Journal of Linear and Topological Algebra
Vol. 11*, No.* 04*,* 2022*,* 225- 229
DOR: 20.1001.1.22520201.2022.11.04.2.9
DOI: 10.30495/JLTA.2022.698655

A new characterization of chevalley groups $\mathbf{G_2(q)}$

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Received 5 September 2022; Revised 12 December 2022; Accepted 14 December 2022.
Communicated by Shervin Sahebi

Abstract. In this paper, we prove that chevalley groups $G_2(q)$, where $q \equiv \pm 2 (\mathrm{mod}\ 5)$ and q^2+q+1 is a prime number, can be uniquely determined by the order of the group and the second largest element order.

Keywords: Element order, the largest element order, the second largest element order, chevalley group.

2010 AMS Subject Classification: 20D06, 20D60.

1. Introduction and preliminaries

Let G be a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of elements order in G. We denote the largest element order of G by $k_1(G)$ and also the second largest element of G by $k_2(G)$. Also we denote a sylow p-subgroup of G by G_p and the number of sylow p-subgroups of G by $n_p(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two distinct vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, ..., t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1$.

In 1987, Thompson posed a question as follows:

Thompsons Problem. Suppose G_1 and G_2 are the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

Group characterization is one of the issues that have been considered by researchers, where this characterization is done by using properties such as element order, number

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.ctb.iau.ir

of elements, order, etc. One of this methods, is group characterization by using the largest element order and the order of the group. In other words, we say the group G is characterizable by using the order of the group and the largest element order of G, if for every group H, so that $k_1(G) = k_1(H)$ and $|G| = |H|$, then $G \cong H$.

However, the authors proved that some finite simple groups are characterizable by using the order of the group and the largest element order of G. For example, the authors in ([2–4, 6–9, 11, 13]) proved that the sporadic simple groups, the projective special linear groups $L_2(q)$, where $q = p^n < 125$, the simple groups $L_3(q)$ and $U_3(q)$ where q is some special power of prime, the projective special unitary group $PSU_3(3^n)$, the symplectic groups $PSP(8, q)$, the simple K_4 -group of type $L_2(p)$ where p is a prime but not 2^n-1, the sympelectic group $C_4(q)$ and ${}^2D_8((2^n)^2)$, where $2^{8n} + 1$ is a prime number are characterizable by the largest element order and the order of the group.

In this paper, we prove that chevalley groups $G_2(q)$, where $q \equiv \pm 2 (\bmod\ 5)$ and $q^2 + q + 1$ is a prime number, can be uniquely determined by the order of the group and the second largest element order. In fact, we prove the following main theorem.

Main Theorem. Let G be a group with $|G| = |G_2(q)|$ and $k_2(G) = k_2(G_2(q))$, where $q \equiv \pm 2 (\bmod\ 5)$ and $q^2 + q + 1$ is a prime number. Then $G \cong G_2(q)$.

In this section, we describe some preliminary results which will be used later.

Lemma 1.1 [10, Theorem 3.1] Let G be a Frobenius group of even order with kernel K and complement H. Then

(1) $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;
(2) $|H|$ divides $|K| - 1$;
(3) K is nilpotent.

Definition 1.2 [1] A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively.

Lemma 1.3 [1, Theorem 2] Let G be a 2-Frobenius group of even order. Then,

(1) $t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;
(2) G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.

Lemma 1.4 [15, Theorem A] Let G be a finite group with $t(G) \geqslant 2$. Then one of the following statements holds:

(1) G is a Frobenius group;
(2) G is a 2-Frobenius group;
(3) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.

Lemma 1.5 [16, Lemma 6] Let q, k, l be natural numbers. Then

(1) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.
(2) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q + 1) & \text{otherwise.} \end{cases}$
(3) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q + 1) & \text{otherwise.} \end{cases}$

In particular, the inequality $(q^k - 1, q^k + 1) \leqslant 2$ holds for every $q \geqslant 2$ and $k \geqslant 1$.

