

Digital cohomology groups of certain minimal surfaces

I. Karaca^a, O. Ege^{a,*}

^a*Department of Mathematics, Faculty of Science, Ege University, 35100 Izmir, Turkey.*

Received 6 August 2018; Revised 26 September 2018, Accepted 1 October 2018.

Communicated by Ghasem Soleimani Rad

Abstract. In this study, we compute simplicial cohomology groups with different coefficients of a connected sum of certain minimal simple surfaces by using the universal coefficient theorem for cohomology groups. The method used in this paper is a different way to compute digital cohomology groups of minimal simple surfaces. We also prove some theorems related to degree properties of a map on digital spheres.

© 2018 IAUCTB. All rights reserved.

Keywords: Digital image, universal coefficient theorem, digital cohomology group.

2010 AMS Subject Classification: 55N99, 68R99, 68U05, 68U10.

1. Introduction

Digital topology introduced in [19] is an area of great theoretical interest having the additional bonus of significant applications in imaging science and related areas. It continues to rise in many fields of science and engineering such as mathematics, image processing, biology, information systems, and computer science with a great number of applications. There are topological invariants (connectedness, homology, cohomology, Euler characteristics, etc.) which are extremely useful in digital images and geometric modeling. Computing topological invariants of objects has significant impacts in digital images. Homology is a powerful topological invariant which characterizes an object by its n -dimensional holes. Cohomology is an important algebraic invariant in classification of topological spaces because it has a structure called the cup product. Although basic properties of cohomology groups are similar to homology groups, there are some differences

*Corresponding author.

E-mail address: ismet.karaca@ege.edu.tr (I. Karaca); ozgur.ege@ege.edu.tr (O. Ege).

between them. One of the differences is that cohomology groups are contravariant functors while homology groups are covariant. Additional structures in cohomology feature contravariance. They are finer invariants of homotopy type and enable us to distinguish between topological spaces.

Homology and cohomology theory have significant applications in science and engineering. The progress of computational approach to homology and cohomology theory is activated by problems in geometric modeling, digital image processing, dynamical systems and material science (see [12, 13, 17]).

In recent years, there have been many developments in digital topology. Boxer [6] gives the digital versions of some notions in topology. Arslan et al. [1] introduce the simplicial homology groups of n -dimensional digital images. Boxer et al. [9] study the simplicial homology groups of certain minimal simple surfaces and compute the Euler characteristics of certain minimal simple surfaces. Homology groups of some minimal simple surfaces are computed in [10]. In [11], simplicial cohomology theory is given for digital images. Karaca and Burak [18] determine relative cohomology groups of digital images algebraically by the relative homology groups of digital images. They give a method for computing cohomology rings of digital images.

This paper is organized as follows. In preliminaries, we review necessary backgrounds on digital images and digital cohomology groups. Then the simplicial cohomology groups with various coefficients of connected sums of certain minimal simple surfaces are computed by using a universal coefficient theorem. Moreover, we deal with some degree properties of a map on digital spheres.

2. Preliminaries

Let \mathbb{Z}^n be the set of lattice points in the n -dimensional Euclidean space where \mathbb{Z} is the set of integers. We say that (X, κ) is a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation for the members of X . We use a variety of adjacency relations in the study of digital images.

Definition 2.1 [7] For a positive integer l with $1 \leq l \leq n$ and two distinct points $p = (p_1, p_2, \dots, p_n)$, $q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$, p and q are c_l -adjacent, if

- (1) there are at most l indices i such that $|p_i - q_i| = 1$, and
- (2) for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

The notation c_l represents the number of points $q \in \mathbb{Z}^n$ that are adjacent to a given point $p \in \mathbb{Z}^n$. Thus, in \mathbb{Z} , we have $c_1 = 2$ -adjacency; in \mathbb{Z}^2 , we have $c_1 = 4$ -adjacency and $c_2 = 8$ -adjacency; in \mathbb{Z}^3 , we have $c_1 = 6$ -adjacency, $c_2 = 18$ -adjacency and $c_3 = 26$ -adjacency [7].

Given a natural number l in conditions (1) and (2) with $1 \leq l \leq n$, l determines each of the κ -adjacency relations of \mathbb{Z}^n in terms of (1) and (2) as follows [14].

$$\kappa \in \left\{ 2n \ (n \geq 1), \ 3^n - 1 \ (n \geq 2), \ 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 \ (2 \leq r \leq n-1, n \geq 3) \right\}$$

where $C_t^n = \frac{n!}{(n-t)!t!}$.

Let κ be an adjacency relation defined on \mathbb{Z}^n . A κ -neighbor of $p \in \mathbb{Z}^n$ is a point of \mathbb{Z}^n that is κ -adjacent to p . A digital image $X \subset \mathbb{Z}^n$ is κ -connected [16] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \dots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ -neighbors where $i = 0, 1, \dots, r-1$.

Let $a, b \in \mathbb{Z}$ with $a < b$. A digital interval [6] is a set of the form $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} | a \leq z \leq b\}$. Also, let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. A function $f : X \rightarrow Y$ is said to be (κ_0, κ_1) -continuous if for every κ_0 -connected subset U of X , $f(U)$ is a κ_1 -connected subset of Y [4].

A $(2, \kappa)$ -continuous function $f : [0, m]_{\mathbb{Z}} \rightarrow X$ such that $f(0) = x$ and $f(m) = y$ is called a digital κ -path from x to y in a digital image X [8]. A simple closed κ -curve of $m \geq 4$ points in a digital image X is a sequence $\{f(0), f(1), \dots, f(m-1)\}$ of images of the κ -path $f : [0, m-1]_{\mathbb{Z}} \rightarrow X$ such that $f(i)$ and $f(j)$ are κ -adjacent if and only if $j = i \pm 1 \pmod m$.

