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On new types of contraction mappings in bipolar metric spaces and applications

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Abstract. Our aim is to present some common fixed point theorems in bipolar metric spaces via certain contractive conditions. Some examples have been provided to illustrate the effectiveness of new results. At the end, we give two applications dealing with homotopy theory and integral equations.

Keywords: λ-admissible mapping, $\lambda-(\chi,\zeta)$-type contraction mapping, completeness and fixed point.

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1. Introduction and Preliminaries

Fixed point theory has been gained a vital role because of its wide applications in homotopy theory, integral, integro-differential and impulsive differential equations, obtaining solutions of optimization problems, approximation theory and nonlinear analysis.

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Further, many of fixed point theorems are used not only in various mathematical investigations, but also problems in economics, game theory, computer science and digital image processing. Due to its applications in mathematics and other related disciplines, Banach contraction principle has been generalized in many directions. Extensions of Banach contraction principle have been obtained either by generalizing the domain of the mapping or by extending the contractive condition on the mappings (see, [1, 2, 8–13, 16, 21–23]). Among them, Mutlu and Gürdal [18] initiated the notion of bipolar metric spaces and gave variant related (coupled) fixed point results for covariant and contravariant contractive mappings. See also ([14, 15, 19, 20]).

The notion of α-admissibility has been introduced by Samet et al. [25] and has been generalized by Salimi et al. [24]. For other related papers, see ([3–7, 17]) and references cited therein.

In what follows, we collect relevant definitions needed in our subsequent discussions.

Definition 1.1 [18] Let $\mathcal{P}$ and $\mathcal{Q}$ be a two non-empty sets. If the function $d : \mathcal{P} \times \mathcal{Q} \to [0, +\infty)$ verifies:

- (B_1) $d(p, q) = 0$ implies that $p = q$;
- (B_2) $p = q$ implies that $d(p, q) = 0$;
- (B_3) if $(p, q) \in (\mathcal{P}, \mathcal{Q})$, then $d(p, q) = d(q, p)$;
- (B_4) $d(p_1, q_2) \leqslant d(p_1, q_1) + d(p_2, q_1) + d(p_2, q_2)$,

for all $p, p_1, p_2 \in \mathcal{P}$ and $q, q_1, q_2 \in \mathcal{Q}$, then d is said to be a bipolar metric on $(\mathcal{P}, \mathcal{Q})$. Note that $(\mathcal{P}, \mathcal{Q}, d)$ is said to be a bipolar metric space.

***Example* 1.2** [18] Let $A = (1, +\infty)$ and $B = [-1, 1]$. Define $d : A \times B \to [0, +\infty)$ as $d(a, b) = |a^2 - b^2|$, for all $(a, b) \in (A, B)$. Then the triple (A, B, d) is a bipolar metric space.

***Example* 1.3** [18] Let $A = \{f \mid f : \mathbb{R} \to [1, 3]\}$ be the set of all functions and $B = \mathbb{R}$. Define $d : A \times B \to [0, +\infty)$ as $d(f, a) = f(a)$, for all $(f, a) \in (A, B)$. Then the triple (A, B, d) is a disjoint bipolar metric space.

Definition 1.4 [18] Let $(\mathcal{P}_1, \mathcal{Q}_1)$ and $(\mathcal{P}_2, \mathcal{Q}_2)$ be two pairs of sets. Given $S : \mathcal{P}_1 \cup \mathcal{Q}_1 \to \mathcal{P}_2 \cup \mathcal{Q}_2$ is called

- (i) covariant if $S(\mathcal{P}_1) \subseteq \mathcal{P}_2$ and $S(\mathcal{Q}_1) \subseteq \mathcal{Q}_2$. This is denoted as $S : (\mathcal{P}_1, \mathcal{Q}_1) \rightrightarrows (\mathcal{P}_2, \mathcal{Q}_2)$;
- (ii) contravariant if $S(\mathcal{P}_1) \subseteq \mathcal{Q}_2$ and $S(\mathcal{Q}_1) \subseteq \mathcal{P}_2$. It is denoted as $S : (\mathcal{P}_1, \mathcal{Q}_1) \leftrightarrows (\mathcal{P}_2, \mathcal{Q}_2)$.

Particularly, if d_1 and d_2 are bipolar metrics on $(\mathcal{P}_1, \mathcal{Q}_1)$ and $(\mathcal{P}_2, \mathcal{Q}_2)$, respectively, we often write $S : (\mathcal{P}_1, \mathcal{Q}_1, d_1) \rightrightarrows (\mathcal{P}_2, \mathcal{Q}_2, d_2)$ and $S : (\mathcal{P}_1, \mathcal{Q}_1, d_1) \leftrightarrows (\mathcal{P}_2, \mathcal{Q}_2, d_2)$.

Definition 1.5 [18] Given a bipolar metric space $(\mathcal{P}, \mathcal{Q}, d)$ and $\xi \in \mathcal{P} \cup \mathcal{Q}$.

- (i) Such ξ is a left point if $\xi \in \mathcal{P}$;
- (ii) Such ξ is a right point if $\xi \in \mathcal{Q}$;
- (iii) Such ξ is a central point if it is both left and right.

Also, $\{p_n\}$ in $\mathcal{P}$ is a left sequence. $\{q_n\}$ in $\mathcal{Q}$ is a right sequence. In a bipolar metric space, we call a sequence, a left or a right one. A sequence $\{u_n\}$ is said to be convergent to u iff either $\{u_n\}$ is a left sequence, u is a right point and $\lim\limits_{n\to\infty} d(u_n, u) = 0$, or $\{u_n\}$ is a right sequence, u is a left point and $\lim\limits_{n\to\infty} d(u, u_n) = 0$. The bisequence $(\{p_n\}, \{q_n\})$ on $(\mathcal{P}, \mathcal{Q}, d)$ is a sequence on $\mathcal{P} \times \mathcal{Q}$. In the case where $\{p_n\}$ and $\{q_n\}$ are both convergent, then

$(\{p_n\},\{q_n\})$ is convergent. $(\{p_n\},\{q_n\})$ is a Cauchy bisequence if $\lim_{n,m\to\infty} d(p_n,q_m)=0$.

Note that every convergent Cauchy bisequence is biconvergent. The bipolar metric space is complete, if each Cauchy bisequence is convergent (and so it is biconvergent).

Definition 1.6 [14] Let $S,T:(\mathcal{P},\mathcal{Q})\rightrightarrows(\mathcal{P},\mathcal{Q})$ be two covariant mappings on a bipolar metric space $(\mathcal{P},\mathcal{Q},d)$. The pair $\{S,T\}$ is compatible iff $\lim_{n\to\infty} d(STp_n,TSq_n)=\lim_{n\to\infty} d(TSp_n,STq_n)=0$, whenever $(\{p_n\},\{q_n\})$ is a sequence in $(\mathcal{P},\mathcal{Q})$ so that

$$\lim_{n\to\infty} Sp_n=\lim_{n\to\infty} Sq_n=\lim_{n\to\infty} Tp_n=\lim_{n\to\infty} Tq_n=\tau,$$

for some $\tau\in\mathcal{P}\cup\mathcal{Q}$.