Lemma 1.6 [14, Lemma 6] Let G be a non-abelian simple group such that $(5, |G|) = 1$. Then G is isomorphic to one of the following groups:

(1) $L_n(q)$, $n = 2, 3$, $q \equiv \pm 2 \pmod 5$(Projective special linear group);
(2) $G_2(q)$, $q \equiv \pm 2 \pmod 5$(Chevalley group);
(3) $U_3(q)$, $q \equiv \pm 2 \pmod 5$(Projective special unitary group);
(4) ${}^3D_4(q)$, $q \equiv \pm 2 \pmod 5$(Steinberg group);
(5) ${}^2G_2(q)$, $q = 3^{2m+1}$, $m \geq 1$(Ree group).

2. Main results

In this section, we prove that the chevalley groups $G_2(q)$ are characterizable by using the order of the group and the second largest element order. In fact, we prove that if G is a group with $|G| = |G_2(q)|$ and $k_2(G) = k_2(G_2(q))$, where $q \equiv \pm 2 (\mathrm{mod}\ 5)$ and $q^2 + q + 1$ is a prime number, then $G \cong G_2(q)$. From now on, we denote the chevalley groups $G_2(q)$ and prime number $q^2 + q + 1$ by R and p, respectively. Suppose that G is a group with $|G| = |R| = q^6(q^6 - 1)(q^2 - 1)$ and $k_2(G) = k_2(R) = q^2 + q$, (See [5, 12]).

Claim 1. p is an isolated vertex of $\Gamma(G)$.
proof. We, prove that p is an isolated vertex of $\Gamma(G)$. Suppose the opposite, then there is a prime number $t \in \pi(G) - \{p\}$, so that $tp \in \pi_e(G)$. So, we deduce $tp \geq 2p = 2(q^2 + q + 1) \geq q^2 + q + 1 > q^2 + q$. Therefore $k_2(G) > q^2 + q$, which is a contradiction.

Claim 2. The group G is neither a Frobenius group nor a 2-Frobenius group.
proof. Let G be a Frobenius group with kernel K and complement H. Then by lemma 1.1, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now by Claim1 p is an isolated vertex of $\Gamma(G)$. Thus, we deduce that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| = |G|/p$ and $|K| = p$. Now we prove that $|H| = p$ and $|K| = |G|/p$. For this purpose, we assume $\pi(H) = p$, then we show $|H| = p$. Since p is an isolated vertex and p be a set of prime divisor of H. Hence, $H = \{p, p^2, ..., p^n\}$ so $|H| = p^n$. Now, we prove that only $n = 1$ is satiesfied. For this purpose assume $n > 1$. The least value $n = 2$. Now, since G be a Frobenius group by kernel K and compelement H. On the other hand $G = KH$. As a result $|K| = \frac{|G|}{|H|}$ so $|K| = \frac{q^6(q^6-1)(q^2-1)}{(q^2+q+1)^2}$. Thus, $|K| = \frac{q^{14}-q^{12}-q^8+q^6}{(q^2+q+1)^2}$. It follows that $|K| = (q^4+2q^3+3q^2+2q+1)(q^{10}-2q^9+4q^7-5q^6+6q^4-6q^3+6q-6)+(6q^2+6q+6)$. Now, since $|H| \mid |K|-1$, so $(q^4+2q^3+3q^2+2q+1) \mid (q^4+2q^3+3q^2+2q+1)(q^{10}-2q^9+4q^7-5q^6+6q^4-6q^3+6q-6)+(6q^2+6q+5)$. As a result $(q^4+2q^3+3q^2+2q+1) \mid (6q^2+6q+5)$, which is a contradiction, so only $n = 1$ is satiesfied. Now, assume $\pi(K) = p$, then we prove that $|K| = p$. Since, p is an isolated vertex and p be a set of prime divisor of H so $H = \{p, p^2, ..., p^n\}$ it follows that $|H| = p^n$. Now, we prove that only $n = 1$ is satiesfied. For this purpose, assume $n > 1$. In the least value $n = 2$. Now, since G be a Frobenius group by kernel K and compelement H. On the other hand, $G = KH$. As a result $|H| = \frac{|G|}{|K|}$ so $|H| = \frac{q^6(q^6-1)(q^2-1)}{(q^2+q+1)^2}$. Thus $|H| = \frac{q^{14}-q^{12}-q^8+q^6}{q^4+2q^3+3q^2+2q+1}$. Since $|H|$ divides $|K| - 1$, so$\frac{q^{14}-q^{12}-q^8+q^6}{q^4+2q^3+3q^2+2q+1} \mid (q^4 + 2q^3 + 3q^2 + 2q + 1) - 1$ it follows that $(q^4 + 2q^3 + 3q^2 + 2q + 1)(q^{10} - 2q^9 + 4q^7 - 5q^6 + 6q^4 - 6q^3 + 6q - 6) + (6q^2 + 6q + 6) \mid (q^4 + 2q^3 + 3q^2 + 2q)$, which this is a contradiction. Thus $|K| = p$. Now, since $|H| = |G|/p \nmid p - 1$, we conclude that the last case (ii) can not occur. Thus, $|H| = p$ and

