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Fixed points of generalized α-Meir-Keeler type contractions and Meir-Keeler contractions through rational expression in b-metric-like spaces

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Abstract. In this paper, we first introduce some types of generalized α-Meir-Keeler contractions in b-metric-like spaces and then we establish some fixed point results for these types of contractions. Also, we present a new fixed point theorem for a Meir-Keeler contraction through rational expression. Finally, we give some examples to illustrate the usability of the obtained results.

Keywords: α-admissible, fixed point, Meir-Keeler contraction, b-metric-like.

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1. Introduction and Preliminaries

The Banach contraction principle [5] which is useful and classical tool in nonlinear analysis, has many generalizations. In 1969, Meir and Keeler [11] published their paper in which an interesting and general contraction for self-maps in metric spaces was considered.

Theorem 1.1 [11] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following condition:

$$\forall \varepsilon > 0, \exists \delta > 0;\ \varepsilon \leqslant d(x, y) < \varepsilon + \delta(\varepsilon) \Rightarrow d(Tx, Ty) < \varepsilon$$

for all $x, y \in X$. Then T has a unique fixed point.

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In recent years, Samet et al. [13] introduced the concept of α-admissible mappings in a metric space and obtained some fixed point results for these mappings. There are many researchers who improved and generalized fixed point results by using the concept of α-admissible mappings for single-valued and multi-valued mappings ([2, 9, 10]).

Alsulami et al. [2] defined two types of generalized α-admissible Meir-Keeler contractions and proved some fixed point theorems for these kinds of mappings (for other works, see [4, 7, 9, 14]). On the other hand, Alghamdi et al. [1] introduced the concept of b-metric-like spaces and established the existence and uniqueness of fixed points in a b-metric-like space as well as in a partially ordered b-metric-like space.

In this work, by using the concepts of Meir-Keeler contractions, α-admissible mappings, and b-metric-like spaces, we define the concept of generalized α-Meir-Keeler contraction mappings in b-metric-like spaces. Then we investigate some fixed point results for these classes of contractions. Also, we present a new fixed point theorem for a Meir-Keeler contraction through rational expression. Some examples are given to support the usability of our results. In [8], Gholamian and Khanehgir investigated some fixed point results for generalized Meir-Keeler contractions on a b-metric-like space. Note that our definition of generalized α-Meir-Keeler contractions is different from that of [8].

It will be helpful to recall some basic definitions and facts which will be used further on. We denote by $\mathbb{R}$ the set of real numbers and $\mathbb{R}^+$ the set of non-negative real numbers.

Definition 1.2 [16] A partial b-metric on a nonempty set X is a function $p_b : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

Pb1) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$,
Pb2) $p_b(x,x) \leqslant p_b(x,y)$,
Pb3) $p_b(x,y) = p_b(y,x)$,
Pb4) there exists a real number $s \geqslant 1$ such that $p_b(x,y) \leqslant s[p_b(x,z) + p_b(z,y)] - p_b(z,z)$.

A partial b-metric space is a pair (X, p_b), where X is a nonempty set and p_b is a partial b-metric on X. The real number s is called the coefficient of (X, p_b).

***Example* 1.3** Let $X = \mathbb{R}^+$, $q > 1$ be a constant number and $p_b^1, p_b^2 : X \times X \to \mathbb{R}^+$ be defined by

$$p_b^1(x,y) = \left(\max\{x,y\}\right)^q, \; p_b^2(x,y) = (x+y)^2.$$

Then (X, p_b^i), with $i = 1,2$ are partial b-metric spaces with coefficients 2^{q-1} and 2, respectively.

Definition 1.4 [3] A metric-like on a nonempty set X is a mapping $\sigma : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(σ1) $\sigma(x,y) = 0$ implies $x = y$,
(σ2) $\sigma(x,y) = \sigma(y,x)$,
(σ3) $\sigma(x,y) \leqslant \sigma(x,z) + \sigma(z,y)$.

The pair (X, σ) is called a metric-like space.

***Example* 1.5** [15] Let $X = \mathbb{R}$. Then the mappings $\sigma_i : X \times X \longrightarrow \mathbb{R}^+$, $i = 1,2,3$ defined by

$$\sigma_1(x,y) = |x| + |y| + a, \quad \sigma_2(x,y) = |x-b| + |y-b|, \quad \sigma_3(x,y) = x^2 + y^2$$

are metrics-like on X, where $a \geqslant 0$ and $b \in \mathbb{R}$.

Definition 1.6 [1] Let X be a nonempty set and $s \geqslant 1$ be a given real number. A function $\sigma_b : X \times X \to \mathbb{R}^+$ is a b-metric-like if for all $x, y, z \in X$ the following conditions are satisfied:

(σ_b1) $\sigma_b(x, y) = 0$ implies $x = y$,
(σ_b2) $\sigma_b(x, y) = \sigma_b(y, x)$,
(σ_b3) $\sigma_b(x, y) \leqslant s[\sigma_b(x, z) + \sigma_b(z, y)]$.

A b-metric-like space is a pair (X, σ_b) such that X is a nonempty set and σ_b is a b-metric-like on X. The number s is called the coefficient of (X, σ_b).

Some examples of b-metric-like spaces can be constructed with the help of following proposition.

Proposition 1.7 [12] Let (X, σ) be a metric-like space and $\sigma_b(x, y) = [\sigma(x, y)]^l$, where $l > 1$. Then σ_b is a b-metric-like with coefficient $s = 2^{l-1}$.

Every partial b-metric space is a b-metric-like space with the same coefficient s. However, the converse of this fact need not hold. For this, take $p > 1$. According to Proposition 1.7 and Example 1.5, σ_3^p is a b-metric-like, but it is not a partial b-metric.

Every b-metric-like σ_b on a nonempty set X generates a topology τ_{σ_b} on X whose base is the family of open σ_b-balls $\{B_{\sigma_b}(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_{\sigma_b}(x, \varepsilon) = \{y \in X : |\sigma_b(x, y) - \sigma_b(x, x)| < \varepsilon\}$ for all $x \in X$ and all $\varepsilon > 0$.

Definition 1.8 [1] Let (X, σ_b) be a b-metric-like space with coefficient s, $\{x_n\}$ be any sequence in X and $x \in X$. Then,

(i) the sequence $\{x_n\}$ is said to be convergent to x with respect to τ_{σ_b} if $\lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x)$.
(ii) the sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, σ_b), if $\lim_{n,m\to\infty} \sigma_b(x_n, x_m)$ exists and is finite.
(iii) (X, σ_b) is said to be a complete b-metric-like space if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that

$$\lim_{n,m\to\infty} \sigma_b(x_n, x_m) = \lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x).$$

Note that in a b-metric-like space the limit of convergent sequence may not be unique (since already partial metric spaces share this property).

Definition 1.9 [6] Suppose that (X, σ_b) is a b-metric-like space. A mapping $T : X \to X$ is said to be continuous at a point $x \in X$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $T(B_{\sigma_b}(x, \delta)) \subseteq B_{\sigma_b}(Tx, \varepsilon)$. The mapping T is continuous on X if it is continuous at all $x \in X$.