2. Common fixed points for single-valued admissible mappings

Let Ω be the set of increasing continuous functions $\chi:[0,+\infty)\to[0,+\infty)$. Denote by Υ be the collection of lower-semicontinuous functions $\zeta:[0,+\infty)\to[0,+\infty)$ so that $\zeta(\nu)=0$ iff $\nu=0$.

Definition 2.1 Let $F:\mathcal{P}\cup\mathcal{Q}\to\mathcal{P}\cup\mathcal{Q}$ be a covariant mapping and given $\lambda:\mathcal{P}\cup\mathcal{Q}\to[0,+\infty)$. Such F is called λ-admissible, if $\xi\in\mathcal{P}\cup\mathcal{Q}$ with $\lambda(\xi)\geqslant 1$, implies $\lambda(F\xi)\geqslant 1$.

Theorem 2.2 Let $S,T:(\mathcal{P},\mathcal{Q})\rightrightarrows(\mathcal{P},\mathcal{Q})$ be λ-admissible mappings on a complete bipolar metric space $(\mathcal{P},\mathcal{Q},d)$ so that $S(\mathcal{P}\cup\mathcal{Q})\subseteq T(\mathcal{P}\cup\mathcal{Q})$. Assume that the following assertions hold:

- (i) there exists $p_0\in\mathcal{P}\cup\mathcal{Q}$ so that $\lambda(p_0)\geqslant 1$;
- (ii) either T is continuous, or;
- (iii) if $(\{a_n\},\{b_n\})$ is a bisequence in $(\mathcal{P},\mathcal{Q})$ so that $(a_n,b_n)\to(\kappa,\kappa)$ with $\lambda(a_n)\geqslant 1,\lambda(b_n)\geqslant 1$ for each n, then $\lambda(\kappa)\geqslant 1$;
- (iv) $\{S,T\}$ is compatible;
- (v) $\lambda(p)\lambda(q)\geqslant 1\Rightarrow\chi(d(Sp,Sq))\leqslant\chi(d(Tp,Tq))-\zeta(d(Tp,Tq))$ for all $p\in\mathcal{P},q\in\mathcal{Q}$ where $\chi\in\Omega$ and $\zeta\in\Upsilon$.

Then $S,T:\mathcal{P}\cup\mathcal{Q}\to\mathcal{P}\cup\mathcal{Q}$ have a common fixed point. Moreover, if $\lambda(p)\geqslant 1$ and $\lambda(q)\geqslant 1$ for all $p,q\in\mathcal{P}\cup\mathcal{Q}$ are fixed points of S and T, then such common fixed point is unique.

Proof. Let $p_0\in\mathcal{P}$ and $q_0\in\mathcal{Q}$. As $S(\mathcal{P}\cup\mathcal{Q})\subseteq T(\mathcal{P}\cup\mathcal{Q})$, there is $p_1\in\mathcal{P}$ and $q_1\in\mathcal{Q}$ such that $Sp_0=Tp_1$ and $Sq_0=Tq_1$. Continuing in same process, we get p_n, p_{n+1} in $\mathcal{P}$ and q_n, q_{n+1} in $\mathcal{Q}$ in order that $Sp_n=Tp_{n+1}$ and $Sq_n=Tq_{n+1}$. Define the bisequence $(\{\omega_n\},\{\xi_n\})$ in $(\mathcal{P},\mathcal{Q})$ as $\omega_n=Sp_n=Tp_{n+1}$ and $\xi_n=Sq_n=Tq_{n+1}$ for $n\geqslant 0$.
Since S and T are λ-admissible mappings and $\lambda(p_0)\geqslant 1$, one has $\lambda(\omega_0)=\lambda(Sp_0)=\lambda(Tp_1)\geqslant 1$ and since $\lambda(q_0)\geqslant 1$, then $\lambda(\xi_0)=\lambda(Sq_0)=\lambda(Tq_1)\geqslant 1$. By continuing this process, we get that $\lambda(\omega_n)\geqslant 1$, $\lambda(\xi_n)\geqslant 1$ for all $n\in N\cup\{0\}$. Equivalently, $\lambda(\omega_n)\lambda(\xi_{n-1})\geqslant 1$, $\lambda(\omega_{n-1})\lambda(\xi_n)\geqslant 1$ and $\lambda(\omega_n)\lambda(\xi_n)\geqslant 1$ for all $n\geqslant 1$. By using the condition (v), we get

$$\begin{aligned}\chi(d(\omega_n,\xi_{n+1}))&=\chi(d(Sp_n,Sq_{n+1}))\\&\leqslant\chi(d(Tp_n,Tq_{n+1}))-\zeta(d(Tp_n,Tq_{n+1}))\\&=\chi(d(\omega_{n-1},\xi_n))-\zeta(d(\omega_{n-1},\xi_n))\\&\leqslant\chi(d(\omega_{n-1},\xi_n)),\end{aligned}\tag{1}$$

and thus

$$d\left(\omega_n, \xi_{n+1}\right) \leqslant d\left(\omega_{n-1}, \xi_n\right). \tag{2}$$

Also, one writes

$$\begin{aligned}
\chi\left(d\left(\omega_{n+1}, \xi_n\right)\right) &= \chi\left(d\left(Sp_{n+1}, Sq_n\right)\right) \\
&\leqslant \chi\left(d\left(Tp_{n+1}, Tq_n\right)\right) - \zeta\left(d\left(Tp_{n+1}, Tq_n\right)\right) \\
&= \chi\left(d\left(\omega_n, \xi_{n-1}\right)\right) - \zeta\left(d\left(\omega_n, \xi_{n-1}\right)\right) \\
&\leqslant \chi\left(d\left(\omega_n, \xi_{n-1}\right)\right). \qquad (3)
\end{aligned}$$

We deduce that

$$d\left(\omega_{n+1}, \xi_n\right) \leqslant d\left(\omega_n, \xi_{n-1}\right). \tag{4}$$

Moreover,

$$\begin{aligned}
\chi\left(d\left(\omega_n, \xi_n\right)\right) &= \chi\left(d\left(Sp_n, Sq_n\right)\right) \\
&\leqslant \chi\left(d\left(Tp_n, Tq_n\right)\right) - \zeta\left(d\left(Tp_n, Tq_n\right)\right) \\
&= \chi\left(d\left(\omega_{n-1}, \xi_{n-1}\right)\right) - \zeta\left(d\left(\omega_{n-1}, \xi_{n-1}\right)\right) \\
&\leqslant \chi\left(d\left(\omega_{n-1}, \xi_{n-1}\right)\right). \qquad (5)
\end{aligned}$$

Consequently,

$$d\left(\omega_n, \xi_n\right) \leqslant d\left(\omega_{n-1}, \xi_{n-1}\right). \tag{6}$$

