $|h_0(t) - g_1(t)| \leqslant C_{g_1}\varphi(t)$ for all $t \in I$. Thus, we can write that $d(h_0, g_1) < \infty$ for any $g_1 \in Y$. Therefore, we get that $\{g_1 \in Y|d(h_0, g_1) < \infty\} = Y$. From Theorem 1.3 (b), we conclude that v_0, given by equation (7), is the unique continuous function. Finally, Theorem 1.3 (c) implies that

$$d(v, v_0) \leqslant \frac{1}{1 - (KL_1 + L_2)} d(\Theta v, v) \leqslant \frac{1}{1 - (KL_1 + L_2)}, \tag{13}$$

since inequality (5) means that $d(\Theta v, v) \leqslant 1$. In view of (9), we can conclude that the inequality (8) holds for all $t \in I$. ■

In the previous theorem, we have examined the Hyers–Ulam–Rassias stability of the impulsive Volterra integral equation (2) defined on compact domains. Now, we will prove the last theorem for the case of unbounded domains.

Theorem 2.2 Suppose I denote either $\mathbb{R}$ or $(-\infty, a]$ or $[a, \infty)$ and K, L_1, L_2 are positive constants with $0 < KL_1+L_2 < 1$. Let $f : I\times\mathbb{C} \to \mathbb{C}$ is continuous function which satisfies the Lipschitz condition (3) for all $t \in I$ and $u_1, v_1 \in \mathbb{C}$. Moreover, $I_k : \mathbb{C} \to \mathbb{C}$ be impulse satisfying the condition (4) with constant L_2, for all $u_1, v_1 \in \mathbb{C}$. Let $v : I \to \mathbb{C}$ be a continuous function satisfies inequality (5) for all $t \in I$ and for some $c \in I$, where $\varphi : I \to (0, \infty)$ is a continuous function satisfying condition (6) for any $t \in I$, then there exists a unique continuous function $v_0 : I \to \mathbb{C}$ which satisfies (7) and (8) for all $t \in I$.

Proof. First, we will prove our theorem for the case $I = \mathbb{R}$. For any $n \in \mathbb{N}$, we define the interval $I_n = [c - n, c + n]$. In accordance with Theorem (2.1) exists a unique continuous function $v_{0,n} : I_n \to \mathbb{C}$ such that

$$v_{0,n}(t) = \int_c^t f(\alpha, v_{0,n}(\alpha))d\alpha + \sum_{c<t_k<t} I_k(v_{0,n}(t_k^-)) \tag{14}$$

and

$$|v(t) - v_0(t)| \leqslant \frac{1}{1 - (KL_1 + L_2)}\varphi(t), \text{for all}\, t \in I. \tag{15}$$

The uniqueness of $v_{0,n}$ implies that if $t \in I_n$, then

$$v_{0,n}(t) = v_{0,n+1}(t) = v_{0,n+2}(t) = \cdots . \tag{16}$$

For $t \in \mathbb{R}$. Define $n(t) \in \mathbb{N}$ as $n(t) := \min\{n \in \mathbb{N} | \, t \in I_n\}$. Next, we define a function $v_0 : \mathbb{R} \to \mathbb{C}$ by

$$v_0(t) = v_{0,n(t)}(t) \tag{17}$$

and claiming that v_0 is continuous. For any $t_1 \in \mathbb{R}$ we take the integer $n_1 = n(t_1)$. Then, t_1 belongs to the interior of the interval I_{n_1+1} and there exists positive $\varepsilon > 0$ such that $v_0(t) = v_{0,n_1+1}(t)$ for all t with $t_1 - \varepsilon < t < t_1 + \varepsilon$. Since v_{0,n_1+1} is continuous at t_1, so is v_0. That is v_0 is continuous at t_1 for any $t_1 \in \mathbb{R}$.