Note that if $T : X \to X$ is a continuous mapping and $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x)$, then $\lim_{n\to\infty} \sigma_b(Tx_n, Tx) = \sigma_b(Tx, Tx)$.

Definition 1.10 Let X be a nonempty set, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. we say that if for all $x, y \in X$

$$\alpha(x, y) \geqslant 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geqslant 1.$$

Definition 1.11 A mapping $T : X \to X$ is called triangular α-admissible if it is α-

admissible and satisfies the following condition:

$$\alpha(x,y) \geqslant 1, \quad \alpha(y,z) \geqslant 1 \quad \Rightarrow \quad \alpha(x,z) \geqslant 1,$$

where $x, y, z \in X$.

The following lemma is useful in proving our main results which is stated and proved according to [9, Lemma 7].

Lemma 1.12 Let X be a nonempty set, $T : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$, $\alpha(Tx_0, x_0) \geqslant 1$. If $x_n = T^n x_0$, then $\alpha(x_m, x_n) \geqslant 1$ for all $m, n \in \mathbb{N}$.

2. Main results

In this section, first we introduce the concept of generalized α-Meir-Keeler contraction mappings in b-metric-like spaces which can be regarded as an extension of the Meir-Keeler contractions defined in [11]. Then we establish some fixed point theorems for these classes of contractions.

Definition 2.1 Let (X, σ_b) be a b-metric-like space with coefficient s. A triangular α-admissible mapping $T : X \to X$ is said to be α-admissible Meir-Keeler contraction (or shortly α-Meir-Keeler contraction) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leqslant \sigma_b(x,y) < s(\varepsilon + \delta) \quad \text{implies} \quad \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon$$

for all $x, y \in X$.

Applying definition of α-Meir-Keeler contraction, it is clear that

$$\alpha(x,y)\sigma_b(Tx,Ty) < \sigma_b(x,y)$$

for all $x, y \in X$ when $x \neq y$.

Remark 1 *Note that our definition of α-Meir-Keeler contraction is different from that of [8, Definition 2.1]. For this, take $X = \{0,1,2,3\}$ and $\sigma_b : X \times X \to \mathbb{R}^+$ defined by $\sigma_b(x,y) = 1$, if $x \neq y$ and 0, otherwise. Then (X, σ_b) is a b-metric-like space with $s = 2$. Also, consider the mapping $T : X \to X$ defined by $T0 = 0$, $T1 = T3 = 1$ and $T2 = 2$, and functions $\beta : [0,\infty) \to (0, \frac{1}{s})$ and $\alpha : X \times X \to [0,\infty)$ defined by*

$$\beta(t) = \frac{1}{t+1}, \quad \alpha(x,y) = \begin{cases} \frac{1}{5}, & x+y = 1 \text{ or } 3 \\ 0, & x = y = 0 \\ 1, & x = y = 1 \\ \frac{1}{2x+y+2}, & \text{otherwise.} \end{cases}$$

It is easily can be checked that T is an α-Meir-Keeler contraction. According to [8, Definition 2.1], for $x = 0$, $y = 3$ and $\varepsilon = \frac{1}{6}$ we have $\varepsilon \leqslant \beta(\sigma_b(0,3))\sigma_b(0,3) = \frac{1}{2} < \varepsilon + \delta$ which does not imply that $\alpha(0,3)\sigma_b(T0,T3) < \varepsilon$, Since $\alpha(0,3)\sigma_b(T0,T3) = \frac{1}{5}$.

From now on, for convenience, we denote by $\mathcal{B}_s$ the set of all functions $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to (0, \frac{1}{s})$ for a real number $s \geqslant 1$.

We now define generalized α-Meir-Keeler contractions on b-metric-like spaces, say type (I) and type (II).

Definition 2.2 Let (X, σ_b) be a b-metric-like space with coefficient s. A triangular α-admissible mapping $T : X \to X$ is said to be a generalized α-Meir-Keeler contraction of type (I) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leqslant M_\beta(x,y) < s(\varepsilon+\delta) \quad \text{implies} \quad \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon, \tag{1}$$

where

$$M_\beta(x,y) = \max\{\sigma_b(x,y), \beta(x,Tx)\sigma_b(x,Tx), \beta(y,Ty)\sigma_b(y,Ty)\} \tag{2}$$

for all $x, y \in X$.

Definition 2.3 Let (X, σ_b) be a b-metric-like space with coefficient s. A triangular α-admissible mapping $T : X \to X$ is said to be a generalized α-Meir-Keeler contraction of type (II) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leqslant N_\beta(x,y) < s(\varepsilon+\delta) \quad \text{implies} \quad \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon, \tag{3}$$

where

$$N_\beta(x,y) = \max\left\{\sigma_b(x,y), \frac{1}{2}[\beta(x,Tx)\sigma_b(x,Tx) + \beta(y,Ty)\sigma_b(y,Ty)]\right\} \tag{4}$$

for all $x, y \in X$.

Remark 2 *Suppose that $T : X \to X$ is a generalized α-Meir-Keeler contraction of type (I) (respectively, type (II)). Then for all $x, y \in X$ with $M_\beta(x,y) > 0$ (respectively, $N_\beta(x,y) > 0$) we have*

$$\alpha(x,y)\sigma_b(Tx,Ty) < M_\beta(x,y) \quad (\textit{respectively}, \ \ N_\beta(x,y)).$$

Remark 3 *It is clear that $N_\beta(x,y) \leqslant M_\beta(x,y)$ for all $x, y \in X$.*

Now, we present the existence of fixed point of mappings satisfying generalized α-Meir-Keeler contractions of type (I) in the setup of b-metric-like spaces.

Theorem 2.4 Let (X, σ_b) be a complete b-metric like space and $T : X \to X$ be a mapping. Suppose that the following conditions hold:
(a) T is a continuous generalized α-Meir-Keeler contraction of type (I),
(b) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\alpha(Tx_0, x_0) \geqslant 1$,
(c) if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_m) \geqslant 1$ for all $n, m \in \mathbb{N}$, then $\alpha(z,z) \geqslant 1$.
Then T has a fixed point in X.

Proof. Choose $x_0 \in X$ such that condition (b) holds and define a sequence $\{x_n\}$ in X so that $x_1 = Tx_0$, $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. We may assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$, otherwise T has trivially a fixed point. Taking into account α-admissibly of T, we deduce that

$$\alpha(x_n, x_{n+1}) \geqslant 1, \quad \forall n \in \mathbb{N}. \tag{5}$$

Replace x by x_n and y by x_{n+1} in (1), then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leqslant M_\beta(x_n, x_{n+1}) < s(\varepsilon + \delta) \quad \Rightarrow \quad \alpha(x_n, x_{n+1})\sigma_b(Tx_n, Tx_{n+1}) < \varepsilon, \tag{6}$$

where

$$M_\beta(x_n, x_{n+1}) = \max\{\sigma_b(x_n, x_{n+1}), \beta(x_n, x_{n+1})\sigma_b(x_n, x_{n+1}), \beta(x_{n+1}, x_{n+2})\sigma_b(x_{n+1}, x_{n+2})\}.$$