Combining (2), (4) and (6) yields that the bisequence $(\{\omega_n\}, \{\xi_n\})$ is non-increasing, so it biconverges to $\delta \geqslant 0$. When $n \to \infty$ in equations (1), (3) and (5), we get

$$\chi(\delta) \leqslant \chi(\delta) - \zeta(\delta),$$

that is, $\zeta(\delta) = 0$, so $\delta = 0$. Therefore,

$$\lim_{n\to\infty} d(\omega_n, \xi_{n+1}) = 0. \tag{7}$$

Now, we shall show $(\{\omega_n\}, \{\xi_n\})$ is a Cauchy bisequence. Suppose there is $\epsilon > 0$, for which there are $\{\omega_{n_k}\}$, $\{\omega_{m_k}\}$ of $\{\omega_n\}$ and $\{\xi_{n_k}\}$, $\{\xi_{m_k}\}$ of $\{\xi_n\}$ with $n_k > m_k \geqslant k$ so that

$$\begin{aligned}
&d\left(\omega_{n_k}, \xi_{m_k}\right) \geqslant \epsilon, \\
&d\left(\omega_{n_k-1}, \xi_{m_k}\right) < \epsilon, \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
&d\left(\omega_{m_k}, \xi_{n_k}\right) \geqslant \epsilon, \\
&d\left(\omega_{m_k}, \xi_{n_k-1}\right) < \epsilon. \qquad (9)
\end{aligned}$$

By view of (8) and triangle inequality, we get

$$\begin{aligned}\epsilon &\leqslant d\left(\omega_{n_k}, \xi_{m_k}\right) \\ &\leqslant d\left(\omega_{n_k}, \xi_{n_k-1}\right) + d\left(\omega_{n_k-1}, \xi_{n_k-1}\right) + d\left(\omega_{n_k-1}, \xi_{m_k}\right) \\ &< d\left(\omega_{n_k}, \xi_{n_k-1}\right) + d\left(\omega_{n_k-1}, \xi_{n_k-1}\right) + \epsilon.\end{aligned}$$

Letting $k \to \infty$ and using (7),

$$\lim_{k\to\infty} d\left(\omega_{n_k}, \xi_{m_k}\right) = \epsilon. \tag{10}$$

Again, by means of triangle inequality, we have

$$d\left(\omega_{n_k}, \xi_{m_k}\right) \leqslant d\left(\omega_{n_k}, \omega_{n_k+1}\right) + d\left(\xi_{m_k+1}, \omega_{n_k+1}\right) + d\left(\xi_{m_k+1}, \xi_{m_k}\right),$$

and

$$d\left(\xi_{m_k+1}, \omega_{n_k+1}\right) \leqslant d\left(\xi_{m_k+1}, \xi_{m_k}\right) + d\left(\omega_{n_k}, \xi_{m_k}\right) + d\left(\omega_{n_k}, \omega_{n_k+1}\right).$$

Taking $k \to \infty$ and using (7) and (10),

$$\lim_{k\to\infty} d\left(\omega_{n_k+1}, \xi_{m_k+1}\right) = \lim_{k\to\infty} d\left(\xi_{m_k+1}, \omega_{n_k+1}\right) = \epsilon. \tag{11}$$

Similarly, using (9), we can prove

$$\lim_{k\to\infty} d\left(\omega_{m_k}, \xi_{n_k}\right) = \epsilon, \qquad \lim_{k\to\infty} d\left(\omega_{m_k+1}, \xi_{n_k+1}\right) = \epsilon. \tag{12}$$

Since $\lambda(\omega_{n_k})\lambda(\omega_{m_k}) \geqslant 1$ for all $k \in N$, by (v), we get

$$\chi\left(d\left(\omega_{n_k+1}, \xi_{m_k+1}\right)\right) \leqslant \chi\left(d\left(\omega_{n_k}, \xi_{m_k}\right)\right) - \zeta\left(d\left(\omega_{n_k}, \xi_{m_k}\right)\right), \tag{13}$$

and

$$\chi\left(d\left(\omega_{m_k+1}, \xi_{n_k+1}\right)\right) \leqslant \chi\left(d\left(\omega_{m_k}, \xi_{n_k}\right)\right) - \zeta\left(d\left(\omega_{m_k}, \xi_{n_k}\right)\right). \tag{14}$$

Taking the limsup on (13), (14) and applying (10), (11) and (12), we have $\chi(\epsilon) \leqslant \chi(\epsilon) - \zeta(\epsilon)$. That is, $\epsilon = 0$, which is a contradiction. Hence $(\{\omega_n\}, \{\xi_n\})$ is a Cauchy bisequence in $(\mathcal{P}, \mathcal{Q})$. Therefore,

$$\lim_{n,m\to\infty} (\omega_n, \xi_m) = 0.$$

Since $(\mathcal{P}, \mathcal{Q}, d)$ is complete, (ω_n, ξ_n) converges. So it biconverges to some $\kappa \in \mathcal{P} \cap \mathcal{Q}$ so that

$$\lim_{n\to\infty} \omega_{n+1} = \kappa = \lim_{n\to\infty} \xi_{n+1}. \tag{15}$$

That is,

$$\lim_{n\to\infty} Sp_{n+1} = \lim_{n\to\infty} Tp_{n+2} = \lim_{n\to\infty} Sq_{n+1} = \lim_{n\to\infty} Tq_{n+2} = \kappa.$$

The continuity of T leads to

$$\lim_{n\to\infty} T^2 p_{n+2} = T\kappa,\ \lim_{n\to\infty} TSp_{n+1} = T\kappa,\ \lim_{n\to\infty} T^2 q_{n+2} = T\kappa,\ \lim_{n\to\infty} TSq_{n+1} = T\kappa.$$

The compatibility of $\{S, T\}$ implies that

$$\lim_{n\to\infty} d(STp_{n+2}, TSq_{n+1}) = \lim_{n\to\infty} d(TSp_{n+1}, STq_{n+2}) = 0.$$

So that $\lim_{n\to\infty} TSq_{n+1} = \lim_{n\to\infty} STp_{n+2} = T\kappa$, $\lim_{n\to\infty} STq_{n+2} = \lim_{n\to\infty} TSp_{n+1} = T\kappa$.
Choosing $p = Tp_{2n+2}$ and $q = q_{2n+1}$ in (v) and assuming that (iii) holds, that is, $\lambda(Tp_{n+2})\lambda(\kappa) \geqslant 1$, we have

$$\chi\left(d(STp_{n+2}, Sp_{n+1})\right) \leqslant \chi\left(d(TTp_{n+2}, Tq_{n+1})\right) - \zeta\left(d(TTp_{n+2}, Tq_{n+1})\right).$$

At the limit,

$$\chi\left(d(T\kappa, \kappa)\right) \leqslant \chi\left(d(T\kappa, \kappa)\right) - \zeta\left(d(T\kappa, \kappa)\right).$$