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 -adjacency and κ_1 -adjacency respectively. A function $f : X \rightarrow Y$ is (κ_0, κ_1) -isomorphism if f is (κ_0, κ_1) -continuous and bijective and also $f^{-1} : Y \rightarrow X$ is (κ_1, κ_0) -continuous [5].

Definition 2.2 [4] Let $(X, \kappa_1) \subset \mathbb{Z}^{n_1}$ and $(Y, \kappa_2) \subset \mathbb{Z}^{n_2}$ be digital images and $f, g : X \rightarrow Y$ be two (κ_1, κ_2) -continuous functions. f and g are called digitally (κ_1, κ_2) -homotopic in Y if there is a positive integer m and a function $H : X \times [0, m]_{\mathbb{Z}} \rightarrow Y$ such that

- for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$,
- for all $x \in X$, $H_x : [0, m]_{\mathbb{Z}} \rightarrow Y$ defined by $H_x(t) = H(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is $(2, \kappa_2)$ -continuous,
- for all $t \in [0, m]_{\mathbb{Z}}$, $H_t : X \rightarrow Y$ defined by $H_t(x) = H(x, t)$ for all $x \in X$ is (κ_1, κ_2) -continuous.

The function H is called a digital (κ_1, κ_2) -homotopy between f and g .

A digital image (X, κ) is said to be κ -contractible if the identity map on X is (κ, κ) -homotopic to the constant map with value a for some $a \in X$ [6]. For a digital image (X, κ) and its subset (A, κ) , we call (X, A) a digital image pair with κ -adjacency. Moreover, if A is a singleton set $\{x_0\}$, then (X, x_0) is called a pointed digital image.

A point $x \in X$ is called a κ -corner, if x is κ -adjacent to two and only two points $y, z \in X$ such that y and z are κ -adjacent to each other [3]. Moreover, the κ -corner x is called simple, if y, z are not κ -corners and if x is the only point κ -adjacent to both y, z [2]. X is called a generalized simple closed κ -curve [2] if what is obtained by removing all simple κ -corners of X is a simple closed κ -curve [3].

If (X, κ) is a κ -connected digital image in \mathbb{Z}^n , $n \geq 3$, $|X|^x = N_{3^n-1}^*(x) \cap X$, where $N_{3^n-1}^*(x) = \{x' : x \text{ and } x' \text{ are } (3^n - 1)\text{-adjacent}\}$ [15]. Let c^* be a closed κ -curve in \mathbb{Z}^2 . A point x in $\mathbb{Z}^2 \setminus c^*$ is said to be in the interior of c^* if it belongs to the bounded $\hat{\kappa}$ -connected component of $\mathbb{Z}^2 \setminus c^*$. The set of all interior points of c^* is denoted by $Int(c^*)$.

Definition 2.3 [15] Let (X, κ) be a digital image in \mathbb{Z}^n , $n \geq 3$ and $\hat{\kappa}$ be an adjacency relation for $\mathbb{Z}^n \setminus X$. Then X is called a closed κ -surface if it satisfies the following.

(1) In case that $(\kappa, \hat{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$, where the κ -adjacency is taken from Definition 2.1 with $\kappa \neq 3^n - 2^n - 1$, then

- (a) for each point $x \in X$, $|X|^x$ has exactly one κ -component κ -adjacent to x ;
- (b) $|\mathbb{Z}^n \setminus X|^x$ has exactly two $\hat{\kappa}$ -components $\hat{\kappa}$ -adjacent to x ; we denote by C^{xx} and D^{xx} these two components;
- (c) for any point $y \in N_{\kappa}(x) \cap X$, $N_{\hat{\kappa}}(y) \cap C^{xx} \neq \emptyset$ and $N_{\hat{\kappa}}(y) \cap D^{xx} \neq \emptyset$, where $N_{\kappa}(x)$ means the κ -neighbors of x .

Further, if a closed κ -surface X does not have a simple κ -point, then X is called simple.

(2) In case that $(\kappa, \hat{\kappa}) = (3^n - 2^n - 1, 2n)$, then

- (a) X is κ -connected;
- (b) for each point $x \in X$, $|X|^x$ is a generalized simple closed κ -curve.

Further, if the image $|X|^x$ is a simple closed κ -curve, then the closed κ -surface X is called

simple.

A point x of $\mathbb{Z}^n \setminus S_\kappa$ is said to be interior of S_κ if it belongs to a bounded $\hat{\kappa}$ -connected component of $\mathbb{Z}^n \setminus S_\kappa$. The set of all interior points of S_κ is denoted by $int(S_\kappa)$.

The 3-dimensional digital images MSS_6^* and MSS_{18}^* which are obtained from the minimal simple closed curves MSC_4 and MSC_8 in \mathbb{Z}^2 (see Figure 1), respectively, are essentially used in establishing the notion of a connected sum [15].

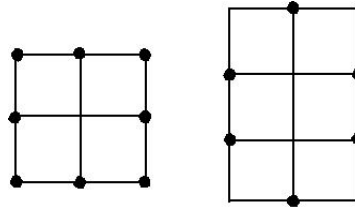


Figure 1. Minimal simple closed curves MSC_4 and MSC_8 [15]