Clearly, we have $\beta(x_n, x_{n+1})\sigma_b(x_n, x_{n+1}) < \sigma_b(x_n, x_{n+1})$. So we shall consider the following two cases:
Case 1. Assume that $M_\beta(x_n, x_{n+1}) = \beta(x_{n+1}, x_{n+2})\sigma_b(x_{n+1}, x_{n+2})$. In this case

$$\begin{aligned}\sigma_b(x_{n+1}, x_{n+2}) \leqslant \alpha(x_n, x_{n+1})\sigma_b(Tx_n, Tx_{n+1}) &< \beta(x_{n+1}, x_{n+2})\sigma_b(x_{n+1}, x_{n+2}) \\ &< \sigma_b(x_{n+1}, x_{n+2}),\end{aligned}$$

which gives a contradiction.
Case 2. Assume that $M_\beta(x_n, x_{n+1}) = \sigma_b(x_n, x_{n+1})$. Then (6) becomes

$$\varepsilon \leqslant \sigma_b(x_n, x_{n+1}) < s(\varepsilon + \delta) \quad \Rightarrow \alpha(x_n, x_{n+1})\sigma_b(Tx_n, Tx_{n+1}) < \varepsilon.$$

It enforces that

$$\sigma_b(x_{n+1}, x_{n+2}) \leqslant \alpha(x_n, x_{n+1})\sigma_b(Tx_n, Tx_{n+1}) < \varepsilon \leqslant \sigma_b(x_n, x_{n+1})$$

for all n; that is, $\{\sigma_b(x_n, x_{n+1})\}$ is a strictly decreasing positive sequence in $\mathbb{R}^+$ and it converges to some $r \geqslant 0$. We will show that $r = 0$. To support the claim, let it be untrue. Then we have

$$0 < r \leqslant \sigma_b(x_n, x_{n+1}) \quad \text{for all} \quad n \in \mathbb{N}. \tag{7}$$

In view of (6), we may choose $\varepsilon = r$. Hence there exists $\delta = \delta(r) > 0$ satisfying (6). In other words,

$$r \leqslant \sigma_b(x_n, x_{n+1}) < s(r + \delta) \quad \Rightarrow \alpha(x_n, x_{n+1})\sigma_b(Tx_n, Tx_{n+1}) < r.$$

On the other hand, there exists sufficiently large N such that $r < \sigma_b(x_N, x_{N+1}) < r+\delta < s(r + \delta)$. Therefore,

$$\sigma_b(x_{N+1}, x_{N+2}) \leqslant \alpha(x_N, x_{N+1})\sigma_b(Tx_N, Tx_{N+1}) < r,$$

which leads to a contradiction with the condition (7). Thus, $\lim_{n\to\infty} \sigma_b(x_n, x_{n+1}) = 0$. Next, we claim that the sequence $\{x_n\}$ is a Cauchy sequence in (X, σ_b). To this aim, we prove that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sigma_b(x_l, x_{l+k}) < \varepsilon \tag{8}$$

for all $l \geqslant N$ and $k \in \mathbb{N}$. Since the sequence $\{\sigma_b(x_n, x_{n+1})\}$ converges to 0 as $n \to \infty$, then for each $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$\sigma_b(x_n, x_{n+1}) < \delta \quad \text{for all} \quad n \geqslant N.$$

Choose δ such that $\delta < \varepsilon$. We will proceed using induction on k in order to prove (8). For $k = 1$, (8) becomes $\sigma_b(x_l, x_{l+1}) < \varepsilon$, and clearly holds for all $l \geqslant N$ (due to the choice of δ). Assume that the inequality (8) holds for some $k = m$; that is, $\sigma_b(x_l, x_{l+m}) < \varepsilon$ for all $l \geqslant N$. We will show that $\sigma_b(x_l, x_{l+m+1}) < \varepsilon$ for all $l \geqslant N$. First, suppose that $\sigma_b(x_{l-1}, x_{l+m}) \geqslant \varepsilon$. Using the condition ($\sigma_b$3), we get

$$\sigma_b(x_{l-1}, x_{l+m}) \leqslant s[\sigma_b(x_{l-1}, x_l) + \sigma_b(x_l, x_{l+m})] < s(\delta + \varepsilon)$$

for all $l \geqslant N$. Then we deduce

$$\begin{aligned}\varepsilon &\leqslant \sigma_b(x_{l-1}, x_{l+m})\\ &\leqslant M_\beta(x_{l-1}, x_{l+m})\\ &= \max\{\sigma_b(x_{l-1}, x_{l+m}), \beta(x_{l-1}, x_l)\sigma_b(x_{l-1}, x_l),\\ &\quad \beta(x_{l+m}, x_{l+m+1})\sigma_b(x_{l+m}, x_{l+m+1})\}\\ &< \max\{s(\varepsilon + \delta), \frac{1}{s}\delta, \frac{1}{s}\delta\}\\ &= s(\varepsilon + \delta),\end{aligned}$$

and according to Lemma 1.12, on using the contractive condition (1) with $x = x_{l-1}$, $y = x_{l+m}$ one yields

$$\begin{aligned}&\varepsilon \leqslant M_\beta(x_{l-1}, x_{l+m}) < s(\delta + \varepsilon)\\ &\Rightarrow\\ &\sigma_b(x_l, x_{l+m+1}) \leqslant \alpha(x_{l-1}, x_{l+m})\sigma_b(x_l, x_{l+m+1})\\ &= \alpha(x_{l-1}, x_{l+m})\sigma_b(Tx_{l-1}, Tx_{l+m}) < \varepsilon,\end{aligned}$$

and hence (8) holds for $k = m + 1$. Next, suppose that $\sigma_b(x_{l-1}, x_{l+m}) < \varepsilon$, then

$$\begin{aligned}M_\beta(x_{l-1}, x_{l+m}) &= \max\{\sigma_b(x_{l-1}, x_{l+m}), \beta(x_{l-1}, x_l)\sigma_b(x_{l-1}, x_l),\\ &\quad \beta(x_{l+m}, x_{l+m+1})\sigma_b(x_{l+m}, x_{l+m+1})\}\\ &< \max\{\varepsilon, \frac{1}{s}\delta, \frac{1}{s}\delta\} = \varepsilon.\end{aligned}$$

Note that $M_\beta(x_{l-1}, x_{l+m}) > 0$, otherwise $\sigma_b(x_l, x_{l-1}) = 0$, and hence $x_l = x_{l-1}$ which is a contradiction. In view of Remark *2*, we have

$$\sigma_b(x_l, x_{l+m+1}) \leqslant \alpha(x_{l-1}, x_{l+m})\sigma_b(Tx_{l-1}, Tx_{l+m}) < M_\beta(x_{l-1}, x_{l+m}) < \varepsilon;$$

that is, (8) holds for $k = m+1$. Thus, $\sigma_b(x_l, x_{l+k}) < \varepsilon$ for all $l \geqslant N$ and $k \geqslant 1$. It means that $\sigma_b(x_n, x_m) < \varepsilon$ for all $m \geqslant n \geqslant N$. Consequently, $\lim_{n\to\infty} \sigma_b(x_n, x_m) = 0$ and so $\{x_n\}$ is a Cauchy sequence in complete b-metric like space (X, σ_b). Therefore, there exists $z \in X$ such that

$$\lim_{n,m\to\infty} \sigma_b(x_n, x_m) = \lim_{n\to\infty} \sigma_b(x_n, z) = \sigma_b(z, z) = 0.$$