$|K| = |G|/p$ it follows that $q^2 + q + 1 \mid q^6(q^6 - 1)(q^2 - 1)/(q^2 + q + 1) - 1$. Hence, we have $q^2 + q + 1 \mid (q^2 + q + 1)(q^{10} - 2q^9 + 4q^7 - 5q^6 + 6q^4 - 6q^3 + 6q - 6) + 5$. As a result $p \mid 5$, which is impossible.
We now show that G is not a 2-Frobenius group. Suppose the opposite, assume G be a 2-Frobenius group, so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups by kernels K/H and H respectively. Now, since p is an isolated vertex of $\Gamma(G)$, it follows that $\pi_2(G) = p$ and also$|K/H| = p$. On the other hand, $|G/K|$ divides $|Aut(K/H)|$, we deduce that $|G/K||(p-1)$. On the other hand, we have $(q^2 + q + 1, q^2 + q - 1) = 1$. Now, since $|G/K| \mid (p-1)$, we deduce that $q^2 + q - 1 \mid |H|$. Let H_1 be a subgroup of H of order $q^2 + q - 1$. On the other hand, H is nilpotent, therefore $H_1 \rtimes K/H$ is a Frobenius group with kernel H_1 and complementK/H. It follows that, $|K/H|$divides $|H_1| - 1$, so we have $q^2 + q + 1 \leq (q^2 + q - 1) - 1$, but this is a contradiction.

Claim 3. The group G is isomorphic to the group R.
proof. By Claim 1, p is an isolated vertex of $\Gamma(G)$. Thus, $t(G) > 1$ and G satisfies one of the cases of Lemma 1.4. Furthermore, Claim 2 implies that G is neither a Frobenius group nor a 2-Frobenius group. Thus only the case (c) of lemma 1.4 occurs. So, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group. Since, p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$. On the other hand, we know that $5 \nmid |G|$. Thus K/H is isomorphic to one of the groups in Lemma 1.6. Hence, we consider the following cases:

(1) If $K/H \cong {}^2G_2(q')$, where $q' = 3^{2m+1}$, then by ([12, Table A.7]), $k_2({}^2G_2(q')) = q' - \sqrt{3q'} + 1$. On the other hand, we know $|{}^2G_2(q')| \mid |G|$, in other words $q'^3(q'^3+1)(q'-1) \mid |G|$. For this purpose, we consider $q^2+q = q' - \sqrt{3q'}+1$. It follows that $3^{m+1}(3^m - 1) = (q - (\frac{-1+\sqrt{5}}{2}))(q - (\frac{-1-\sqrt{5}}{2}))$. Since $(3^{m+1}, 3^m - 1) = 1$, so we deduce $q - (\frac{-1-\sqrt{5}}{2}) = 3^m - 1$ and $q - (\frac{-1+\sqrt{5}}{2}) = 3^{m+1}$. Then, we can see easily this equations don't have any solution in natural number $\mathbb{N}$, which is a contradiction.

(2) If $K/H \cong {}^3D_4(q')$, then by ([12, Table A.7]), $k_2({}^3D_4(q')) = q'^4 - q'^2 + 1$. On the other hand we know$|{}^3D_4(q')| \mid |G|$, as $q'^{12}(q'^8 + q'^4 + 1)(q'^6 - 1)(q'^2 - 1) \mid |G|$. For this purpose, we consider $q^2 + q = q'^4 - q'^2 + 1$. As a result $(q - (\frac{-1+\sqrt{5}}{2}))(q - (\frac{-1-\sqrt{5}}{2})) = q'^2(q'^2 - 1)$ and hence $q - (\frac{-1+\sqrt{5}}{2}) = q'^2$ and $q'^2 - 1 = q - (\frac{-1-\sqrt{5}}{2})$. Then, we can see easily this equations don't have any solution in natural number $\mathbb{N}$, which this is a contradiction.