- $MSS_6^* := MSS_6 \cup Int(MSS_6)$ where

$$MSS_6 \approx_{(6,6)} (MSC_4 \times [0, 2]_{\mathbb{Z}}) \cup (Int(MSC_4) \times \{0, 2\})$$

and MSC_4 is 4-isomorphic to the set

$$\{(1, 0), (1, 1), (0, 1), (-1, 1), (-1, 0), (-1, -1), (0, -1), (1, -1)\}.$$

- $MSS_{18}^* := MSS_{18} \cup Int(MSS_{18})$ where

$$MSS_{18} \approx_{(18,18)} (MSC_8 \times \{1\}) \cup (Int(MSC_8) \times \{0, 2\})$$

and MSC_8 is 8-isomorphic to the set

$$\{(0, 0), (-1, 1), (-2, 0), (-2, -1), (-1, -2), (0, -1)\}.$$

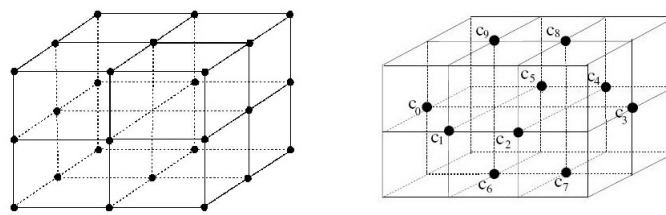


Figure 2. MSS_6 and MSS_{18} [15]

Definition 2.4 [15] Let S_{κ_0} be a closed κ_0 -surface in \mathbb{Z}^{n_0} and S_{κ_1} be a closed κ_1 -surface in \mathbb{Z}^{n_1} for $n_0, n_1 \geq 3$. Consider $A'_{\kappa_0} \subset A_{\kappa_0} \subset S_{\kappa_0}$ such that $A'_{\kappa_0} \approx_{(\kappa_0,8)} Int(MSC_8^*)$, $A'_{\kappa_0} \approx_{(\kappa_0,4)} Int(MSC_4^*)$ or $A'_{\kappa_0} \approx_{(\kappa_0,8)} Int(MSC_8^*)$. Let $f : A_{\kappa_0} \rightarrow f(A_{\kappa_0}) \subset S_{\kappa_1}$ be a (κ_0, κ_1) -isomorphism. Let $S'_{\kappa_i} = S_{\kappa_i} \setminus A'_{\kappa_i}$, $i \in \{0, 1\}$. Then the connected sum, denoted by $S_{\kappa_0} \# S_{\kappa_1}$, is the quotient space $S'_{\kappa_0} \cup S'_{\kappa_1} / \sim$, where $i : A_{\kappa_0} \setminus A'_{\kappa_0} \rightarrow S'_{\kappa_0}$ is the inclusion map and $i(x) \sim f(x)$ for $x \in A_{\kappa_0} \setminus A'_{\kappa_0}$.

3. Digital Homology and Cohomology Groups

In this section, we recall some notions about the digital simplicial homology and cohomology theory.

Definition 3.1 [20]. Let S be a set of nonempty subsets of a digital image (X, κ) . Let the following statements hold:

- If p and q are distinct points of $s \in S$, then p and q are κ -adjacent,
- If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

Then the members of S are called simplexes of (X, κ) .

An m -simplex is a simplex S such that $|S| = m + 1$. For a digital m -simplex P , if P' is a nonempty proper subset of P , then P' is called a *face* of P . Note that we use the same notation for both the digital image and the associated simplicial complex.

Definition 3.2 [1]. Let (X, κ) be a finite collection of digital m -simplices, $0 \leq m \leq d$ for some non-negative integer d . If the followings hold, then (X, κ) is called a finite digital simplicial complex:

- If P belongs to X , then every face of P also belongs to X ,
- If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q .

The dimension of a digital simplicial complex X is the largest integer m such that X has an m -simplex.

Definition 3.3 [1]. $C_q^\kappa(X)$ is a free abelian group with basis all digital (κ, q) -simplices in X where κ is the adjacency relation and q is the dimension of simplex in X .

Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m . Then for all $q > m$, $C_q^\kappa(X)$ is a trivial group. The homomorphism $\partial_q : C_q^\kappa(X) \rightarrow C_{q-1}^\kappa(X)$ defined by

$$\partial_q(\langle p_0, p_1, \dots, p_q \rangle) = \begin{cases} \sum_{i=0}^q (-1)^i \langle p_0, p_1, \dots, \hat{p}_i, \dots, p_q \rangle, & q \leq m \\ 0, & q > m \end{cases}$$

is called a boundary homomorphism, where \hat{p}_i means delete the point p_i .

For a digital image (X, κ) , a digital chain complex $C_*^\kappa(X)$ is a sequence of maps

$$\dots \xrightarrow{\partial_{i+1}} C_i^\kappa(X) \xrightarrow{\partial_i} C_{i-1}^\kappa(X) \xrightarrow{\partial_{i-1}} \dots$$

where $\partial_{i-1} \circ \partial_i = 0$. For all $1 \leq q \leq m$, we have $\partial_{q-1} \circ \partial_q = 0$. Arslan et al. [1] conclude that the sequence

$$C_*^\kappa(X) : 0 \xrightarrow{\partial_{m+1}} C_m^\kappa(X) \xrightarrow{\partial_m} C_{m-1}^\kappa(X) \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_1} C_0^\kappa(X) \xrightarrow{\partial_0} 0$$

is a digital chain complex. Let (X, κ) be a digital simplicial complex.

- $Z_q^\kappa(X) = \text{Ker } \partial_q$ is called the group of digital simplicial q -cycles.
- $B_q^\kappa(X) = \text{Im } \partial_{q+1}$ is called the group of digital simplicial q -boundaries.
- $H_q^\kappa(X) = Z_q^\kappa(X)/B_q^\kappa(X)$ is called the q th digital simplicial homology group.

Theorem 3.4 [9] Let (X, κ) be a digital simplicial complex of dimension m .

- (1) $H_q^\kappa(X)$ is a finitely generated abelian group for every $q \geq 0$.
- (2) $H_q^\kappa(X)$ is a trivial group for all $q > m$.
- (3) $H_q^\kappa(X)$ is a free abelian group, possibly zero.