We will now prove that the continuous function v_0 satisfies (7) and (8) for all $t \in \mathbb{R}$. Assume that $n(t)$ be an integer for any $t \in \mathbb{R}$. Then it holds that $t \in I_{n(t)}$ and it follows from (14) that

$$\begin{aligned} v_0(t) = v_{0,n(t)}(t) &= \int_c^t f(\alpha, v_{0,n(t)}(\alpha))d\alpha + \sum_{c<t_k<t} I_k(v_{0,n}(t_k^-)) \\ &= \int_c^t f(\alpha, v_0(\alpha))d\alpha + \sum_{c<t_k<t} I_k(v_0(t_k^-)), \end{aligned}$$

where the last condition holds true because $n(\alpha) < n(t)$ for any $\alpha \in I_{n(t)}$ and it follows from (16) that

$$v_0(\alpha) = v_{0,n(\alpha)}(\alpha) = v_{0,n(t)}(\alpha). \tag{18}$$

Since $v_0(t) = v_{0,n(t)}(t)$ and $t \in I_{n(t)}$ for all $t \in \mathbb{R}$, (15) implies that

$$|v(t) - v_0(t)| = |v(t) - v_{0,n(t)}(t)| \leqslant \frac{1}{1 - (KL_1 + L_2)}\varphi(t). \tag{19}$$

Finally, we are going to prove that v_0 is unique. To do this, we consider another continuous function $v_1 : \mathbb{R} \to \mathbb{C}$ which satisfies (7) and (8), with v_1 instead of v_0 for all $t \in \mathbb{R}$. Suppose that $t \in \mathbb{R}$ be optional number. Since the restrictions $v_0|_{I_{n(t)}}$ and $v_1|_{I_{n(t)}}$ both satisfies (7) and (8) for each $t \in I_{n(t)}$, the uniqueness of $v_{0,n(t)} = v_0|_{I_{n(t)}}$ suggest that

$$v_0(t) = v_0|_{I_{n(t)}}(t) = v_1|_{I_{n(t)}}(t) = v_1(x)$$

are required. We can prove similarly for the cases $I = (-\infty, a]$ and $I = [a, \infty)$. ■

3. Hyers–Ulam stability

In this section, we prove the Hyers–Ulam stability of impulsive Volterra integral equation (2).

Theorem 3.1 Given $p \in \mathbb{R}$ and $q > 0$, suppose that $I(p\,; q)$ denote a closed interval $\{t \in \mathbb{R} | p - q \leqslant t \leqslant p + q\}$. Let $f : I(p\,; q) \times \mathbb{C} \to \mathbb{C}$ is continues function which satisfies the Lipschitz condition (3) for all $t \in I$, $u_1, v_1 \in \mathbb{C}$ where L_1 and L_2 are constants with $0 < L_1 q + L_2 < 1$ and $I_k : \mathbb{C} \to \mathbb{C}$ with constant L_2 satisfy the Lipschitz condition (4). If $\sigma \geqslant 0$ and a continuous function $v : I(p\,;q) \to \mathbb{C}$ which satisfies

$$\Big|v(t) - b - \int_p^t f(\alpha, v(\alpha))d\alpha - \sum_{c<t_k<t} I_k(v(t_k^-))\Big| \leqslant \sigma$$

for all $t \in I(p;q)$, where b is complex number, then there exists a unique continuous function $v_0 : I(p;q) \to \mathbb{C}$ such that

$$v_0(t) = b + \int_p^t f(\alpha, v_0(\alpha))d\alpha + \sum_{c<t_k<t} I_k(v_0(t_k^-)) \tag{20}$$

and

$$|v(t) - v_0(t)| \leqslant \frac{\sigma}{1 - (L_1 q + L_2)} \tag{21}$$

for all $t \in I(p\,;q)$.

Proof. Let $Y = \{h_1 : I(p\,;q) \to \mathbb{C}|\, h_1 \text{ is continuous}\}$ be a set and we introduce a generalized metric on set Y as:

$$d(g_1, h_1) = \inf\{C \in [0, \infty], |g_1(t) - h_1(t)| \leqslant C, \text{ for all } t \in I(p\,;q)\}. \tag{22}$$

We can see easily that (Y, d) is a complete generalized metric space see [13]. Consider the operator $\Theta : Y \to Y$ defined by

$$(\Theta h_1)(t) = b + \int_c^t f(\alpha, h_1(\alpha))d\alpha + \sum_{c<t_k<t} I_k(h_1(t_k^-)) \tag{23}$$

and for all $h_1 \in Y$ and $t \in I(p\,;q)$. Next, we will to check that the operator Θ is strictly contractive on the set Y. Suppose that $C_{g_1h_1} \in [0, \infty]$ be a constant with $d(g_1, h_1) \leqslant C_{g_1h_1}$ for any $g_1, h_1 \in Y$. We have

$$|g_1(t) - h_1(t)| \leqslant C_{g_1h_1}, \quad \text{for all} \quad t \in I(p\,;q). \tag{24}$$

