Journal of
Linear and Topological Algebra
Vol. 07*, No.* 04*,* 2018*,* 269- 272

A note on spectral mapping theorem

Z. Heydarbeygi[a], B. Moosavi[b,*], M. Shah Hosseini[c]

[a] *Department of Mathematics, Payame Noor Universtiy (PNU), P.O. BOX, 19395-4697, Tehran, Iran.*
[b] *Department of Mathematics, Safadasht Branch, Islamic Azad University, Tehran, Iran.*
[c] *Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran.*

Received 19 June 2018; Revised 24 July 2018; Accepted 12 August 2018.

Communicated by Hamidreza Rahimi

Abstract. This paper aims to present the well-known spectral mapping theorem for multi-variable functions.

Keywords: Banach algebra, spectral mapping theorem.

2010 AMS Subject Classification: 47A60, 47A10.

1. Introduction

Spectral mapping theorem is a basic theorem in functional analysis. This theorem says that, if f is an analytic function, then

$$\sigma(f(x)) = f(\sigma(x)) \tag{1}$$

for an element of a Banach Algebra Ω (see for example [5, Proposition 2.8] and [6, Theorem 2.1.14]). Similar results have been studied for other types of spectrum as the joint spectrum (see [2,3]).
Here, we prove an extension of this theorem for multi-variable functions. We conclude

*Corresponding author.
E-mail address: zheydarbeygi@yahoo.com (Z. Heydarbeygi); baharak_moosavie@yahoo.com (B. Moosavi); mohsen_shahhosseini@yahoo.com (M. Shah Hosseini).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

this short paper by recalling some notations which will be used in the sequel. Let Ω be a commutative C^*-algebra. Then for any x in Ω we have

$$\sigma(x) = \{\varphi(x) : \quad \varphi \in M_\Omega\},$$

where φ is character and M_Ω denotes the maximal ideal space of Ω. In the sequel an essential role is played by bi-analytic functions. A continuous function F on $\mathbb{C}^2$ is said to be bi-analytic, if for all x_0, y_0 in $\mathbb{C}$,

$$F(x) = F(x, y_0), \quad F(y) = F(x_0, y)$$

are analytic on $\mathbb{C}$. For instance

$$F(x, y) = e^{(x+y)}, \quad F(x, y) = e^{(x)} + e^{(y)}$$

are bi-analytic functions.
Assume that A and B are subsets of $\mathbb{C}$ and F is two variable function. We define

$$F(A, B) := \{F(a, b) : \ a \in A, b \in B\}.$$

The next section includes an extension of (1).

2. The main result

In order to derive our main result, we need the following observation.

Lemma 2.1 Every bi-analytic function f, has the representation

$$f(x, y) = \sum_{m,n=0}^{\infty} \alpha_{m,n} x^m y^n,$$

where $\alpha_{m,n}$ are complex numbers.

Proof. First of all, consider $f(x, y)$ as function in y. Since $f(y) = f(x_0, y)$ is analytic, therefore

$$f(x, y) = \sum_n \alpha_n(x) y^n. \tag{2}$$

It remains to prove, $\alpha_n(x)$ is analytic. Since $\alpha_0(x) = f(x, 0)$, so α_0 is analytic. Also $\alpha_1(x) = f_y(x, 0)$, and therefore α_1 is analytic. In a similar manner one can get α_i are analytic. Whence, it has power series of the form

$$\alpha_n(x) = \sum_m \alpha_{m,n} x^m. \tag{3}$$

By replacing (3) into (2) we get the desired result. ■

Now we use Lemma to prove an analogous result but different technique to the main theorem of Harte [4] in an easy fashion as an offshoot of our work.

Theorem 2.2 Let Λ be a commutative C^*-algebra and F be a bi-analytic function. Then for any x, y in Λ we have

$$\sigma(F(x,y)) \subseteq F(\sigma(x), \sigma(y)).$$

Proof. It follows from previous lemma that

$$F(x,y) = \sum_{m,n=0}^{\infty} \alpha_{m,n} x^m y^n.$$

On the other hand, bearing in mind that

$$\sigma(F(x,y)) = \{\varphi(F(x,y)) : \quad \varphi \in M_\Omega\},$$

we have

$$\varphi(F(x,y)) = \varphi(\sum_{m,n=0}^{\infty} \alpha_{m,n} x^m y^n) = \sum_{m,n=0}^{\infty} \alpha_{m,n} \varphi(x)^m \varphi(y)^n = F(\varphi(x), \varphi(y)),$$

where $\varphi(x) \in \sigma(x)$ and $\varphi(y) \in \sigma(y)$. Therefore, $\varphi(F(x,y)) \in F(\varphi(x), \varphi(y))$ and since φ is arbitrary we have $\sigma(F(x,y)) \subseteq F(\sigma(x), \sigma(y))$. This completes the proof. ■

Related to the above theorem, the following remarks are worth mentioning.

(i) If we take $F(x,y) = x + y$ or $F(x,y) = xy$, then we have $\sigma(x+y) \subseteq \sigma(x) + \sigma(y)$ and $\sigma(xy) \subseteq \sigma(x)\sigma(y)$, which are well known results (see, e.g., [1, Corollary 3.2.10]).

(ii) In general, for some fixed x, y it is enough x, y commute together, and then we consider the C^*-algebra generated by x, y.

(iii) We guess the above theorem can be extended in the following way:

$$\sigma(F(x_1, x_2, \dots, x_n)) \subseteq F(\sigma(x_1), \sigma(x_2), \dots, \sigma(x_n)).$$

However, this does not appear to be easy to prove directly.

(iv) For completeness, we also state the extension of our result to spectral radius $r(x)$. More precisely, we have

$$r(F(x,y)) \subseteq F(r(x), r(y))$$

and in general

$$r(F(x_1, x_2, \dots, x_n)) \subseteq F(r(x_1), r(x_2), \dots, r(x_n)).$$

Notice that, the proof is based on the fact that

$$|F(x_1, x_2, \dots, x_n)| \leqslant F(|x_1|, |x_2|, \dots, |x_n|).$$

Acknowledgements

The authors would like to thank the anonymous reviewers for careful reading of the manuscript and giving useful comments, which will help to improve the paper.

References

[1] B. Aupetit, A primer on spectral theory. Springer-Verlag, New York, 1991.
[2] J. Eschmeier, Analytic spectral mapping theorems for joint spectra, Opr. Theory. Adv. Appl. 24 (1987), 167-181.
[3] R. E. Harte, Spectral mapping theorems, Proceedings of the Royal Irish Academy, Section A: Math. Physic. Sci, 1972.
[4] R. E. Harte, The spectral mapping theorems in several variables, Bull. Amer. Math. Soc. 78 (5) (1972), 871-875.
[5] J. G. Murphy, C^*-algebras and operator theory, Academic Press Inc, 1990.
[6] M. Takesaki, Theory of operator algebras. Encyclopaedia of Math. Sci, 2002.