Hence, $\zeta\left(d(T\kappa, \kappa)\right) = 0$ and so $T\kappa = \kappa$.
By using conditions (v) and (iii), we deduce

$$\begin{aligned}\chi\left(d\left(S\kappa, \xi_{n+1}\right)\right) &= \chi\left(d\left(S\kappa, Sq_{n+1}\right)\right)\\ &\leqslant \chi\left(d\left(T\kappa, Tq_{n+1}\right)\right) - \zeta\left(d\left(T\kappa, Tq_{n+1}\right)\right).\end{aligned}$$

At the limit,

$$\chi\left(d(S\kappa, \kappa)\right) \leqslant \chi\left(d(T\kappa, \kappa)\right) - \zeta\left(d(T\kappa, \kappa)\right) \leqslant \chi\left(d(T\kappa, \kappa)\right).$$

Thus, $d(S\kappa, \kappa) \leqslant d(T\kappa, \kappa) = 0$. That is, $d(S\kappa, \kappa) = 0$ implies $S\kappa = \kappa$. Hence $S\kappa = T\kappa = \kappa$.
Now, let ν be so that $S\nu = T\nu = \nu$. Then $\nu \in \mathcal{P} \cap \mathcal{Q}$. Since $\lambda(\kappa)\lambda(\nu) \geqslant 1$, by (v), one has

$$\begin{aligned}\chi\left(d(\kappa, \nu)\right) &= \chi\left(d(S\kappa, S\nu)\right)\\ &\leqslant \chi\left(d(T\kappa, T\nu)\right) - \zeta\left(d(T\kappa, T\nu)\right)\\ &\leqslant \chi\left(d(\kappa, \nu)\right) - \zeta\left(d(\kappa, \nu)\right).\end{aligned}$$

Hence, $\zeta\left(d(\kappa, \nu)\right) = 0$, and so $\kappa = \nu$. That is, we get uniqueness. ■

Corollary 2.3 Let $S : (\mathcal{P}, \mathcal{Q}) \rightrightarrows (\mathcal{P}, \mathcal{Q})$ be an λ-admissible mapping on a complete bipolar metric space $(\mathcal{P}, \mathcal{Q}, d)$. Suppose that the following assertions hold:

- (i) there exists $p_0 \in \mathcal{P} \cup \mathcal{Q}$ so that $\lambda(p_0) \geqslant 1$;
- (ii) either S is continuous, or;
- (iii) if $(\{p_n\}, \{q_n\})$ is a bisequence in $(\mathcal{P}, \mathcal{Q})$ so that $(p_n, q_n) \to (\kappa, \kappa)$ and $\lambda(p_n) \geqslant 1, \lambda(q_n) \geqslant 1$ for all n, then $\lambda(\kappa) \geqslant 1$;
- (iv) $\lambda(p)\lambda(q) \geqslant 1 \Rightarrow \chi\left(d(Sp, Sq)\right) \leqslant \chi\left(d(p, q)\right) - \zeta\left(d(p, q)\right)$ for all $p \in \mathcal{P}, q \in \mathcal{Q}$ where $\chi \in \Omega$ and $\zeta \in \Upsilon$.

Then $S : \mathcal{P} \cup \mathcal{Q} \to \mathcal{P} \cup \mathcal{Q}$ has a fixed point. Moreover, if $\lambda(p) \geqslant 1$ and $\lambda(q) \geqslant 1$ for all $p, q \in \mathcal{P} \cup \mathcal{Q}$ are fixed points of S, then S has a unique fixed point.

Proof. Let us take $T = I_{\mathcal{P}\cup\mathcal{Q}}$ (identity mapping on $\mathcal{P}\cup\mathcal{Q}$), from Theorem 2.2, S has a unique fixed point. ■

***Example* 2.4** Let $U_m(\mathbb{R})$ (resp. $L_m(\mathbb{R})$) be the set of all $m\times m$ real upper (resp. lower) triangular matrices. Define $d : U_m(\mathbb{R})\times L_m(\mathbb{R})\to[0,\infty)$ as $d(A,B)=\sum_{i,j=1}^{m}|a_{ij}-b_{ij}|$ for all $A=(a_{ij})\in U_m(\mathbb{R})$ and $B=(b_{ij})\in L_m(\mathbb{R})$. Then obviously $(U_m(\mathbb{R}), L_m(\mathbb{R}), d)$ is a complete bipolar metric space. Define $S,T : U_m(\mathbb{R})\cup L_m(\mathbb{R})\to U_m(\mathbb{R})\cup L_m(\mathbb{R})$ as $S(A)=\frac{1}{4}(a_{ij})$ and $T(A)=\frac{1}{2}(a_{ij})$ for all $A=(a_{ij})\in U_m(\mathbb{R})\cup L_m(\mathbb{R})$. Given λ : $U_m(\mathbb{R})\cup L_m(\mathbb{R})\to[0,\infty)$ as

$$\lambda(A)=\begin{cases}\sum_{i,j=1}^{m}|a_{ij}|=1, & \text{if } A=(a_{ij})\in U_m(\mathbb{R})\cup L_m(\mathbb{R}),\\ 0, & \text{otherwise.}\end{cases}$$

Choose $\chi(t)=t$ and $\zeta(t)=\frac{t}{3}$.
Here, obviously $S(U_m(\mathbb{R})\cup L_m(\mathbb{R}))=T(U_m(\mathbb{R})\cup L_m(\mathbb{R}))=O_{m\times m}$. Furthermore, we prove $\{S,T\}$ is compatible. Let (A_n,B_n) be a bisequence in $(\mathcal{P},\mathcal{Q})$ so that for some $\kappa=(\kappa_{ij})\in\mathcal{P}\cap\mathcal{Q}$, $\lim_{n\to\infty}d(TA_n,\kappa)=0$, $\lim_{n\to\infty}d(\kappa,TB_n)=0$ and $\lim_{n\to\infty}d(SA_n,\kappa)=0$ and $\lim_{n\to\infty}d(\kappa,SB_n)=0$. Since T and S are continuous, we have

$$\begin{aligned}\lim_{n\to\infty}d(TSA_n,STB_n)&=d(\lim_{n\to\infty}TSA_n,\lim_{n\to\infty}STB_n)=d(T\kappa,S\kappa)\\&=d(\tfrac{1}{2}(\kappa_{ij}),\tfrac{1}{4}(\kappa_{ij}))\\&=\sum_{i,j=1}^{m}|\tfrac{1}{2}\kappa_{ij}-\tfrac{1}{4}\kappa_{ij}|=\sum_{i,j=1}^{m}\tfrac{1}{4}|\kappa_{ij}|.\end{aligned}$$