Theorem 3.5 [9] Let

$$MSS_{18} = \{c_0 = (0, 0, 1), c_1 = (1, 1, 1), c_2 = (1, 2, 1), c_3 = (0, 3, 1), c_4 = (-1, 2, 1), \\ c_5 = (-1, 1, 1), c_6 = (0, 1, 0), c_7 = (0, 2, 0), c_8 = (0, 2, 2), c_9 = (0, 1, 2)\}.$$

Then its digital simplicial homology groups are

$$H_q^{18}(MSS_{18}) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}^3, & q = 1 \\ 0, & q \neq 0, 1. \end{cases}$$

Theorem 3.6 [10] Let MSS_6 be a minimal simple surface. Then its homology groups are

$$H_q^6(MSS_6) = \begin{cases} \mathbb{Z}, & q = 0; \\ \mathbb{Z}^{23}, & q = 1; \\ 0, & q \neq 0, 1. \end{cases}$$

Theorem 3.7 [10] Let $MSS_{18} \# MSS_{18}$ be a connected sum of a minimal simple surface MSS_{18} with itself (see Figure 3). Then we have

$$H_q^{18}(MSS_{18} \# MSS_{18}) = \begin{cases} \mathbb{Z}, & q = 0; \\ \mathbb{Z}^7, & q = 1; \\ 0, & q \geq 2. \end{cases}$$

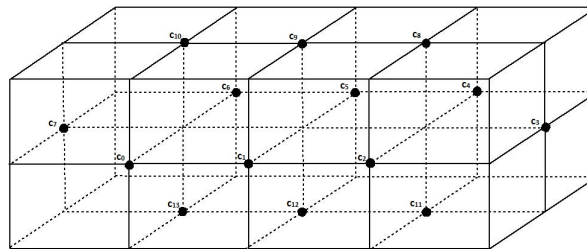


Figure 3. $MSS_{18} \# MSS_{18}$ [15]

Theorem 3.8 [10] The digital simplicial homology groups of $MSS_6 \# MSS_6$ (see Figure 4) are

$$H_q^6(MSS_6 \# MSS_6) = \begin{cases} \mathbb{Z}, & q = 0; \\ \mathbb{Z}^{39}, & q = 1; \\ 0, & q \neq 0, 1. \end{cases}$$

For a digital simplicial complex $(X, \kappa) \subset \mathbb{Z}^n$, the simplicial digital cochain complex $(C^*(X), \delta)$ is defined as follows. For any $q \in \mathbb{Z}$, the q -dimensional digital cochain group [20] is $C^{q,\kappa}(X) = Hom(C_q^\kappa(X), \mathbb{Z})$, where $Hom(G, B)$ is the functor assigning to any abelian group G the group of all homomorphisms from G to B , called the dual of G . Elements of $C^{q,\kappa}(X)$ are called digital cochains and denoted either by c^q or by c^* , if we don't need to specify their dimension q . The value of a digital cochain c^q on a chain d^q is denoted by $\langle c^q, d^q \rangle$. The q th coboundary map [20] $\delta^k : C^{q,\kappa}(X) \rightarrow C^{q+1,\kappa}(X)$ is the dual homomorphism of ∂_{q+1} defined by $\langle \delta^k c^q, d_{q+1} \rangle := \langle c^q, \partial_{q+1} d_{q+1} \rangle$.

Note that $C^{q,\kappa}(X)$ is the free abelian group generated by the dual canonical basis $\{Q^* \mid Q \in C_q^\kappa(X)\}$. Given a digital simplicial complex (X, κ) , the group of q -dimensional

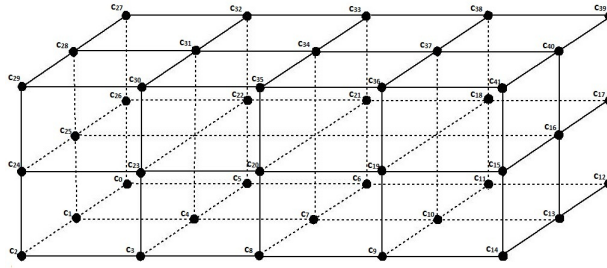


Figure 4. $MSS_6 \# MSS_6$ [15]

cocycles of (X, κ) is $Z^{q,\kappa}(X) := \text{Ker } \delta^q$, and the group of q -dimensional coboundaries of (X, κ) is $B^{q,\kappa} := \text{Im } \delta^{q-1}$. The q th *simplicial cohomology group* [20] of (X, κ) is

$$H^{q,\kappa}(X) := Z^{q,\kappa}(X) / B^{q,\kappa}(X).$$

For a digital simplicial map $f : (X, \kappa_0) \rightarrow (Y, \kappa_1)$ and $q \geq 0$, an induced homomorphism $f^\# : H_q^{\kappa_0}(X) \rightarrow H_q^{\kappa_1}(Y)$ is defined by $f^\#(z + B^{q,\kappa_0}(X)) = f(z) + B^{q,\kappa_1}(Y)$, where $z \in Z_q^{\kappa_0}(X)$.

Theorem 3.9 [11] Let X be a digital image with κ -adjacency. If $i : (X, \kappa) \rightarrow (X, \kappa)$ is the identity, then the induced homomorphism $i^\# : H^{\#, \kappa}(X) \rightarrow H^{\#, \kappa}(X)$ is the identity.

Theorem 3.10 [11] If (X, κ) is a one-point digital image, then

$$H^{q,\kappa}(X; G) = \begin{cases} G, & q = 0 \\ 0, & q > 0. \end{cases}$$

Theorem 3.11 [11] Let X, Y and Z be digital images with κ_0, κ_1 and κ_2 -adjacency, respectively. If $f : (X, \kappa_0) \rightarrow (Y, \kappa_1)$ and $g : (Y, \kappa_1) \rightarrow (Z, \kappa_2)$ are digitally continuous functions, then $(g \circ f)^\# = f^\# \circ g^\#$.

