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Common best proximity points for $(\psi - \phi)-$ generalized weak proximal contraction type mappings

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Abstract. In this paper, we introduce a pair of generalized proximal contraction mappings and prove the existence of a best proximity point for such mappings in a complete metric space. We provide some examples to illustrate the validity of our result. Our results extend some of the results in the literature.

Keywords: Best proximity point, common best proximity points, $(\psi - \phi)$-generalized proximal contraction, lower semi continuous functions.

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1. Introduction

A classical best approximation theorem was introduced by Fan[2], which states that: "if A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$". Afterwards, Prolla [6], Reich [7], and Sehgal and Singh [8] have derived extensions of Fan Theorem in many directions. The common fixed point theorem insists to the authors to investigation on common best proximity point theorem for non-self mappings. The common best proximity point theorem, assures a common optimal solution at which both the real valued multi-objective functions $x \to d(x, Sx)$

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and $x \to d(x, Tx)$ attain the global minimal value $d(A, B)$. A number of authors have improved, generalized and extended this basic result either by defining a new contractive mapping in the context of a complete metric space or extend best proximity results from fixed point theory(see [3, 4, 9, 11]).

In 2015 , Singh et al. [10] introduced a new class of contraction mappings called generalized weak contractions for self mappings. Let Ψ denote the class of all function $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(a) ψ is monotone nondecreasing;
(b) ψ is continuous;
(c) $\psi(t) = 0 \iff t = 0$.

Further, let Φ denotes the class of all functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(a) ϕ is lower semi-continuous function;
(b) $\phi(t) = 0 \iff t = 0$.

The following fixed point theorem is proved by Singh et al. [10].

Theorem 1.1 Let X be a complete metric space. $T : X \to X$ be a self map such that for every $x, y \in X$,

$$\frac{1}{2}d(x, Tx) \leqslant d(x, y) \;\text{ implies }\; \psi(d(Tx, Ty)) \leqslant \psi(M_g(Tx, Ty)) - \phi(M_g(Tx, Ty)), \quad (1)$$

where $\psi \in \Psi, \phi \in \Phi$ and $M_g(Tx, Ty) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$. Then T has a unique fixed point.

In [10], the authors posted a question that whether Theorem 1.1 can be extended to a pair of maps and they speculate the condition in which a pair of maps have a unique fixed point. Inspired by this, the purpose of this paper is to obtain best proximity point theorems for a pair of maps with certain weak contractions in metric spaces. As an application, we proved a fixed point theorem for a pair of self-maps suggested by Singh et al. [10].

2. Preliminaries

Definition 2.1 [5] Let (X, d) be a metric space and A and B be two non-empty subsets of X and $T: A \to B$ be a mapping. A point $x \in A$ is said to be a best proximity point of T if it satisfies the condition that $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y): x \in A, y \in B\}$.

Let A and B be two non-empty subsets of a metric space (X, d). We define A_0 and B_0 as defined by Kirk et al. [5]:

$$A_0 = \{x \in A: d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B: d(x, y) = d(A, B) \text{ for some } x \in A\},$$

where $d(A, B) = \inf\{d(x, y): x \in A, y \in B\}$ is the distance between the sets A and B.

In [5], the authors presented sufficient conditions for the sets A_0 and B_0 are to be non-empty.

Remark 1 *If $A = B$, then $A_0 = A = B = B_0$.*

We denote the set of all best proximity points of T by $P_T(A)$; that is,

$$P_T(A) = \{x \in A : d(x, Tx) = d(A, B)\}.$$

3. Main results

In this section we state and prove the existence and uniqueness theorem of common best proximity point of certain weak contractive maps. This theorem extends, improves and generalizes some of the results in the literature on best proximity points.

Definition 3.1 [1] An element $x \in A$ is said to be a common best proximity point of the non-self mappings $S: A \to B$ and $T: A \to B$ if these satisfy the condition that

$$d(x, Sx) = d(x, Tx) = d(A, B).$$

It should be noted that a common best proximity point is that value at which both the real valued functions $x \to d(x, Sx)$ and $x \to d(x, Tx)$ on A attain global minimum, since $d(x, Sx) \geq d(A, B)$ and $d(x, Tx) \geq d(A, B)$ for all x. Further, if the underlying mappings are self-mappings, the common best proximity point becomes a common fixed point.

We denote the set of all common best proximity points for a pair of maps T and S by $P_{T,S}(A)$; that is, $P_{T,S}(A) = \{x \in A : d(x, Tx) = d(x, Sx) = d(A, B)\}$.