By making the use of (3), (4), (22), (23) and (24) we deduce

$$\begin{aligned}
|(\Theta g_1)(t) - (\Theta h_1)(t)| &= \Big|\int_p^t \Big\{f(\alpha, g_1(\alpha)) - f(\alpha, h_1(\alpha))\Big\}d\alpha\Big| \\
&\quad + \Big|\sum_{c<t_k<t} \Big\{I_k(g_1(t_k^-)) - I_k(h_1(t_k^-))\Big\}\Big| \\
&\leqslant \Big|\int_p^t L_1 \big|g_1(\alpha) - h_1(\alpha)\big|d\alpha + L_2 \sum_{c<t_k<t} \big|g_1(t_k^-) - h_1(t_k^-)\big|\Big| \\
&\leqslant L_1 C_{g_1h_1}|t - p| + L_2 C_{g_1h_1} \\
&\leqslant (L_1 q + L_2) C_{g_1h_1}
\end{aligned}$$

for all $t \in I(p\,;q)$ i.e., $d(\Theta g_1, \Theta h_1) \leqslant (L_1 q + L_2) C_{g_1 h_1}$. Hence, we may conclude that $d(\Theta g_1, \Theta h_1) \leqslant (L_1 q + L_2) d(g_1, h_1)$ for any $g_1, h_1 \in Y$, where $0 < L_1 q + L_2 < 1$. By applying same procedure as in Theorem 2.1, we can choose $h_0 \in Y$ with $d(\Theta h_0, h_0) < \infty$. Hence, from Theorem 1.3 (a) it follows that there exists a continuous function say $v_0 : I(p\,;q) \to \mathbb{C}$ in a way that $\Theta h_0 \to v_0$ in (Y, d) as $n \to \infty$, and such that v_0 satisfies the impulsive Volterra integral equation (20) for any $t \in I(p\,;q)$.
Next, we are going to show that $Y = \{g_1 \in Y | d(h_0, g_1) < \infty\}$. By applying a similar argument to the proof of Theorem 2.1 to this case. Therefore, Theorem 1.3 (b) implies that v_0 is a unique continuous function with property (20). Furthermore, Theorem 1.3 (c) implies that $|v(t) - v_0(t)| \leqslant \frac{\sigma}{1-(L_1 q + L_2)}$ for all $t \in Y$. ■

4. Conclusion

Two kind of novel stability concepts, Hyers–Ulam–Rassias stability and Hyers–Ulam stability, of a impulsive Volterra integral equation are offered. Using Banach's fixed point theorem in a generalized complete metric space, we prove the Hyers–Ulam–Rassias stability on bounded and unbounded intervals and Hyers–Ulam stability results on a finite and closed interval.