But $\sum_{i,j=1}^{m}\frac{1}{4}|\kappa_{ij}|=0\Leftrightarrow\kappa_{ij}=0$. Similarly, $\lim_{n\to\infty}d(STA_n,TSB_n)=0$. That is, $\{S,T\}$ is compatible.
Let $\lambda(A)\geqslant 1$, then $A=(a_{ij})\in U_m(\mathbb{R})\cup L_m(\mathbb{R})$. Also, $S(A)\in U_m(\mathbb{R})\cup L_m(\mathbb{R})$. Therefore, $\lambda(S(A))\geqslant 1$. Then S is λ-admissible (and the same for T). Now, let (A_n,B_n) be in $(\mathcal{P},\mathcal{Q})$ so that $\lambda(A_n)\geqslant 1$ and $\lambda(B_n)\geqslant 1$ and $(A_n,B_n)\to(\kappa,\kappa)$ as $n\to\infty$. Therefore, $\kappa\in\mathcal{P}\cap\mathcal{Q}$, i.e., $\lambda(\kappa)\geqslant 1$.
Let $\lambda(A)\lambda(B)\geqslant 1$, then $A=(a_{ij})\in U_m(\mathbb{R})$ and $B=(b_{ij})\in L_m(\mathbb{R})$ and so

$$\begin{aligned}\chi\left(d(SA,SB)\right)=d(SA,SB)&=\frac{1}{4}\sum_{i,j=1}^{m}|a_{ij}-b_{ij}|\\&\leqslant\frac{1}{3}\sum_{i,j=1}^{m}|a_{ij}-b_{ij}|\\&\leqslant\frac{1}{2}\sum_{i,j=1}^{m}|a_{ij}-b_{ij}|-\frac{1}{6}\sum_{i,j=1}^{m}|a_{ij}-b_{ij}|\\&\leqslant\chi\left(d(TA,TB)\right)-\zeta\left(d(TA,TB)\right).\end{aligned}$$

Thus, all conditions of Theorem 2.2 hold, and $O_{m\times m}$ is the unique common fixed point of S and T.

Theorem 2.5 Let $S, T : (\mathcal{P}, \mathcal{Q}) \rightrightarrows (\mathcal{P}, \mathcal{Q})$ be λ-admissible mappings on a complete bipolar metric space $(\mathcal{P}, \mathcal{Q}, d)$ so that $S(\mathcal{P} \cup \mathcal{Q}) \subseteq T(\mathcal{P} \cup \mathcal{Q})$. Suppose that the following assertions hold:

(*i*) there exists $p_0 \in \mathcal{P} \cup \mathcal{Q}$ such that $\lambda(p_0) \geqslant 1$;
(*ii*) either T is continuous, or;
(*iii*) if $(\{p_n\}, \{q_n\})$ is a bisequence in $(\mathcal{P}, \mathcal{Q})$ such that $(p_n, q_n) \to (\kappa, \kappa)$ and $\lambda(p_n) \geqslant 1, \lambda(q_n) \geqslant 1$ for all n, then $\lambda(\kappa) \geqslant 1$;
(*iv*) the pair $\{S, T\}$ is compatible;
(*v*) $\lambda(p)\lambda(q)\chi(d(Sa, Sb)) \leqslant \chi(d(Ta, Tb)) - \zeta(d(Ta, Tb))$ for all $p \in \mathcal{P}, q \in \mathcal{Q}$ where $\chi \in \Omega$ and $\zeta \in \Upsilon$.

Then $S, T : \mathcal{P} \cup \mathcal{Q} \to \mathcal{P} \cup \mathcal{Q}$ have a common fixed point. Moreover, if $\lambda(p) \geqslant 1$ and $\lambda(q) \geqslant 1$ for all $p, q \in \mathcal{P} \cup \mathcal{Q}$ are fixed points of S and T, then S and T have a unique common fixed point.

Proof. Let $\lambda(p)\lambda(q) \geqslant 1$ for $p \in \mathcal{P}, q \in \mathcal{Q}$. Then by (v), we get

$$\chi(d(Sp, Sq)) \leqslant \chi(d(Tp, Tq)) - \zeta(d(Tp, Tq)).$$

Thus, the condition (v) of Theorem 2.2 holds. From Theorem 2.2, we get the proof. ■

Corollary 2.6 Let $S : (\mathcal{P}, \mathcal{Q}) \rightrightarrows (\mathcal{P}, \mathcal{Q})$ be an λ-admissible mapping on a complete bipolar metric space $(\mathcal{P}, \mathcal{Q}, d)$. Suppose that the following assertions hold:

(*i*) there exists $p_0 \in \mathcal{P} \cup \mathcal{Q}$ such that $\lambda(p_0) \geqslant 1$;
(*ii*) either S is continuous; or
(*iii*) if $(\{p_n\}, \{q_n\})$ is a bisequence in $(\mathcal{P}, \mathcal{Q})$ so that $(p_n, q_n) \to (\kappa, \kappa)$ and $\lambda(p_n) \geqslant 1, \lambda(q_n) \geqslant 1$ for all n, then $\lambda(\kappa) \geqslant 1$,
(*iv*) $\lambda(p)\lambda(q)\chi(d(Sp, Sq)) \leqslant \chi(d(p, q)) - \zeta(d(p, q))$ for all $p \in \mathcal{P}, q \in \mathcal{Q}$ where $\chi \in \Omega$ and $\zeta \in \Upsilon$.

Then $S : \mathcal{P} \cup \mathcal{Q} \to \mathcal{P} \cup \mathcal{Q}$ has a fixed point. Moreover, if $\lambda(p) \geqslant 1$ and $\lambda(q) \geqslant 1$ for all $p, q \in \mathcal{P} \cup \mathcal{Q}$ are fixed points of S, then such fixed point is unique.

Theorem 2.7 Let $S, T : (\mathcal{P}, \mathcal{Q}) \rightrightarrows (\mathcal{P}, \mathcal{Q})$ be λ-admissible mappings on a complete bipolar metric space $(\mathcal{P}, \mathcal{Q}, d)$ so that $S(\mathcal{P} \cup \mathcal{Q}) \subseteq T(\mathcal{P} \cup \mathcal{Q})$. Suppose that the following assertions hold:

(*i*) there exists $p_0 \in \mathcal{P} \cup \mathcal{Q}$ such that $\lambda(p_0) \geqslant 1$;
(*ii*) either T is continuous, or;
(*iii*) if $(\{p_n\}, \{q_n\})$ is a bisequence in $(\mathcal{P}, \mathcal{Q})$ so that $(p_n, q_n) \to (\kappa, \kappa)$ and $\lambda(p_n) \geqslant 1, \lambda(q_n) \geqslant 1$ for all n, then $\lambda(\kappa) \geqslant 1$;
(*iv*) the pair $\{S, T\}$ is compatible;
(*v*) $(\lambda(p)\lambda(q) + 1)^{\chi(d(Sp,Sq))} \leqslant 2^{\chi(d(Tp,Tq)) - \zeta(d(Tp,Tq))}$ for all $p \in \mathcal{P}, q \in \mathcal{Q}$, where $\chi \in \Omega$ and $\zeta \in \Upsilon$.

Then S, T have a common fixed point. Moreover, if $\lambda(p) \geqslant 1$ and $\lambda(q) \geqslant 1$ for all $p, q \in \mathcal{P} \cup \mathcal{Q}$ are common fixed points of S and T, then such common fixed point is unique.