We show that z is a fixed point of T. To see this, it is enough to prove that $\sigma_b(z, Tz) = 0$. Assume that $\sigma_b(z, Tz) > 0$. Thus we have $M_\beta(z, z) \geqslant \beta(z, Tz)\sigma_b(z, Tz) > 0$ and using Remark *2*, we realize that

$$\begin{aligned}\sigma_b(Tz, Tz) \leqslant \alpha(z, z)\sigma_b(Tz, Tz) &< M_\beta(z, z)\\ &= \max\{\sigma_b(z, z), \beta(z, Tz)\sigma_b(z, Tz)\}\\ &= \beta(z, Tz)\sigma_b(z, Tz) < \frac{1}{s}\sigma_b(z, Tz). \qquad (9)\end{aligned}$$

Employing the property ($\sigma_b 3$) we get

$$\sigma_b(z, Tz) \leqslant s[\sigma_b(z, x_{n+1}) + \sigma_b(x_{n+1}, Tz)].$$

Letting $n \to \infty$ in the above inequality, and using the continuity of T it follows that

$$\sigma_b(z, Tz) \leqslant s\sigma_b(Tz, Tz).$$

From (9), we deduce that $\sigma_b(z, Tz) < s \times \frac{1}{s}\sigma_b(z, Tz) = \sigma_b(z, Tz)$, which is a contradiction. Hence $\sigma_b(z, Tz) = 0$ and so $Tz = z$. ■

By Remark *3* we know $N_\beta(x, y) \leqslant M_\beta(x, y)$, so a slight change in the proof of Theorem 2.4 shows that the following theorem holds.

Theorem 2.5 Let (X, σ_b) be a complete b-metric-like space and $T : X \to X$ be a mapping. Suppose that the following conditions hold:
(a) T is a continuous generalized α-Meir-Keeler contraction of type (II),
(b) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\alpha(Tx_0, x_0) \geqslant 1$,
(c) if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_m) \geqslant 1$ for all $n, m \in \mathbb{N}$, then $\alpha(z, z) \geqslant 1$.
Then T has a fixed point in X.

There is an analogous result for α-Meir-Keeler contraction. The proof is an easy adaptation of the one given in Theorem 2.4.

Proposition 2.6 Consider a particular case of Theorem 2.4, whenever T is a generalized α-Meir-Keeler contraction, then T has a fixed point in X.

It is useful to seek a suitable replacement for the continuity of the contraction T. The next two theorems indicate how this can be achieved. In fact, with the aid of α-admissibility of the contraction, we will show that continuity assumption is not required whenever the following condition is satisfied.
(A) If $\{x_n\}$ is a sequence in X which converges to z with respect to τ_{σ_b}, and satisfies $\alpha(x_{n+1}, x_n) \geqslant 1$ and $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(z, x_{n_k}) \geqslant 1$ or $\alpha(x_{n_k}, z) \geqslant 1$ for all k.

Theorem 2.7 Let (X, σ_b) be a complete b-metric-like space and $T : X \to X$ be a generalized α-Meir-Keeler contraction of type (I). If condition (A) holds and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\alpha(Tx_0, x_0) \geqslant 1$, then T has a fixed point in X.

Proof. As the proof of Theorem 2.4, we know that the sequence $\{x_n\}$ defined by $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ ($n \in \mathbb{N}$) converges to some $z \in X$ with $\sigma_b(z, z) = 0$. We prove that z is a fixed point of T. To this end, we show that $\sigma_b(Tz, z) = 0$. On the contrary, suppose that $\sigma_b(z, Tz) > 0$. Applying condition (A), without loss of generality, suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(z, x_{n_k}) \geqslant 1$ for all k. According to Remark *2*, for all $k \in \mathbb{N}$, we have

$$\sigma_b(Tz, x_{n_{k+1}}) = \sigma_b(Tz, Tx_{n_k}) \leqslant \alpha(z, x_{n_k})\sigma_b(Tz, Tx_{n_k}) < M_\beta(z, x_{n_k}), \tag{10}$$

where

$$M_\beta(z, x_{n_k}) = \max\{\sigma_b(z, x_{n_k}), \beta(z, Tz)\sigma_b(z, Tz), \beta(x_{n_k}, Tx_{n_k})\sigma_b(x_{n_k}, Tx_{n_k})\} > 0.$$

Using $(\sigma b3)$, we obtain that

$$\lim_{k\to\infty} M_\beta(z, x_{n_k}) = \beta(z, Tz)\sigma_b(z, Tz).$$

Applying again $(\sigma b3)$ and the relation (10), we get

$$\begin{aligned}\sigma_b(z, Tz) &\leqslant s\sigma_b(z, x_{n_{k+1}}) + s\sigma_b(x_{n_{k+1}}, Tz)\\ &< s\sigma_b(z, x_{n_{k+1}}) + sM_\beta(z, x_{n_k}).\end{aligned}$$

Letting k tends to infinity, we have

$$\sigma_b(z, Tz) \leqslant s\beta(z, Tz)\sigma_b(z, Tz) < \sigma_b(z, Tz),$$

which leads to a contradiction. Thus $\sigma_b(Tz, z) = 0$ and so $Tz = z$. ■

Theorem 2.8 Let (X, σ_b) be a complete b-metric-like space and $T : X \to X$ be a generalized α-Meir-Keeler contraction of type (II). If condition (A) holds and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\alpha(Tx_0, x_0) \geqslant 1$, then T has a fixed point in X.

Proof. Following the proof of Theorem 2.4, we observe that the sequence $\{x_n\}$ defined by $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ $(n \in \mathbb{N})$ converges to some $z \in X$ with $\sigma_b(z, z) = 0$. By using the condition (A), we may suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(z, x_{n_k}) \geqslant 1$ for all k. Note that if $N_\beta(z, x_{n_k}) = 0$ for some k, then $Tz = z$ and the proof is complete. Then we assume that $N_\beta(z, x_{n_k}) > 0$ for all $k \in \mathbb{N}$. Regarding Remark *2*, we get

$$\sigma_b(Tz, x_{n_{k+1}}) = \sigma_b(Tz, Tx_{n_k}) \leqslant \alpha(z, x_{n_k})\sigma_b(Tz, Tx_{n_k}) < N_\beta(z, x_{n_k}),$$

where

$$N_\beta(z, x_{n_k}) = \max\big\{\sigma_b(z, x_{n_k}), \frac{1}{2}[\beta(z, Tz)\sigma_b(z, Tz) + \beta(x_{n_k}, Tx_{n_k})\sigma_b(x_{n_k}, Tx_{n_k})]\big\}.$$

Letting $k \to \infty$ and using $(\sigma b3)$, we obtain

$$\lim_{k\to\infty} N_\beta(z, x_{n_k}) = \frac{1}{2}\beta(z, Tz)\sigma_b(z, Tz).$$

It follows that

$$\lim_{k\to\infty} \sigma_b(Tz, x_{n_{k+1}}) \leqslant \frac{1}{2}\beta(z, Tz)\sigma_b(z, Tz).$$