(3) If $K/H \cong L_2(q')$, where $q' \equiv \pm 2 (\mathrm{mod}\ 5)$, $q' = p'^m$, then by ([12, Table A.1]) $k_2(L_2(q')) = q' - 1$, $\frac{q'+1}{2}$, where q' be even and odd respectively. On the other hand, we know $|L_2(q')| \mid |G|$, in other words $\frac{q'(q'^2-1)}{(2,q'-1)} \mid |G|$. Now, for this purpose, assume q' be even, then $k_2(L_2(q')) = q' - 1$, so we have $q^2 + q = q' - 1$. Then $q^2 - q + 1 = q'$. Since $|L_2(q')| \nmid |G|$, which is a contradiction. Now if q' odd, then $k_2(L_2(q')) = \frac{q'+1}{2}$, so we have $q^2+q = \frac{q'+1}{2}$. Then $2q^2+2q-1 = q'$. But this is a contradiction, because $q' = p'^m$.

(4) If $K/H \cong L_3(q')$, where $q' \equiv \pm 2\ (\mathrm{mod}\ 5)$, then by ([12, Table A.1]), $k_2(L_3(q')) = \frac{q'^2-1}{(3,q'-1)}$. On the other hand, we know $|L_3(q')| \mid |G|$, as $\frac{q'^3(q'^3-1)(q'^2-1)}{(3,q'-1)} \mid |G|$. For this purpose, we consider two cases. First we assume $(3, q' - 1) = 1$, then $q^2 + q = q'^2 - 1$. As a result $q(q + 1) = (q' - 1)(q' + 1)$, now since $(q, q + 1) = 1$, we deduce $q' - 1 = q$ and also $q' + 1 = q + 1$. So, $q' = q + 1$ and $q' = q - 1$, but $|L_3(q')| \nmid |G|$, which this is a contradiction. Now, if $q^2 + q = \frac{q'^2-1}{3}$, then $3q^2 + 3q = q'^2 - 1$. Therefore, $3q(q + 1) = (q' - 1)(q' + 1)$. On the other hand, $(q' - 1, q' + 1) = 1$ or 2. Now, if $(q' - 1, q' + 1) = 1$, then $q + 1 = q' - 1$ and $3q = q' + 1$. So $q' = q + 2$ and $q' = 3q - 1$ but $|L_3(q')| \nmid |G|$, which is a contradiction. The other case is impossible.

(5) $K/H \cong U_3(q')$, where $q' \equiv \pm 2 \pmod 5$, then by ([12, Table A.2]), $k_2(U_3(q')) = \frac{q'^2-1}{(3,q'+1)}$. On the other hand, we know$|U_3(q')| \mid |G|$, in other words $\frac{q'^3(q'^3+1)(q'^2-1)}{(3,q'+1)} \mid |G|$. For this purpose, we consider two cases. First, we assume $(3, q'+1) = 1$, then $q^2+q = q'^2-1$. As a result $q(q + 1) = (q' - 1)(q' + 1)$, now since $(q, q + 1) = 1$, we deduce $q' - 1 = q$ and also $q' + 1 = q + 1$. It follows that $q' = q + 1$ and $q' = q - 1$, but $|U_3(q')| \nmid |G|$, which this is a contradiction. Now if $q^2 + q = \frac{q'^2-1}{3}$, then $3q^2 + 3q = (q' - 1)(q' + 1)$. So, $3q(q + 1) = (q' - 1)(q' + 1)$ it follows that $3q = q' + 1$ and $q + 1 = q' - 1$. So, $q' = 3q - 1$ and $q' = q + 2$ but $|U_3(q')| \nmid |G|$, which is a contradiction. Hence, we have the following isomorphic:

(6) $K/H \cong G_2(q')$, where $q' \equiv \pm 2 \pmod 5$, as a result $|K/H| = |R|$. Now, since p is an isolated vertex and $p \mid |K/H|$ and also $k_2(K/H) \mid k_2(G)$. Hence, we consider $q^2 + q = q'^2 + q'$ as a result $q = q'$. Now, since $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H = 1$, so $G = K \cong R$.

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