***Example* 2.8** Let $\mathcal{P} = (1, +\infty)$ and $\mathcal{Q} = [-1, 1]$. Define $d : \mathcal{P} \times \mathcal{Q} \to [0, +\infty)$ as $d(p, q) = |p^2 - q^2|$ for all $(p, q) \in (\mathcal{P}, \mathcal{Q})$. Then the triple $(\mathcal{P}, \mathcal{Q}, d)$ is a complete bipolar metric space. Consider
$S, T : \mathcal{P} \cup \mathcal{Q} \to \mathcal{P} \cup \mathcal{Q}$ as $S(p) = \frac{1}{3}p$ and $T(p) = p$ for all $p \in (-1, 1]$. Given $\lambda : \mathcal{P} \cup \mathcal{Q} \to$

$[0, +\infty)$ as

$$\lambda(p) = \begin{cases} 1, & \text{if } p \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Take $\chi(t) = t$ and $\zeta(t) = \frac{t}{2}$. For all $p \in (-1, 1]$ and $q \in [0, 1]$, we have

$$\begin{aligned} (\lambda(p)\lambda(q) + 1)^{\chi(d(Sp,Sq))} &= 2^{|\frac{1}{9}p^2 - \frac{1}{9}q^2|} \\ &= 2^{\frac{1}{9}|p^2-q^2|} \leqslant 2^{\frac{1}{2}|p^2-q^2|} \\ &\leqslant 2^{|p^2-q^2| - \frac{1}{2}|p^2-q^2|} \\ &= 2^{\chi(d(Tp,Tq)) - \zeta(d(Tp,Tq))}. \end{aligned}$$

Otherwise, $\lambda(p)\lambda(q) = 0$ and so

$$(\lambda(p)\lambda(q) + 1)^{\chi(d(Sp,Sq))} = 1 \leqslant 2^{\chi(d(Tp,Tq)) - \zeta(d(Tp,Tq))}.$$

All conditions of Theorem 2.7 hold, and 0 is the unique common fixed point of S and T.

3. Application to Homotopy

In this section, we study the existence of a unique solution to homotopy theory.

Theorem 3.1 Let $(\mathcal{P}, \mathcal{Q}, d)$ be a complete bipolar metric space, (A, B) be an open subset of $(\mathcal{P}, \mathcal{Q})$ so that $(\overline{A}, \overline{B})$ is a closed subset of $(\mathcal{P}, \mathcal{Q})$ and $(A, B) \subseteq (\overline{A}, \overline{B})$. Suppose $L : (\overline{A} \cup \overline{B}) \times [0, 1] \to \mathcal{P} \cup \mathcal{Q}$ is λ-admissible so that

- (i) $\sigma \neq L(\sigma, \rho)$ for each $\sigma \in \partial A \cup \partial B$ and $\rho \in [0, 1]$ and $\lambda(\sigma) \geqslant 1$. Here, ($\partial A \cup \partial B$ is the boundary of $A \cup B$ in $\mathcal{P} \cup \mathcal{Q}$)
- (ii) $\lambda(\sigma)\lambda(\varsigma) \geqslant 1 \Rightarrow \chi(d(H(\sigma, \kappa), L(\varsigma, \kappa))) \leqslant \chi(d(\sigma, \varsigma)) - \zeta(d(\sigma, \varsigma))$ for all $\sigma \in \overline{A}$, $\varsigma \in \overline{B}$, where $\rho \in [0, 1]$, $\lambda : \overline{A} \cup \overline{B} \to [0, \infty)$ where $\chi \in \Omega$ and $\zeta \in \Upsilon$.
- (iii) there is $M > 0$, $d(L(\sigma, \chi), L(y, \zeta)) \leqslant M|\chi - \zeta|$ for all $\sigma \in \overline{U}$ and $\varsigma \in \overline{B}$ and $\chi, \zeta \in [0, 1]$,
- (iv) if $(\{x_n\}, \{y_n\})$ is in $(\overline{A}, \overline{B})$ so that $(x_n, y_n) \to (\xi, \xi)$ and $\lambda(x_n) \geqslant 1, \lambda(y_n) \geqslant 1$ for all n, then $\lambda(\xi) \geqslant 1$.

Then $H(., 0)$ has a fixed point $\iff$ $H(., 1)$ has a fixed point.

Proof. Take

$$\begin{aligned} X &= \{\chi \in [0, 1] : \sigma = L(\sigma, \chi),\ \sigma \in A\}, \\ Y &= \{\zeta \in [0, 1] : \varsigma = H(\varsigma, \zeta),\ \varsigma \in B\}. \end{aligned}$$

Since $L(., 0)$ has a fixed point in $A \cup B$, we have $0 \in X \cap Y$. So that $X \cap Y$ is nonempty set. We claim that $X \cap Y$ is both closed and open in $[0, 1]$. The connectedness yields that $X = Y = [0, 1]$. Let $(\{\chi_n\}_{n=1}^{\infty}, \{\zeta_n\}_{n=1}^{\infty}) \subseteq (X, Y)$ with $(\chi_n, \zeta_n) \to (\vartheta, \vartheta) \in [0, 1]$ as $n \to \infty$. We claim that $\vartheta \in X \cap Y$.
Since $(\chi_n, \zeta_n) \in (X, Y)$ for $n = 0, 1, 2, 3, \cdots$, there is a bisequence (x_n, y_n) with $x_{n+1} = L(x_n, \chi_n)$, $y_{n+1} = L(y_n, \zeta_n)$. Since H is λ-admissible and $\lambda(x_0) \geqslant 1$, we get $\lambda(L(x_0, \chi_0)) \geqslant 1$ and $\lambda(y_0) \geqslant 1$. Hence $\lambda(L(y_0, \zeta_0)) \geqslant 1$. Continuing in same direction, $\lambda(x_{n+1}) \geqslant 1$ and

$\lambda(y_{n+1}) \geqslant 1$ for $n \geqslant 0$. That is, $\lambda(x_n) \geqslant 1$ and $\lambda(y_n) \geqslant 1$ for all $n \geqslant 0$. Namely, $\lambda(x_{n+1})\lambda(y_n) \geqslant 1$, $\lambda(x_n)\lambda(y_{n+1}) \geqslant 1$ and $\lambda(x_{n+1})\lambda(y_{n+1}) \geqslant 1$. Therefore, by (ii), we have

$$\begin{aligned}\chi\left(d(x_n, y_{n+1})\right) &= \chi\left(d(L(x_{n-1}, \chi_{n-1}), L(y_n, \zeta_n))\right)\\ &\leqslant \chi\left(d(x_{n-1}, y_n)\right) - \zeta\left(d(x_{n-1}, y_n)\right)\\ &\leqslant \chi\left(d(x_{n-1}, y_n)\right). \end{aligned} \tag{16}$$