Applying again (σ_b3), we get

$$\sigma_b(z,Tz) \leqslant s\sigma_b(z,x_{n_{k+1}}) + s\sigma_b(x_{n_{k+1}},Tz)$$

and passing the limit as $k \to \infty$, we obtain

$$\sigma_b(z,Tz) \leqslant \frac{1}{2}\beta(z,Tz)\sigma_b(z,Tz) < \frac{1}{2s}\sigma_b(Tz,z).$$

It enforces that $\sigma_b(z,Tz) = 0$ and hence $Tz = z$, which completes the proof. ∎

***Example* 2.9** Let $X = [0,2]$ equipped with the b-metric-like $\sigma_b(x,y) = [\max\{x,y\}]^q$, where $q \geqslant 1$. Then (X,σ_b) is a complete b-metric-like space with $s = 2^{q-1}$ (see Proposition 1.7). Consider the mapping $T : X \to X$ and the functions $\beta : X \times X \to (0, \frac{1}{2^{q-1}})$ and $\alpha : X \times X \to [0,\infty)$ defined by

$$\mathrm{T}(x) = \frac{x}{2}, \quad \beta(x,y) = \begin{cases} \frac{1}{2^q}, & x,y \in [0,1], \\ \frac{1}{2^{q+1}}, & \text{otherwise}, \end{cases}, \quad \alpha(x,y) = \begin{cases} 1, & x,y \in [0,1], \\ \frac{1}{2^{2q}}, & \text{otherwise}. \end{cases}$$

It easily can be shown that T is triangular α-admissible and continuous. In order to check the condition (1) without loss of generality, we may take $x \leqslant y$. Let $\varepsilon > 0$ be given. Consider the following two cases.

Case 1. If $0 \leqslant x \leqslant y \leqslant 1$, then we have $\sigma_b(Tx,Ty) = (\frac{y}{2})^q$ and $M_\beta(x,y) = y^q$. We choose $\delta = \varepsilon$ so that $\varepsilon \leqslant M_\beta(x,y) = y^q < s(\varepsilon + \delta) = 2s\varepsilon$. It implies that $\alpha(x,y)\sigma_b(Tx,Ty) = (\frac{y}{2})^q < \varepsilon$.

Case 2. If $0 \leqslant x \leqslant 1$, $1 \leqslant y \leqslant 2$ or $1 < x \leqslant y \leqslant 2$, then we have

$$\sigma_b(Tx,Ty) = (\frac{y}{2})^q, \quad M_\beta(x,y) = y^q.$$

We choose again $\delta = \varepsilon$ so that $\varepsilon \leqslant M_\beta(x,y) = y^q < s(\varepsilon + \delta) = 2s\varepsilon$. It follows that

$$\alpha(x,y)\sigma_b(Tx,Ty) < (\frac{y}{2})^q < \varepsilon.$$

Therefore, the map T is a generalized α-Meir-Keeler contraction of type (I). Note that $\alpha(0,T0) \geqslant 1$ and $\alpha(T0,0) \geqslant 1$. Now, all conditions of Theorem 2.4 are satisfied and so T has a fixed point.

***Example* 2.10** Let $X = [0,\infty)$ equipped with the b-metric-like $\sigma_b : X \times X \to \mathbb{R}^+$ defined by

$$\sigma_b(x,y) = \begin{cases} 0, & x = y, \\ (x+y)^2, & x \neq y. \end{cases}$$

It is easy to see that (X,σ_b) is a complete b-metric-like space with the coefficient $s = 2$. If we define the mapping $T : X \to X$ and the functions $\beta : X \times X \to (0, \frac{1}{2})$ and $\alpha : X \times X \to [0,\infty)$ by

$$\mathrm{T}(x) = \begin{cases} \frac{x}{4}, & x \in [0,1], \\ \ln(x^2+1), & x \in (1,\infty), \end{cases}, \quad \alpha(x,y) = \begin{cases} 1, & x,y \in [0,1], \\ 0, & \text{otherwise}, \end{cases},$$

and

$$\beta(x,y)=\begin{cases}\frac{1}{4}, & x,y\in[0,1],\\ \frac{1}{x+y+2}, & \text{otherwise},\end{cases},$$

then the mapping T is triangular α-admissible, which is not continuous. On the other hand, the condition (A) holds. Indeed, if the sequence $\{x_n\}\subseteq X$ satisfies $\alpha(x_n,x_{n+1})\geqslant 1$ or $\alpha(x_{n+1},x_n)\geqslant 1$, and $\lim_{n\to\infty}x_n=x$, then $\{x_n\}\subseteq[0,1]$, and $x=0$. Hence $\alpha(x_n,0)\geqslant 1$ and $\alpha(0,x_n)\geqslant 1$. Next, assume that $x,y\in[0,1]$ with $x<y$. Then, for $\varepsilon>0$, we choose $\delta=\varepsilon$ so that $\varepsilon\leqslant\beta(x,y)\sigma_b(x,y)=\frac{1}{4}(x+y)^2<2(\varepsilon+\delta)$. It implies that

$$\alpha(x,y)\sigma_b(Tx,Ty)=(\frac{x}{4}+\frac{y}{4})^2=\frac{1}{16}(x+y)^2<\varepsilon.$$

Other cases are obvious by the definition of α. Therefore, the mapping T is a generalized α-Meir-Keeler contraction. Also, notice that $\alpha(0,T0)\geqslant 1$ and $\alpha(T0,0)\geqslant 1$. Then, we conclude that all of the assumptions of Proposition 2.6 are satisfied. Moreover, T has a fixed point $x=0$.

3. A new fixed point theorem through rational expression

In this section, we establish a new fixed point theorem through rational expression.

Theorem 3.1 Let (X,σ_b) be a complete b-metric-like space, $T:X\to X$ be a triangular α-admissible mapping and $\beta\in\mathcal{B}_s$. Suppose that the following conditions hold:
(a) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geqslant 1$ and $\alpha(Tx_0,x_0)\geqslant 1$,
(b) if $\{x_n\}$ is a sequence in X such that $x_n\to z$ as $n\to\infty$ and $\alpha(x_n,x_m)\geqslant 1$ for all $n,m\in\mathbb{N}$, then $\alpha(x_n,z)\geqslant 1$ for all $n\in\mathbb{N}$,
(c) for each $\varepsilon>0$, there exists $\delta>0$ satisfying the following condition

$$\begin{aligned}4s\varepsilon\leqslant\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)}+\sigma_b(x,Tx)+\sigma_b(y,Ty)+N_\beta(x,y)<s(4\varepsilon+\delta)\\ \Rightarrow\alpha(x,y)\sigma_b(Tx,Ty)<\varepsilon.\end{aligned}\tag{11}$$

Then T has a fixed point in X.