4. Main Results

In this section we will firstly give the basic properties of $Ext(A, B)$ below [20]. For any abelian groups A and B , $Ext(A, B)$ is also an abelian group. A sequence of two homomorphisms of abelian groups $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if $Im f = Ker g$. A short exact sequence of abelian groups, written $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence whose end groups are trivial. A short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is said to be split if g has a right inverse. A free abelian group is an abelian group which has a basis in that every element of the group could be written in one way as a finite linear combination of elements of the basis with integer coefficients.

For each abelian group A , choose a short exact sequence $0 \rightarrow R \xrightarrow{i} F \xrightarrow{j} A \rightarrow 0$ with F free abelian and i inclusion map. Since it is an exact sequence, i is injective and j is surjective. Moreover, the group R can be considered as a subgroup of F and A is isomorphic to F/R [20]. For any abelian group G , if we apply the contravariant functor $Hom(-, G)$ to the above exact sequence, then we obtain

$$0 \rightarrow Hom(A, G) \rightarrow Hom(F, G) \xrightarrow{i^\#} Hom(R, G) \rightarrow 0.$$

Thus $Ext(A, G)$ can be defined as

$$Ext(A, G) = \text{coker } i^\# = Hom(R, G)/i^\#(Hom(F, G)).$$

Corollary 4.1 [20] We have the followings:

- (1) If A is free abelian, then $Ext(A, B) = 0$ for any group B .
- (2) If B is divisible, then $Ext(A, B) = 0$ for any group A .
- (3) $Ext(\sum A_i, B) \cong \prod Ext(A_j, B)$.
- (4) $Ext(A, \prod B_j) \cong \prod Ext(A, B_j)$.
- (5) $Ext(\mathbb{Z}/n\mathbb{Z}, B) = B/nB$.

We are ready to state the universal coefficient theorem for cohomology of a digital simplicial complex. Note that the proof of the Theorem 4.2 is similar to the proof of the analogous statement in [20].

Theorem 4.2 Let (X, κ) be a digital simplicial complex. For any abelian group G , there is a short exact and split sequence

$$0 \rightarrow Ext(H_{q-1}^\kappa(X, \mathbb{Z}), G) \rightarrow H^{q, \kappa}(X, G) \rightarrow Hom(H_q^\kappa(X, \mathbb{Z}), G) \rightarrow 0,$$

hence

$$H^{q, \kappa}(X, G) \cong Hom(H_q^\kappa(X, \mathbb{Z}), G) \oplus Ext(H_{q-1}^\kappa(X, \mathbb{Z}), G).$$

Proof. Consider the following short exact sequence of digital simplicial complexes

$$0 \rightarrow Z \rightarrow X \rightarrow B \rightarrow 0 \quad (1)$$

where $Z_q^\kappa = Z_q^\kappa(X)$ and $B_q^\kappa = B_q^\kappa(X)$. Since B is free, the sequence (1) is split. Therefore there is an exact sequence

$$\dots \rightarrow H^{q, \kappa}(B, G) \rightarrow H^{q, \kappa}(X, G) \rightarrow H^{q, \kappa}(Z, G) \xrightarrow{\delta^\#} H^{q+1, \kappa}(B, G) \rightarrow \dots$$

Z and B have trivial boundary operators. Thus $H^{q, \kappa}(Z, G) = Hom(Z_q^\kappa(X), G)$, $H^{q, \kappa}(B, G) = Hom(B_{q-1}^\kappa(X), G)$ and $\delta^\# = Hom(\alpha_q, 1) : Hom(Z_q^\kappa(X), G) \rightarrow Hom(B_{q-1}^\kappa(X), G)$ where $\alpha_q : B_q^\kappa(X) \rightarrow Z_q^\kappa(X)$ is an inclusion map. So we have a short exact sequence

$$0 \rightarrow \text{coker } (Hom(\alpha_{q-1}, 1)) \rightarrow H^{q, \kappa}(X, G) \rightarrow Ker (Hom(\alpha_q, 1)) \rightarrow 0 \quad (2)$$

On the other hand, consider the following short exact sequence

$$0 \rightarrow B_q^\kappa(X) \xrightarrow{\alpha_q} Z_q^\kappa(X) \rightarrow H_q^\kappa(X) \rightarrow 0.$$

If we apply the functor $Hom(-, G)$ and use the Ext functor, we get an exact sequence

$$0 \rightarrow Hom(H_q^\kappa(X, \mathbb{Z}), G) \rightarrow Hom(Z_q^\kappa(X), G) \xrightarrow{Hom(\alpha_q, 1)} Hom(B_q^\kappa(X), G) \rightarrow Ext(H_q^\kappa(X, \mathbb{Z}), G) \rightarrow 0.$$

As a result, we obtain the followings:

$$\begin{aligned} Ker (Hom(\alpha_q, 1)) &\cong Hom(H_q^\kappa(X, \mathbb{Z}), G) \\ \text{coker } (Hom(\alpha_{q-1}, 1)) &\cong Ext(H_q^\kappa(X, \mathbb{Z}), G). \end{aligned}$$

If we substitute these in (2), we get the required short exact sequence

$$0 \rightarrow Ext(H_{q-1}^\kappa(X, \mathbb{Z}), G) \rightarrow H^{q,\kappa}(X, G) \rightarrow Hom(H_q^\kappa(X, \mathbb{Z}), G) \rightarrow 0.$$

This sequence is also split because the sequence in (1) is split. ■

Example 4.3 Let MSS_6 be as above. Then we have

$$H^{q,6}(MSS_6; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q = 0 \\ \mathbb{Z}_2^{23}, & q = 1 \\ 0, & q \neq 0, 1. \end{cases}$$