Since χ is increasing, we get

$$d(x_n, y_{n+1}) \leqslant d(x_{n-1}, y_n). \tag{17}$$

Also, we have

$$\begin{aligned}\chi\left(d(x_n, y_n)\right) &= \chi\left(d(L(x_{n-1}, \chi_{n-1}), L(y_{n-1}, \zeta_{n-1}))\right)\\ &\leqslant \chi\left(d(x_{n-1}, y_{n-1})\right) - \zeta\left(d(x_{n-1}, y_{n-1})\right)\\ &< \chi\left(d(x_{n-1}, y_{n-1})\right). \end{aligned} \tag{18}$$

Similarly,

$$d(x_n, y_n) \leqslant d(x_{n-1}, y_{n-1}). \tag{19}$$

The inequalities (17) and (19) yield that the bisequence $\{d_n := (d(x_n, y_n)\}$ is non-increasing, so it converges to $\vartheta_1 \geqslant 0$. Assume that $\vartheta_1 > 0$. Taking $n \to \infty$ in equations (16) and (18), we get a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{20}$$

We will prove $(\{x_n\}, \{y_n\})$ is a Cauchy bisequence. Assume there are $\epsilon > 0$ and $\{m_k\}$, $\{n_k\}$ so that for $n_k > m_k > k$,

$$\begin{aligned}&d\left(x_{n_k}, y_{m_k}\right) \geqslant \epsilon,\\ &d\left(x_{n_k-1}, y_{m_k}\right) < \epsilon, \end{aligned} \tag{21}$$

and

$$\begin{aligned}&d\left(x_{m_k}, y_{n_k}\right) \geqslant \epsilon,\\ &d\left(x_{m_k}, y_{n_k-1}\right) < \epsilon. \end{aligned} \tag{22}$$

By view of (21) and triangle inequality, we get

$$\begin{aligned}\epsilon &\leqslant d\left(x_{n_k}, y_{m_k}\right)\\ &\leqslant d\left(x_{n_k}, y_{n_k-1}\right) + d\left(x_{n_k-1}, y_{n_k-1}\right) + d\left(x_{n_k-1}, y_{m_k}\right)\\ &\leqslant d\left(x_{n_k}, y_{n_k-1}\right) + d\left(L(x_{n_k-2}, \chi_{n_k-2}), L(y_{n_k-2}, \zeta_{n_k-2})\right) + \epsilon\\ &\leqslant d\left(x_{n_k}, y_{n_k-1}\right) + M|\chi_{n_k-2} - \zeta_{n_k-2}| + \epsilon. \end{aligned}$$

Letting $k \to \infty$ and using (20), we obtain

$$\lim_{k\to\infty} d\left(x_{n_k}, y_{m_k}\right) = \epsilon. \tag{23}$$

Using (22), one can prove

$$\lim_{k\to\infty} d\left(x_{m_k}, y_{n_k}\right) = \epsilon. \tag{24}$$

Since $\lambda(x_{n_k})\lambda(y_{m_k}) \geqslant 1$ for all $k \in N$, by (ii), we get

$$\chi\left(d\left(x_{n_k+1}, y_{m_k+1}\right)\right) \leqslant \chi\left(d\left(x_{n_k}, y_{m_k}\right)\right) - \zeta\left(d\left(x_{n_k}, y_{m_k}\right)\right), \tag{25}$$

and

$$\chi\left(d\left(x_{m_k+1}, y_{n_k+1}\right)\right) \leqslant \chi\left(d\left(x_{m_k}, y_{n_k}\right)\right) - \zeta\left(d\left(x_{m_k}, y_{n_k}\right)\right). \tag{26}$$

Applying (23) and (24), we get at the limit, $\chi(\epsilon) \leqslant \chi(\epsilon) - \zeta(\epsilon)$. That is, $\epsilon = 0$, which is a contradiction. Hence $(\{x_n\}, \{y_n\})$ is a Cauchy bisequence in (A, B). By completeness, there is $\gamma \in A \cap B$ with

$$\lim_{n\to\infty} x_n = \gamma = \lim_{n\to\infty} y_n. \tag{27}$$

Now, consider

$$\begin{aligned}\chi\left(d(L(\gamma,\chi), y_{n+1})\right) &= \chi\left(d(L(\gamma,\chi), L(y_n,\zeta_n))\right)\\ &\leqslant \chi\left(d(\gamma, y_n)\right) - \zeta\left(d(\gamma, y_n)\right)\\ &\leqslant \chi\left(d(\gamma, y_n)\right).\end{aligned}$$

Since χ is increasing, we get $d(L(\gamma,\chi), y_{n+1}) \leqslant d(\gamma, y_n)$.
By taking the limsup on both sides, we get $d(L(\gamma,\chi),\gamma) = 0$, which implies $L(\gamma,\chi) = \gamma$. Similarly, $L(\gamma,\zeta) = \gamma$. Therefore, $\chi = \zeta \in X \cap Y$. Clearly, $X \cap Y$ is a closed in [0, 1].
Let $(\chi_0, \zeta_0) \in (X, Y)$. Then there is a bisequence (x_0, y_0) so that $x_0 = L(x_0, \chi_0)$, $y_0 = L(y_0, \zeta_0)$. Since $A \cup B$ is open, there is $r > 0$ so that $B_d(x_0, r) \subseteq U \cup V$ and $B_d(r, y_0) \subseteq A \cup B$. Choose $\chi \in (\zeta_0 - \epsilon, \zeta_0 + \epsilon)$ and $\zeta \in (\chi_0 - \epsilon, \chi_0 + \epsilon)$ so that $|\chi - \zeta_0| \leqslant \frac{1}{M^n} < \frac{\epsilon}{2}$, $|\zeta - \chi_0| \leqslant \frac{1}{M^n} < \frac{\epsilon}{2}$ and $|\chi_0 - \zeta_0| \leqslant \frac{1}{M^n} < \frac{\epsilon}{2}$. Then for $y \in \overline{B_{X\cup Y}(x_0, r)} = \{y, y_0 \in B \mid d(x_0, y) \leqslant r + d(x_0, y_0)\}$ and $x \in \overline{B_{X\cup Y}(y_0, r)} = \{x, x_0 \in A \mid d(x, y_0) \leqslant r + d(x_0, y_0)\}$. Also,

$$\begin{aligned}d(L(x,\chi), y_0) &= d(L(x,\chi), L(y_0,\zeta_0))\\ &\leqslant d(L(x,\chi), L(y,\zeta_0)) + d(L(x_0,\chi), L(y,\zeta_0))\\ &\quad + d(L(x_0,\chi), L(y_0,\zeta_0))\\ &\leqslant 2M|\chi - \zeta_0| + d(L(x_0,\chi), L(y,\zeta_0))\\ &< \tfrac{2}{M^{n-1}} + d(L(x_0,\chi), L(y,\zeta_0)).\end{aligned}$$

Letting $n \to \infty$, we get $d(L(x,\chi), y_0) \leqslant d(L(x_0,\chi), L(y,\zeta_0))$. By (ii), we have

$$\begin{aligned}\chi\left(d(L(x,\chi), y_0)\right) &\leqslant \chi\left(d(L(x_0,\chi), L(y,\zeta_0))\right)\\ &\leqslant \chi\left(d(x_0, y)\right) - \zeta\left(d(x_0, y)\right)\\ &\leqslant \chi\left(d(x_0, y)\right).\end{aligned}$$