Proof. Let $x,y\in X$ be given. If $x\neq y$ or $y\neq Ty$ or $x\neq Tx$, then implication (11) gives us

$$\begin{aligned}\alpha(x,y)\sigma_b(Tx,Ty)<\frac{1}{4s}\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)}\\ +\frac{1}{4s}\sigma_b(x,Tx)+\frac{1}{4s}\sigma_b(y,Ty)+\frac{1}{4s}N_\beta(x,y).\end{aligned}\tag{12}$$

Now, let $x_0\in X$ be such that condition (a) holds and define a sequence $\{x_n\}$ in X such that $x_1=Tx_0$, $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$. We may suppose that $x_{n+1}\neq x_n$ for all $n\in\mathbb{N}\cup\{0\}$, otherwise T has trivially a fixed point. Since T is α-admissible, then

$$\alpha(x_0,x_1)=\alpha(x_0,Tx_0)\geqslant 1\quad\Rightarrow\quad\alpha(Tx_0,Tx_1)=\alpha(x_1,x_2)\geqslant 1.\tag{13}$$

Repeating the above procedure, we obtain

$$\alpha(x_n, x_{n+1}) \geqslant 1 \tag{14}$$

for each $n \in \mathbb{N}$. Take $c_n = \sigma_b(x_{n+1}, x_{n+2})$ $(n \in \mathbb{N})$, and replace x by x_n and y by x_{n+1} in (12). Applying the relation (14), we deduce

$$\begin{aligned}
c_n &= \sigma_b(Tx_n, Tx_{n+1}) \\
&\leqslant \alpha(x_n, x_{n+1})\sigma_b(Tx_n, Tx_{n+1}) \\
&< \frac{1}{4s}\sigma_b(x_{n+1}, x_{n+2})\frac{1+\sigma_b(x_n, x_{n+1})}{1+M_\beta(x_n, x_{n+1})} + \frac{1}{4s}\sigma_b(x_n, x_{n+1}) \\
&\quad + \frac{1}{4s}\sigma_b(x_{n+1}, x_{n+2}) + \frac{1}{4s}N_\beta(x_n, x_{n+1}),
\end{aligned}$$

where

$$\begin{aligned}
M_\beta(x_n, x_{n+1}) = \max\{&\sigma_b(x_n, x_{n+1}), \beta(x_n, x_{n+1})\sigma_b(x_n, x_{n+1}), \\
&\beta(x_{n+1}, x_{n+2})\sigma_b(x_{n+1}, x_{n+2})\}.
\end{aligned}$$

We consider two following cases:
Case 1. Assume that $M_\beta(x_n, x_{n+1}) = \beta(x_{n+1}, x_{n+2})\sigma_b(x_{n+1}, x_{n+2})$. Regarding (12) together with Remark *3*, we have

$$\begin{aligned}
c_n &= \sigma_b(Tx_n, Tx_{n+1}) \\
&\leqslant \alpha(x_n, x_{n+1})\sigma_b(Tx_n, Tx_{n+1}) \\
&< \frac{1}{4s}\sigma_b(x_{n+1}, x_{n+2})\frac{1+\sigma_b(x_n, x_{n+1})}{1+\beta(x_{n+1}, x_{n+2})\sigma_b(x_{n+1}, x_{n+2})} + \frac{1}{4s}\sigma_b(x_{n+1}, x_{n+2}) \\
&\quad + \frac{1}{4s}\sigma_b(x_n, x_{n+1}) + \frac{1}{4s}\beta(x_{n+1}, x_{n+2})\sigma_b(x_{n+1}, x_{n+2}) \\
&\leqslant \sigma_b(x_{n+1}, x_{n+2}) = c_n,
\end{aligned}$$

which gives a contradiction.
Case 2. Assume that $M_\beta(x_n, x_{n+1}) = \sigma_b(x_n, x_{n+1})$. Then $N_\beta(x_n, x_{n+1}) = \sigma_b(x_n, x_{n+1})$, too. Applying Remark *3* and the relations (12) and (14), we observe that

$$\begin{aligned}
c_n &= \sigma_b(Tx_n, Tx_{n+1}) \\
&\leqslant \alpha(x_n, x_{n+1})\sigma_b(Tx_n, Tx_{n+1}) \\
&< \frac{1}{4s}\sigma_b(x_{n+1}, x_{n+2})\frac{1+\sigma_b(x_n, x_{n+1})}{1+\sigma_b(x_n, x_{n+1})} + \frac{1}{4s}\sigma_b(x_n, x_{n+1}) + \\
&\quad \frac{1}{4s}\sigma_b(x_{n+1}, x_{n+2}) + \frac{1}{4s}\sigma_b(x_n, x_{n+1}) \\
&\leqslant \frac{1}{2s}\sigma_b(x_{n+1}, x_{n+2}) + \frac{1}{2s}\sigma_b(x_n, x_{n+1}) \\
&\leqslant \frac{1}{2}c_n + \frac{1}{2}c_{n-1}.
\end{aligned}$$

Therefore $c_n < c_{n-1}$ for all n; that is, the sequence $\{c_n\}$ is a strictly decreasing positive sequence in $\mathbb{R}^+$ and it converges to some $r \geqslant 0$. We will show that $r = 0$. Suppose in contrary $r > 0$. We assert that

$$0 < r \leqslant c_n \quad \text{for all} \quad n \in \mathbb{N}. \tag{15}$$

Since the condition (11) holds for every $\varepsilon > 0$, we may choose $\varepsilon = \frac{r}{s}$. Let $\delta = \delta(\frac{r}{s})$ be such that satisfying (11). We know that $2c_n + 2c_{n-1} \downarrow 4r$ as $n \to \infty$. Then there exists $N_0 \in \mathbb{N}$ such that $4r < 2\sigma_b(x_{N_0+1}, x_{N_0+2}) + 2\sigma_b(x_{N_0}, x_{N_0+1}) < 4r + \delta$. Consequently,

$$\begin{aligned}
4s\varepsilon &< 2\sigma_b(x_{N_0+1}, x_{N_0+2}) + 2\sigma_b(x_{N_0}, x_{N_0+1}) \\
&= \sigma_b(x_{N_0+1}, Tx_{N_0+1}) \frac{1 + \sigma_b(x_{N_0}, Tx_{N_0})}{1 + M_\beta(x_{N_0}, x_{N_0+1})} \\
&\quad + \sigma_b(x_{N_0}, x_{N_0+1}) + \sigma_b(x_{N_0+1}, x_{N_0+2}) + N_\beta(x_{N_0}, x_{N_0+1}) \\
&< 4s\varepsilon + \delta \\
&\leqslant s(4\varepsilon + \delta),
\end{aligned}$$

and hence using (11) and (14), we get

$$c_{N_0} = \sigma_b(x_{N_0+1}, x_{N_0+2}) \leqslant \alpha(x_{N_0}, x_{N_0+1})\sigma_b(Tx_{N_0}, Tx_{N_0+1}) < \frac{r}{s} \leqslant r,$$

which leads to a contradiction with the condition (15). Thus, $r = 0$; that is,

$$\lim_{n\to\infty} \sigma_b(x_n, x_{n+1}) = 0. \tag{16}$$

We claim that the sequence $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given and $\delta = \delta(\frac{4\varepsilon}{7})$ be such that satisfying (11). Take $\delta' = \min\{\delta, \frac{4\varepsilon}{7}, 1\}$. From (16), there exists $k \in \mathbb{N}$ such that

$$\sigma_b(x_m, x_{m+1}) < \frac{\delta'}{8}, \quad \forall m \geqslant k. \tag{17}$$