Proof. By Theorem 4.2, Theorem 3.6 and Corollary 4.1, we have the following for $q = 0$,

$$\begin{aligned} H^{0,6}(MSS_6; \mathbb{Z}_2) &\cong Hom(H_0^6(MSS_6, \mathbb{Z}), \mathbb{Z}_2) \oplus Ext(H_{-1}^6(MSS_6, \mathbb{Z}), \mathbb{Z}_2) \\ &\cong Hom(\mathbb{Z}, \mathbb{Z}_2) \oplus Ext(0, \mathbb{Z}_2) \cong \mathbb{Z}_2, \end{aligned}$$

for $q = 1$,

$$\begin{aligned} H^{1,6}(MSS_6; \mathbb{Z}_2) &\cong Hom(H_1^6(MSS_6, \mathbb{Z}), \mathbb{Z}_2) \oplus Ext(H_0^6(MSS_6, \mathbb{Z}), \mathbb{Z}_2) \\ &\cong Hom(\mathbb{Z}^{23}, \mathbb{Z}_2) \oplus Ext(\mathbb{Z}, \mathbb{Z}_2) \cong \mathbb{Z}_2^{23}, \end{aligned}$$

and for $q = 2$,

$$\begin{aligned} H^{2,6}(MSS_6; \mathbb{Z}_2) &\cong Hom(H_2^6(MSS_6, \mathbb{Z}), \mathbb{Z}_2) \oplus Ext(H_1^6(MSS_6, \mathbb{Z}), \mathbb{Z}_2) \\ &\cong Hom(0, \mathbb{Z}_2) \oplus Ext(\mathbb{Z}^{23}, \mathbb{Z}_2) \cong 0. \end{aligned}$$

For all $q > 2$, we have $H^{q,6}(MSS_6; \mathbb{Z}_2) \cong 0$. As a result, we get

$$H^{q,6}(MSS_6; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q = 0 \\ \mathbb{Z}_2^{23}, & q = 1 \\ 0, & q \neq 0, 1. \end{cases}$$

■

Example 4.4 The digital cohomology groups with the coefficient \mathbb{Z}_3 of minimal simple surface $MSS_6 \# MSS_6$ are

$$H^{q,6}(MSS_6 \# MSS_6; \mathbb{Z}_3) = \begin{cases} \mathbb{Z}_3, & q = 0 \\ \mathbb{Z}_3^{39}, & q = 1 \\ 0, & q \neq 0, 1. \end{cases}$$

Proof. By using Theorem 4.2, Theorem 3.8 and Corollary 4.1 (since 0 and \mathbb{Z} are free abelian groups, Ext groups are trivial), we compute for $q = 0$,

$$\begin{aligned} H^{0,6}(MSS_6 \# MSS_6; \mathbb{Z}_3) &\cong Hom(H_0^6(MSS_6 \# MSS_6, \mathbb{Z}), \mathbb{Z}_3) \\ &\quad \oplus Ext(H_{-1}^6(MSS_6 \# MSS_6, \mathbb{Z}), \mathbb{Z}_3) \\ &\cong Hom(\mathbb{Z}, \mathbb{Z}_3) \oplus Ext(0, \mathbb{Z}_3) \cong \mathbb{Z}_3, \end{aligned}$$

for $q = 1$,

$$\begin{aligned} H^{1,6}(MSS_6 \# MSS_6; \mathbb{Z}_3) &\cong Hom(H_1^6(MSS_6 \# MSS_6, \mathbb{Z}), \mathbb{Z}_3) \\ &\quad \oplus Ext(H_0^6(MSS_6 \# MSS_6, \mathbb{Z}), \mathbb{Z}_3) \\ &\cong Hom(\mathbb{Z}^{39}, \mathbb{Z}_3) \oplus Ext(\mathbb{Z}, \mathbb{Z}_3) \cong \mathbb{Z}_3^{39}, \end{aligned}$$

and for $q = 2$,

$$\begin{aligned} H^{2,6}(MSS_6 \# MSS_6; \mathbb{Z}_3) &\cong Hom(H_2^6(MSS_6 \# MSS_6, \mathbb{Z}), \mathbb{Z}_3) \\ &\quad \oplus Ext(H_1^6(MSS_6 \# MSS_6, \mathbb{Z}), \mathbb{Z}_3) \\ &\cong Hom(0, \mathbb{Z}_3) \oplus Ext(\mathbb{Z}^{39}, \mathbb{Z}_3) \cong 0. \end{aligned}$$

For all $q > 2$, it's clear that $H^{q,6}(MSS_6 \# MSS_6; \mathbb{Z}_3) \cong 0$. Therefore, we get the desired result. \blacksquare

Example 4.5 The digital cohomology groups with \mathbb{Z}_5 -coefficient of $MSS_{18} \# MSS_{18}$ are given as follows:

$$H^{q,18}(MSS_{18} \# MSS_{18}; \mathbb{Z}_5) = \begin{cases} \mathbb{Z}_5, & q = 0 \\ \mathbb{Z}_5^7, & q = 1 \\ 0, & q \neq 0, 1. \end{cases}$$