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References

[1] M. Akkouchi, Hyers–Ulam–Rassias stability of nonlinear Volterra integral equations via a fixed point approach, Acta. Univ. Apulen. Math. Inform. 26 (2011), 257-266.
[2] K. Balachandran, S. Kiruthika, J. J. Trujillo, Existence results for fractional impulsive integrodifferential equations in Banach spaces, Commun. Nonlinear Sci. Numer. Simulat. (16) (2011), 1970-1977.
[3] M. Benchohra, A. Ouahab, Impulsive neutral functional differential inclusions with variable times, Electron. J. Diff. Equ. 2003 (2003), 1-12.
[4] M. Benchohra, D. Seba, Impulsive fractional differential equations in Banach spaces, Electron. J. Qual. Theo. Diff. Equa. Spec. Edit. (8) (2009), 1-14.
[5] M. Benchohra, B. A. Slimani, Existence and uniqness of solutions to impulsive fractional differential equations, Electron. J. Diff. Equ. (10) (2009), 1-11.
[6] L. Cdariu, V. Radu, On the stability of the Cauchy functional equation a fixed point approach, Grazer. Math. Ber. 346 (2004), 43-52.
[7] L. P. Castro, A. Ramos, Hyers–Ulam–Rassias stability for a class of nonlinear Volterra integral equations, Banach J. Math. Anal. 3 (2009), 36-43.
[8] J. B. Diaz, B. Margolis, A fixed point theorem of the alternative, for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. 74 (1968), 305-309.
[9] M. Gachpazan, O. Baghani, Hyers–Ulam stability of nonlinear integral equation, Fixed Point Theory Appl. (2010), 2010:927640.
[10] M. Gachpazan, O. Baghani, Hyers–Ulam stability of Volterra integral equation, J. Nonlinear Anal. Appl. 1 (2010), 19-25.
[11] D. Guo, Nonlinear impulsive Volterra integral equations in Banach spaces and applications, J. Appl. Math. Stoch. Anal. 6 (1) (1993), 35-48.
[12] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U. S. A. 27 (1941), 222-224.
[13] S. M. Jung, A fixed point approach to the stability of a Volterra integral equation, Fixed Point Theory Appl. (2007), 2007: 057064.
[14] S. M. Jung, A fixed point approach to the stability of differential equations $y' = F(x, y)$, Bull. Malays. Math. Sci. Soc. 33 (2010), 47-56.
[15] S. M. Jung, S. Sevgin, H. Sevli, On the perturbation of Volterra integro-differential equations, Appl. Math. Lett. 26 (2013), 665-669.

[16] T. Li, A. Zada, Connections between Hyers–Ulam stability and uniform exponential stability of discrete evolution families of bounded linear operators over Banach spaces, Adv. Difference Equ. (2016), 2016:153.
[17] T. Li, A. Zada, S. Faisal, Hyers–Ulam stability of nth order linear differential equations, J. Nonlinear Sci. Appl. 9 (2016), 2070-2075.
[18] T. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.
[19] R. Shah, A. Zada, A fixed point approach to the stability of a nonlinear Volterra integro-differential equation with delay, Hacet. J. Math. Stat. 47 (3) (2018), 615-623.
[20] S. Tang, A. Zada, S. Faisal, M. M. A. El-Sheikh, T. Li, Stability of higher–order nonlinear impulsive differential equations, J. Nonlinear Sci. Appl. 9 (2016), 4713-4721.
[21] S. M. Ulam, Problems in modern mathematics, John Wiley & Sons Inc., New York, 1960.
[22] A. Zada, W. Ali, S. Farina, Hyers–Ulam stability of nonlinear differential equations with fractional integrable impulses, Math. Method. Appl. Sci. 40 (2017), 5502-5514.
[23] A. Zada, S. Faisal, Y. Li, On the Hyers–Ulam stability of first-order impulsive delay differential equations, J. Funct. Spaces. (2016), 2016:8164978.
[24] A. Zada, S. Faisal, Y. Li, Hyers–Ulam–Rassias stability of non-linear delay differential equations, J. Nonlinear Sci. Appl. 10 (2017), 504-510.
[25] A. Zada, F. U. Khan, U. Riaz, T. Li, Hyers–Ulam stability of linear summation equations, Punjab U. J. Math. 49 (2017), 19-24.
[26] A. Zada, T. Li, S. Ismail, O. Shah, Exponential dichotomy of linear autonomous systems over time scales, Diff. Equa. Appl. 8 (2016), 123-134.
[27] A. Zada, S. O. Shah, Hyers–Ulam stability of first-order non-linear delay differential equations with fractional integrable impulses, Hacet. J. Math. Stat. 47 (5) (2018), 1196-1205.
[28] A. Zada, S. O. Shah, S. Ismail, T. Li, Hyers–Ulam stability in terms of dichotomy of first order linear dynamic systems, Punjab. Uni. J. Math. 49 (2017), 37-47.
[29] A. Zada, O. Shah, R. Shah, Hyers–Ulam stability of non-autonomous systems in terms of boundedness of Cauchy problems, Appl. Math. Comp. 271 (2015), 512-518.
[30] A. Zada, P. Wang, D. Lassoued, T. Li, Connections between Hyers–Ulam stability and uniform exponential stability of 2–periodic linear nonautonomous systems, Adv. Difference Equ. (2017), 2017:192.