We define the set $\Lambda \subset X$ by

$$\Lambda := \{x_p | p \geqslant k, \sigma_b(x_p, x_k) < s(\frac{4\varepsilon}{7} + \frac{\delta'}{4})\}.$$

We show that $T(\Lambda) \subset \Lambda$. Let $\lambda \in \Lambda$, there exists $p \geqslant k$ such that $\lambda = x_p$ and $\sigma_b(x_p, x_k) < s(\frac{4\varepsilon}{7} + \frac{\delta'}{4})$. If $p = k$, then $T(\lambda) = x_{k+1} \in \Lambda$ by (17). We assume that $p > k$. First, we suppose that $\frac{4s\varepsilon}{7} \leqslant \sigma_b(x_p, x_k)$, so

$$\frac{4s\varepsilon}{7} \leqslant \sigma_b(x_p, x_k) < s(\frac{4\varepsilon}{7} + \frac{\delta'}{4}). \tag{18}$$

Let us prove that

$$\begin{aligned}\frac{\varepsilon}{7} &\leqslant \frac{1}{4s}\sigma_b(x_k,x_{k+1})\frac{1+\sigma_b(x_p,x_{p+1})}{1+M_\beta(x_p,x_k)}\\ &\quad+\frac{1}{4s}\sigma_b(x_p,x_{p+1})\frac{1}{4s}\sigma_b(x_k,x_{k+1})+\frac{1}{4s}N_\beta(x_p,x_k)\\ &< \frac{\varepsilon}{7}+\frac{\delta'}{4}. \end{aligned} \tag{19}$$

Using (17) and since $\frac{4s\varepsilon}{7} \leqslant \sigma_b(x_p, x_k)$, then $N_\beta(x_p, x_k) = \sigma_b(x_p, x_k)$ and $\frac{1+\sigma_b(x_p,x_{p+1})}{1+M_\beta(x_p,x_k)} < 1$. So, from (18), we get

$$\begin{aligned}\frac{\varepsilon}{7} &\leqslant \frac{1}{4s}\sigma_b(x_p,x_k)\\ &\leqslant \frac{1}{4s}\sigma_b(x_k,x_{k+1})\frac{1+\sigma_b(x_p,x_{p+1})}{1+M_\beta(x_p,x_k)}+\frac{1}{4s}\sigma_b(x_p,x_{p+1})\\ &\quad+\frac{1}{4s}\sigma_b(x_k,x_{k+1})+\frac{1}{4s}N_\beta(x_p,x_k).\\ &\leqslant \frac{1}{4s}\sigma_b(x_k,x_{k+1})+\frac{1}{4s}\sigma_b(x_p,x_{p+1})+\frac{1}{4s}\sigma_b(x_k,x_{k+1})+\frac{1}{4s}\sigma_b(x_p,x_k)\\ &\leqslant \frac{3\delta'}{32s}+\frac{1}{4s}\sigma_b(x_p,x_k)\\ &< \frac{3\delta'}{32s}+\frac{s}{4s}(\frac{4\varepsilon}{7}+\frac{\delta'}{4})\\ &= \frac{5\delta'}{32}+\frac{\varepsilon}{7}\\ &\leqslant \frac{\delta'}{4}+\frac{\varepsilon}{7}. \end{aligned}$$

It proves that (19) holds. Then

$$\begin{aligned}\frac{4s\varepsilon}{7} &\leqslant \sigma_b(x_k,Tx_k)\frac{1+\sigma_b(x_p,Tx_p)}{1+M_\beta(x_p,x_k)}\\ &\quad+\sigma_b(x_p,x_{p+1})+\sigma_b(x_k,x_{k+1})+N_\beta(x_p,x_k)\\ &< s(\frac{4\varepsilon}{7}+\delta'), \end{aligned} \tag{20}$$

and according to Lemma 1.12, we conclude that

$$\sigma_b(Tx_p,Tx_k) \leqslant \alpha(x_p,x_k)\sigma_b(Tx_p,Tx_k) < \frac{\varepsilon}{7}. \tag{21}$$

Now, using (σ_b3) together with (17) and (21), we obtain that

$$\sigma_b(Tx_p,x_k) \leqslant s\sigma_b(Tx_p,Tx_k)+s\sigma_b(Tx_k,x_k) < s(\frac{\varepsilon}{7}+\frac{\delta'}{8}) < s(\frac{4\varepsilon}{7}+\frac{\delta'}{4}).$$

This implies that $T\lambda = Tx_p = x_{p+1} \in \Lambda$.
Next, we suppose that $\sigma_b(x_p, x_k) < \frac{4s\varepsilon}{7}$, then $N_\beta(x_p, x_k) < \frac{4s\varepsilon}{7}$. From (12), we derive

$$\begin{aligned}
\sigma_b(Tx_p, x_k) &\leqslant s\sigma_b(Tx_p, Tx_k) + s\sigma_b(Tx_k, x_k) \\
&\leqslant s\alpha(x_p, x_k)\sigma_b(Tx_p, Tx_k) + s\sigma_b(Tx_k, x_k) \\
&< \frac{1}{4}\sigma_b(x_k, x_{k+1})\frac{1+\sigma_b(x_p, x_{p+1})}{1+M_\beta(x_p, x_k)} + \frac{1}{4}\sigma_b(x_p, x_{p+1}) \\
&\quad + \frac{1}{4}\sigma_b(x_k, x_{k+1}) + \frac{1}{4}N_\beta(x_p, x_k) + s\sigma_b(x_{k+1}, x_k).
\end{aligned}$$

We consider two following cases:
(i) If $\sigma_b(x_p, x_{p+1}) \leqslant \sigma_b(x_p, x_k)$, then

$$\begin{aligned}
\sigma_b(Tx_p, x_k) &\leqslant \frac{1}{4}\sigma_b(x_k, x_{k+1}) + \frac{1}{4}\sigma_b(x_p, x_{p+1}) + \frac{1}{4}\sigma_b(x_k, x_{k+1}) \\
&\quad + \frac{1}{4}N_\beta(x_p, x_k) + s\sigma_b(x_{k+1}, x_k) \\
&< \frac{3\delta'}{32} + \frac{s\varepsilon}{7} + \frac{s\delta'}{8} \\
&< s(\frac{7\delta'}{32} + \frac{4\varepsilon}{7}) \\
&< s(\frac{\delta'}{4} + \frac{4\varepsilon}{7}).
\end{aligned}$$

(ii) If $\sigma_b(x_p, x_{p+1}) > \sigma_b(x_p, x_k)$, then

$$\begin{aligned}
\sigma_b(Tx_p, x_k) &\leqslant s(\sigma_b(Tx_p, x_p) + \sigma_b(x_p, x_k)) \\
&< s(\sigma_b(x_{p+1}, x_p) + \sigma_b(x_p, x_{p+1})) \\
&< s\frac{\delta'}{4} \\
&< s(\frac{4\varepsilon}{7} + \frac{\delta'}{4}).
\end{aligned}$$

So $T\lambda = Tx_p = x_{p+1} \in \Lambda$. Hence, $T(\Lambda) \subset \Lambda$. Thus,

$$\sigma_b(x_m, x_k) < s(\frac{4\varepsilon}{7} + \frac{\delta'}{4}), \quad \forall m > k. \tag{22}$$