Proof. From Theorem 4.2, Theorem 3.7 and Corollary 4.1, we obtain for $q = 0$,

$$\begin{aligned} H^{0,18}(MSS_{18} \# MSS_{18}; \mathbb{Z}_5) &\cong Hom(H_0^{18}(MSS_{18} \# MSS_{18}, \mathbb{Z}), \mathbb{Z}_5) \\ &\quad \oplus Ext(H_{-1}^{18}(MSS_{18} \# MSS_{18}, \mathbb{Z}), \mathbb{Z}_5) \\ &\cong Hom(\mathbb{Z}, \mathbb{Z}_5) \oplus Ext(0, \mathbb{Z}_5) \cong \mathbb{Z}_5, \end{aligned}$$

for $q = 1$,

$$\begin{aligned} H^{1,18}(MSS_{18} \# MSS_{18}; \mathbb{Z}_5) &\cong Hom(H_1^{18}(MSS_{18} \# MSS_{18}, \mathbb{Z}), \mathbb{Z}_5) \\ &\quad \oplus Ext(H_0^{18}(MSS_{18} \# MSS_{18}, \mathbb{Z}), \mathbb{Z}_5) \\ &\cong Hom(\mathbb{Z}^7, \mathbb{Z}_5) \oplus Ext(\mathbb{Z}, \mathbb{Z}_5) \cong \mathbb{Z}_5^7, \end{aligned}$$

and for $q = 2$,

$$\begin{aligned} H^{2,18}(MSS_{18} \# MSS_{18}; \mathbb{Z}_5) &\cong Hom(H_2^{18}(MSS_{18} \# MSS_{18}, \mathbb{Z}), \mathbb{Z}_5) \\ &\quad \oplus Ext(H_1^{18}(MSS_{18} \# MSS_{18}, \mathbb{Z}), \mathbb{Z}_5) \\ &\cong Hom(0, \mathbb{Z}_5) \oplus Ext(\mathbb{Z}^7, \mathbb{Z}_5) \cong 0. \end{aligned}$$

For all $q > 2$, it's clear that $H^{q,18}(MSS_{18} \# MSS_{18}; \mathbb{Z}_5) \cong 0$. As a result, we conclude

that

$$H^{q,18}(MSS_{18}\#MSS_{18};\mathbb{Z}_5) = \begin{cases} \mathbb{Z}_5, & q = 0 \\ \mathbb{Z}_5^7, & q = 1 \\ 0, & q \neq 0, 1. \end{cases}$$

■

The boundary $Bd(I^{n+1})$ of an $(n + 1)$ -cube I^{n+1} is homeomorphic to n -sphere S^n . This allows us to represent a digital sphere by using the boundary of a digital cube. We use 0_n to denote the origin of \mathbb{Z}^n . Boxer [5] defines sphere-like digital image as follows:

$$S_n = [-1, 1]_{\mathbb{Z}}^{n+1} \setminus \{0_{n+1}\} \subset \mathbb{Z}^{n+1}.$$

For example, $S_1 = [-1, 1]_{\mathbb{Z}}^2 \setminus \{(0, 0)\}$ is digital 1-sphere with 4-adjacency in \mathbb{Z}^2 and $S_2 = [-1, 1]_{\mathbb{Z}}^3 \setminus \{(0, 0, 0)\}$ is digital 2-sphere with 6-adjacency in \mathbb{Z}^3 (see Figure 5). The following results are given in [1] and [10], respectively:

$$H_q^4(S_1) = \begin{cases} \mathbb{Z}, & q = 0, 1 \\ 0, & q \neq 0, 1, \end{cases} \quad H_q^6(S_2) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}^{23}, & q = 1 \\ 0, & q \neq 0, 1. \end{cases}$$

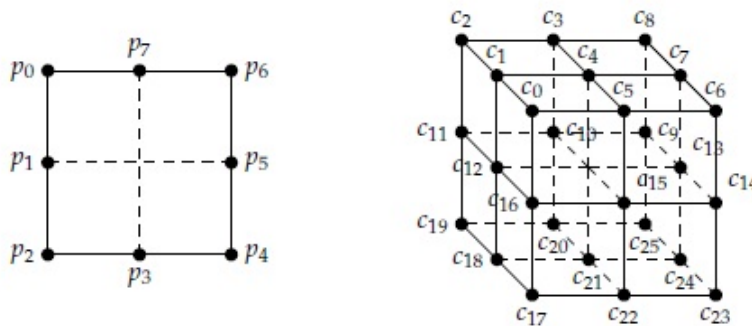


Figure 5. S_1 and S_2 [15]

Definition 4.6 Let $f : (S_n, \kappa_n) \rightarrow (S_n, \kappa_n)$ be a (κ_n, κ_n) -continuous map where (S_n, κ_n) is digital n -sphere, $n \in \{1, 2\}$ and $\kappa_1 = 4, \kappa_2 = 6$. Then f induces homomorphisms

$$f^\# : H^{\#, \kappa_n}(S_n) \rightarrow H^{\#, \kappa_n}(S_n).$$

We see that $f^\#$ must be of the form $f^\#([x]) = m[x]$, for some fixed $m \in \mathbb{Z}$, where $[x]$ is a generator of $H^{\#, \kappa_n}(S_n)$. This m is called the degree of f .

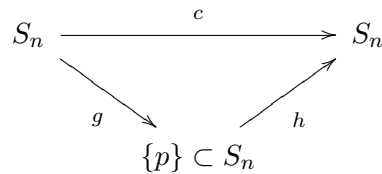
We now would like to deal with some degree properties of a map.

Theorem 4.7 Let $f : S_n \rightarrow S_n$ be a (κ_n, κ_n) -continuous map of a digital n -sphere where $n \in \{1, 2\}, \kappa_1 = 4$ and $\kappa_2 = 6$.

- (1) $\text{deg}(1_{S_n}) = 1$ where 1_{S_n} is the identity map on S_n .
- (2) $\text{deg}(c) = 0$, where c is a constant map on S_n .
- (3) $\text{deg}(f \circ g) = \text{deg}(f) \cdot \text{deg}(g)$, where $f, g : S_n \rightarrow S_n$.

Proof.