Since χ is increasing, we have $d(L(x,\chi),y_0) \leqslant d(x_0,y) \leqslant r + d(x_0,y_0)$.
Similarly, $d(x_0, L(y,\zeta)) \leqslant d(x,y_0) \leqslant r + d(x_0,y_0)$.
On the other hand,

$$\begin{aligned} d(x_0,y_0) &= d(L(x_0,\chi_0), L(y_0,\zeta_0)) \\ &\leqslant M|\chi_0 - \zeta_0| \leqslant \tfrac{1}{M^{n-1}} \to 0 \text{ as } n \to \infty. \end{aligned}$$

So $x_0 = y_0$. Thus, for each fixed $\zeta = \chi \in (\zeta_0 - \epsilon, \zeta_0 + \epsilon)$ and $L(.,\chi) : \overline{B_{X\cup Y}(x_0,r)} \to \overline{B_{X\cup Y}(x_0,r)}$. Hence $L(.,\chi)$ has a fixed point in $\overline{A\cup B}$. But this must be in $A \cup B$. Therefore, $L(.,\chi)$ have a fixed point in $\overline{A}\cap\overline{B}$, which must be in $A\cap B$. Then $\chi = \zeta \in X\cap Y$ for $\zeta \in (\zeta_0 - \epsilon, \zeta_0 + \epsilon)$. Hence $(\zeta_0 - \epsilon, \zeta_0 + \epsilon) \subseteq X \cap Y$. Clearly, $X \cap Y$ is open in [0, 1]. The proof of the reverse could be done similarly. ■

4. Application to Integral Equations

We will apply Corollary 2.6 to resolve the integral equation

$$\gamma(x) = f(x) + \int\limits_{E_1\cup E_2} S(x,y)P(y,\gamma(y))dy, \qquad x \in E_1 \cup E_2, \tag{28}$$

where $E_1 \cup E_2$ is a Lebesgue measurable set.

Let $\mathcal{P} = L^\infty(E_1)$ and $\mathcal{Q} = L^\infty(E_2)$ be two normed linear spaces, where E_1, E_2 are Lebesgue measurable sets with $m(E_1 \cup E_2) < \infty$.

Define $d : \mathcal{P} \times \mathcal{Q} \to [0,+\infty)$ as $d(f,g) = ||f - g||_\infty$ for all $(f,g) \in \mathcal{P} \times \mathcal{Q}$. Note that $(\mathcal{P},\mathcal{Q},d)$ is a complete bipolar metric space.

Define $T : L^\infty(E_1) \cup L^\infty(E_2) \to L^\infty(E_1) \cup L^\infty(E_2)$ by

$$T\gamma(x) = f(x) + \int\limits_{E_1\cup E_2} S(x,y)P(y,\gamma(y))dy, \qquad x \in E_1 \cup E_2.$$

Then T is a covariant mapping.

Theorem 4.1 Assume that

(i) $S : (E_1^2 \cup E_2^2) \to [0,+\infty)$, $P : (E_1 \cup E_2) \times [0,+\infty) \to [0,+\infty)$ and $f : (E_1 \cup E_2) \to [0,+\infty)$;

(ii) there are continuous functions $\theta, \tau : \mathcal{P} \cup \mathcal{Q} \to [0,+\infty)$ so that if $\theta(\gamma)\theta(\beta) \geqslant 0$ for some $\gamma \in \mathcal{P}$, $\beta \in \mathcal{Q}$, then for each $y \in E_1 \cup E_2$,

$$|P(y,\gamma(y)) - P(y,\beta(y))| \leqslant |\tau(\beta)||\gamma(y) - \beta(y)|;$$

(iii)

$$\left\| \int\limits_{E_1\cup E_2} S(x,y)|\tau(\beta)|dy \right\|_\infty < 1;$$

(iv) $\theta(\gamma) \geqslant 0$ for some $\gamma \in \mathcal{P} \cup \mathcal{Q}$ implies $\theta(T\gamma) \geqslant 0$;

(v) if $(\{\gamma_n\},\{\beta_n\})$ is a bisequence in $(\mathcal{P},\mathcal{Q})$ such that $\theta(\gamma_n)\geqslant 0, \theta(\beta_n)\geqslant 0$ for all $n\geqslant 0$ and $(\gamma_n,\beta_n)\to(\kappa,\kappa)$ as $n\to\infty$, then $\theta(\kappa)\geqslant 0$.

Then the integral equation (28) has a solution in $L^\infty(E_1)\cup L^\infty(E_2)$.

Proof. Let $\gamma\in\mathcal{P},\ \beta\in\mathcal{Q}$ be such that $\theta(\gamma)\theta(\beta)\geqslant 0$. By (ii), we deduce that

$$|T\gamma(x)-T\beta(x)| = \left|\int\limits_{E_1\cup E_2} S(x,y)\left[P(y,\gamma(y))-P(y,\beta(y))\right]dy\right|$$

$$\leqslant \int\limits_{E_1\cup E_2} S(x,y)|P(y,\gamma(y))-P(y,\beta(y))|dy$$

$$\leqslant \int\limits_{E_1\cup E_2} S(x,y)|\tau(\beta(y))||\gamma(y)-\beta(y)|dy$$

$$\leqslant \int\limits_{E_1\cup E_2} S(x,y)|\tau(\beta(y))|||\gamma-\beta||_\infty dy$$

$$\leqslant ||\gamma-\beta||_\infty\left(\int\limits_{E_1\cup E_2} S(x,y)|\tau(\beta(y))|dy\right).$$

Then,

$$||T\gamma-T\beta||_\infty \leqslant \left\|\int\limits_{E_1\cup E_2} S(x,y)|\tau(\beta)|dy\right\|_\infty ||\gamma-\beta||_\infty.$$

Choose $\chi(t)=t$ and

$$\zeta(t)=\left(1-\left\|\int\limits_{E_1\cup E_2} S(x,y)|\tau(\beta)|dy\right\|_\infty\right)t.$$

Define $\lambda:\mathcal{P}\cup\mathcal{Q}\to[0,+\infty)$ by

$$\lambda(t)=\begin{cases}1, \theta(t)\geqslant 0,\\ 0, \text{otherwise.}\end{cases}$$

Consequently, for all $\gamma\in\mathcal{P},\beta\in\mathcal{Q}$, we deduce that

$$\lambda(\gamma)\lambda(\beta)\chi\left(d(T\gamma,T\beta)\right)\leqslant\chi\left(d(\gamma,\beta)\right)-\zeta\left(d(\gamma,\beta)\right).$$

Thus, all the hypotheses of Corollary 2.6 are satisfied and hence the mapping T has a fixed point which is a solution of the integral equation (28) in $\mathcal{P} \cup \mathcal{Q}$. ■

5. Conclusion

We ensured the existence and uniqueness of a common fixed point for two covariant mappings in the class of bipolar metric spaces. Two illustrated applications have been provided.

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