Now, for all $m, n \in \mathbb{N}$ such that $m > n > k$ and by (22), we get

$$\sigma_b(x_m, x_n) \leqslant s\sigma_b(x_m, x_k) + s\sigma_b(x_k, x_n) < s^2(\frac{8\varepsilon}{7} + \frac{\delta'}{2}) \leqslant 2s^2\varepsilon.$$

It follows that $\lim_{m,n\to\infty} \sigma_b(x_m, x_n) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in X and since

X is complete, there exists $z \in X$ such that

$$\lim_{n,m\to\infty}\sigma_b(x_n,x_m)=\lim_{n\to\infty}\sigma_b(x_n,z)=\sigma_b(z,z)=0.$$

Finally, from (12), we have

$$\begin{aligned}\sigma_b(Tz,z) &\leqslant s\sigma_b(Tz,Tx_n)+s\sigma_b(x_{n+1},z)\\ &\leqslant s\alpha(z,x_n)\sigma_b(Tz,Tx_n)+s\sigma_b(x_{n+1},z)\\ &< \frac{1}{4}\sigma_b(x_n,x_{n+1})\frac{1+\sigma_b(z,Tz)}{1+M_\beta(z,x_n)}+\frac{1}{4}\sigma_b(z,Tz)+\frac{1}{4}\sigma_b(x_n,Tx_n)\\ &\quad +\frac{1}{4}N_\beta(z,x_n)+s\sigma_b(x_{n+1},z).\end{aligned}$$

Applying the definition of $N_\beta(z,x_n)$, the right hand side of the above inequality tends to $\frac{1}{4}\sigma_b(z,Tz)+\frac{1}{8}\beta(Tz,z)\sigma_b(Tz,z)$ when n tends to infinity. Thus, we get $\sigma_b(Tz,z)<\frac{3}{8}\sigma_b(Tz,z)$. Consequently, $\sigma_b(Tz,z)=0$ and $Tz=z$. ■

Theorem 3.2 Let (X,σ_b) be a b-metric-like space, $T:X\to X$ be an α-admissible mapping and $\beta\in\mathcal{B}_s$. Assume that there exists a function $\theta:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the following conditions:
(a) $\theta(0)=0$ and $\theta(t)>0$ for every $t>0$,
(b) θ is nondecreasing and right continuous,
(c) for every $\varepsilon>0$, there exists $\delta>0$ such that

$$\begin{aligned}&4\varepsilon\leqslant\theta\Big(\frac{1}{s}\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)}+\frac{1}{s}\sigma_b(x,Tx)+\frac{1}{s}\sigma_b(y,Ty)+\frac{1}{s}N_\beta(x,y)\Big)<4\varepsilon+\delta\\ &\quad\Rightarrow\quad \theta(4\alpha(x,y)\sigma_b(Tx,Ty))<4\varepsilon\end{aligned}$$

for all $x,y\in X$. Then (11) is satisfied.

Proof. Fix $\varepsilon>0$. Since $\theta(4\varepsilon)>0$ by (c), there exists $\delta>0$ such that

$$\begin{aligned}\theta(4\varepsilon)&\leqslant\theta\Big(\frac{1}{s}\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)}+\frac{1}{s}\sigma_b(x,Tx)+\frac{1}{s}\sigma_b(y,Ty)+\frac{1}{s}N_\beta(x,y)\Big)\\ &<\theta(4\varepsilon)+\delta\\ &\Rightarrow\quad\theta(4\alpha(x,y)\sigma_b(Tx,Ty))<\theta(4\varepsilon).\end{aligned}\tag{23}$$

From right continuity of θ, there exists $\delta'>0$ such that $\theta(4\varepsilon+\delta')<\theta(4\varepsilon)+\delta$. Fix $x,y\in X$ such that

$$4\varepsilon\leqslant\frac{1}{s}\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)}+\frac{1}{s}\sigma_b(x,Tx)+\frac{1}{s}\sigma_b(y,Ty)+\frac{1}{s}N_\beta(x,y)<4\varepsilon+\delta'.$$

Since θ is nondecreasing, we get

$$\begin{aligned}\theta(4\varepsilon) &\leqslant \theta\Big(\frac{1}{s}\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)}+\frac{1}{s}\sigma_b(x,Tx)+\frac{1}{s}\sigma_b(y,Ty)+\frac{1}{s}N_\beta(x,y)\Big)\\ &< \theta(4\varepsilon+\delta') < \theta(4\varepsilon)+\delta.\end{aligned}$$

Then, by (23), we have

$$\theta(4\alpha(x,y)\sigma_b(Tx,Ty)) < \theta(4\varepsilon).$$

It enforces that $\alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon$, i.e., (11) is satisfied. ■

Corollary 3.3 Let (X,σ_b) be a complete b-metric-like space, $T : X \to X$ be a triangular α-admissible mapping, φ be a locally integrable function from $\mathbb{R}^+$ into itself such that $\int_0^t \varphi(s) > 0$ for all $t > 0$, and $\beta \in \mathcal{B}_s$. Also, suppose that the following conditions hold:
a) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geqslant 1$ and $\alpha(Tx_0,x_0) \geqslant 1$,
b) if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n,x_m) \geqslant 1$ for all $n,m \in \mathbb{N}$, then $\alpha(x_n,z) \geqslant 1$ for all $n \in \mathbb{N}$,
c) for each $x,y \in X$,

$$\int_0^{4\alpha(x,y)\sigma_b(Tx,Ty)}\varphi(t)dt \leqslant c\int_0^{\frac{1}{s}\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)}+\frac{1}{s}\sigma_b(x,Tx)+\frac{1}{s}\sigma_b(y,Ty)+\frac{1}{s}N_\beta(x,y)}\varphi(t)dt,$$

where $c \in (0,\frac{1}{4s})$ is a constant.
Then T has a fixed point.

Proof. As a result of Theorem 3.2, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}4\varepsilon &\leqslant \int_0^{\frac{1}{s}\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)}+\frac{1}{s}\sigma_b(x,Tx)+\frac{1}{s}\sigma_b(y,Ty)+\frac{1}{s}N_\beta(x,y)}\varphi(t)dt < 4\varepsilon+\delta\\ &\Rightarrow \int_0^{4\alpha(x,y)\sigma_b(Tx,Ty)}\varphi(t)dt < 4\varepsilon,\end{aligned}$$

then (11) is satisfied.
Fix $\varepsilon > 0$. Take $\delta = 4\varepsilon(\frac{1}{4c}-1)$. Then

$$\begin{aligned}\int_0^{4\alpha(x,y)\sigma_b(Tx,Ty)}\varphi(t)dt &\leqslant c\int_0^{\frac{1}{s}\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)}+\frac{1}{s}\sigma_b(x,Tx)+\frac{1}{s}\sigma_b(y,Ty)+\frac{1}{s}N_\beta(x,y)}\varphi(t)dt\\ &< c(4\varepsilon+\delta) = \varepsilon < 4\varepsilon.\end{aligned}$$

Now, all conditions of Theorem 3.1 holds. Therefore, f has a fixed point. ■

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