- (1) Let $1_{S_n} : S_n \rightarrow S_n$ be the identity map. Using the Theorem 3.9, we obtain that $1_{S_n}^\# : H^{\#, \kappa_n}(S_n) \rightarrow H^{\#, \kappa_n}(S_n)$ is also the identity map. By the definition of the degree, we conclude that $\deg(1_{S_n}) = 1$ because $1_{S_n}^\#([x]) = [x]$, where $[x]$ is a generator of $H^{\#, \kappa_n}(S_n)$.
- (2) Let c be a constant map on S_n such that $c(x) = p$ for all $x \in S_n$ where $p \in S_n$. We can factor c in the following way:



i.e., $c = h \circ g$. By Theorem 3.10, $H^{\#, \kappa_n}(\{p\}) = 0$. Therefore $c^\# = g^\# \circ h^\# = 0$, so $\deg(c) = 0$.

- (3) For two maps $f, g : S_n \rightarrow S_n$, consider the composition map $f \circ g : S_n \rightarrow S_n$. Using Theorem 3.11, we get

$$\begin{aligned}
 \deg(f \circ g)([x]) &= (f \circ g)^\#([x]) \\
 &= g^\#(f^\#([x])) \\
 &= g^\#(\deg(f) \cdot [x]) \\
 &= \deg(g) \cdot (\deg(f) \cdot [x]) \\
 &= (\deg(g) \cdot \deg(f))([x])
 \end{aligned}$$

for all $[x] \in H^{\#, \kappa_n}(S_n)$. So we have $\deg(f \circ g) = \deg(f) \cdot \deg(g)$. ■

Theorem 4.8 Let $f : S_1 \rightarrow S_1$ be a $(4, 4)$ -continuous map of a digital 1-sphere. If f is not surjective, then $\deg(f) = 0$.

Proof. If $a \notin \text{Im}(f)$ where $a \in S_1$, then f can be factored as $S_1 \xrightarrow{f_1} S_1 \setminus \{a\} \xrightarrow{i} S_1$. Since i is an inclusion map, $f_1(x) = f(x)$ for all $x \in S_1$. Taking digital cohomology groups of the above sequence, we have $H^{n,4}(S_1) \xrightarrow{f_1^\#} H^{n,4}(S_1 \setminus \{a\}) \xrightarrow{i^\#} H^{n,4}(S_1)$. Since $S_1 \setminus \{a\}$ is 4-contractible, $f_1^\# = 0$ and so, by Theorem 4.7, we conclude that $H^{n,4}(S_1 \setminus \{a\}) = 0$. ■

5. Conclusion

Digital topology with various applications will continue to be an important area of mathematics. Algebraic topology which has very effective tools such as homology and cohomology is used to solve some digital imaging problems. So we believe that the results of this paper will make a major contribution for developing digital topology.

Acknowledgments

We would like to express our gratitude to the anonymous referees for their helpful suggestions and corrections.

References

- [1] H. Arslan, I. Karaca, A. Oztel, Homology groups of n -dimensional digital images, XXI. Turkish National Mathematics Symposium, B (2008), 1-13.
- [2] G. Bertrand, Simple points, topological numbers and geodesic neighborhoods in cubic grids, *Pattern Recognit. Lett.* 15 (1994), 1003-1011.
- [3] G. Bertrand, R. Malgouyres, Some topological properties of discrete surfaces, *J. Math. Imaging Vis.* 11 (1999), 207-211.
- [4] L. Boxer, A classical construction for the digital fundamental group, *J. Math. Imaging Vis.* 10 (1999), 51-62.
- [5] L. Boxer, Digital products, wedges and covering spaces, *Appl. Math.* 1 (2010), 377-386.
- [6] L. Boxer, Digitally continuous functions, *Pattern Recognit. Lett.* 15 (1994), 833-839.
- [7] L. Boxer, Homotopy properties of sphere-like digital images, *J. Math. Imaging Vis.* 24 (2006), 167-175.
- [8] L. Boxer, Properties of digital homotopy, *J. Math. Imaging Vis.* 22 (2005), 19-26.
- [9] L. Boxer, I. Karaca, A. Oztel, Topological invariants in digital images, *J. Math. Sci. Adv. Appl.* 11 (2) (2011), 109-140.
- [10] E. U. Demir, I. Karaca, Simplicial homology groups of certain digital surfaces, *Hacet. J. Math. Stat.* 44 (5) (2015), 1011-1022.
- [11] O. Ege, I. Karaca, Cohomology theory for digital images, *Rom. J. Inf. Sci. Tech.* 16 (1) (2013), 10-28.
- [12] R. Gonzalez-Diaz, A. Ion, M. I. Ham, W. G. Kropatsch, Invariant representative cocycles of cohomology generators using irregular graph pyramids, *Comput. Vis. Image Underst.* 115 (7) (2011), 1011-1022.
- [13] R. Gonzalez-Diaz, P. Real, On the cohomology of 3D digital images, *Discrete Appl. Math.* 147 (2-3) (2005), 245-263.
- [14] S. E. Han, An extended digital (k_0, k_1) -continuity, *J. Appl. Math. Comput.* 16 (1-2) (2004), 445-452.
- [15] S. E. Han, Connected sum of digital closed surfaces, *Inform. Sci.* 176 (2006), 332-348.
- [16] G. T. Herman, Oriented surfaces in digital spaces, *CVGIP: Graphical Models and Image Processing* 55 (1993), 381-396.
- [17] T. Kaczynski, M. Mrozek, The cubical cohomology ring: an algorithmic approach, *Found. Comput. Math.* 13 (2013), 789-818.
- [18] I. Karaca, G. Burak, Simplicial relative cohomology rings of digital images, *Appl. Math. Inform. Sci.* 8 (5) (2014), 2375-2387.
- [19] A. Rosenfeld, *Digital topology*, *Amer. Math. Monthly* 86 (1979), 76-87.
- [20] E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.