Definition 3.2 Let (X, d) be a metric space, A, B be nonempty subsets of X, and $T, S : A \to B$ be non-self mappings. We say that (T, S) is a (ψ, ϕ)-generalized weak proximal contraction pair if, for all $x, y, u, v \in A$

$$\left.\begin{array}{l} \frac{1}{2}\min\{d(x,u), d(y,v)\} \leqslant d(x,y) \\ d(u, Tx) = d(A, B) \\ d(v, Sy) = d(A, B) \end{array}\right\} \implies \psi(d(u,v)) \leqslant \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)), \tag{2}$$

where $M(x, y, u, v) = \max\{d(x, y), d(x, u), d(y, v), \frac{d(x,v)+d(y,u)}{2}\}$, $\psi \in \Psi$ and $\phi \in \Phi$.

Now we prove the following theorem.

Theorem 3.3 Let A and B be two nonempty, closed subsets of a complete metric space (X, d). Let $T, S : A \to B$ be a pair of mappings. Suppose that the following conditions are satisfied:

(i) (T, S) is (ψ, ϕ)-generalized weak proximal contraction pair;
(ii) $A_0 \neq \emptyset$, $T(A_0) \subseteq B_0$, and $S(A_0) \subseteq B_0$;
(iii) either T or S is continuous.

Then T and S have a unique common best proximity point.

Proof. We first show that for T and S satisfying the hypothesis, $P_T(A) = P_S(A)$; that is, z is a best proximity point of T if and only if it is a best proximity point of S. Let $z \in P_T(A)$. This implies $d(z, Tz) = d(A, B)$. Hence, $z \in A_0$. Since $z \in A_0$ and $S(A_0) \subseteq B_0$ there exist $w \in A_0$ with $d(w, Sz) = d(A, B)$. Now, $\frac{1}{2}\min\{d(z, z), d(z, w)\} \leqslant d(z, z)$. Thus, from (2), we have

$$\psi(d(z, w)) \leqslant \psi(M(z, z, z, w)) - \phi(M(z, z, z, w)),$$

where

$$M(z,z,z,w) = \max\{d(z,z), d(z,z), d(z,w), \frac{d(z,w)+d(z,z)}{2}\} = d(z,w).$$

This implies that $\psi(d(z,w)) \leqslant \psi(d(z,w)) - \phi(d(z,w))$. Thus, $\phi(d(z,w)) = 0$. By the property of ϕ, we get $d(z,w) = 0$. Hence, $z = w$. Therefore, $z \in P_S(A)$. This shows that $P_T(A) \subseteq P_S(A)$. Similarly, we can show that $P_S(A) \subseteq P_T(A)$. Hence $P_T(A) = P_S(A)$.

Now, we prove the existence of common best proximity points. Since $A_0 \neq \emptyset$, we take arbitrary $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ there exist $x_1 \in A$ such that $d(x_1, Tx_0) = d(A,B)$. Then by the definition of A_0 we have that $x_1 \in A_0$. Again since $S(A_0) \subseteq B_0$ and $x_1 \in A_0$ there exist $x_2 \in A$ such that $d(x_2, Sx_1) = d(A,B)$. Similarly $x_2 \in A_0$. On continuing this process by induction, we construct a sequence $\{x_n\} \subseteq A_0$ such that

$$d(x_{2n+1}, Tx_{2n}) = d(A,B) \text{ and } d(x_{2n+2}, Sx_{2n+1}) = d(A,B) \tag{3}$$

for all $n = 0, 1, 2, \cdots$. Now, for any $n \in \mathbb{N}$, we have

$$\frac{1}{2}\min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \leqslant d(x_{2n}, x_{2n+1}).$$

By (i), there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\begin{aligned}\psi(d(x_{2n+1}, x_{2n+2})) \leqslant\ & \psi(M(x_{2n}, x_{2n+1}, x_{2n+1}, x_{2n+2})) \\ & - \phi(M(x_{2n}, x_{2n+1}, x_{2n+1}, x_{2n+2})),\end{aligned} \tag{4}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}, x_{2n+1}, x_{2n+2}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ &\quad \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2}\}. \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2}\}. \end{aligned}$$

Here by triangular inequality we have that

$$d(x_{2n}, x_{2n+2}) \leqslant 2\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}.$$

Thus, for $n = 0, 1, 2\cdots$, we have

$$M(x_{2n}, x_{2n+1}, x_{2n+1}, x_{2n+2}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}.$$

Case i: Suppose that there exist $n_0 \in \mathbb{N}$ such that

$$M(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}, x_{2n_0+2}) = d(x_{2n_0+1}, x_{2n_0+2}).$$

From (4), we have

$$\psi(d(x_{2n_0+1}, x_{2n_0+2})) \leqslant \psi(d(x_{2n_0+1}, x_{2n_0+2})) - \phi(d(x_{2n_0+1}, x_{2n_0+2})).$$

Hence, $\phi(d(x_{2n_0+1}, x_{2n_0+2})) = 0$. From the property of ϕ, we have $d(x_{2n_0+1}, x_{2n_0+2}) = 0$. This implies $x_{2n_0+1} = x_{2n_0+2}$. From (3), $d(x_{2n_0+2}, Sx_{2n_0+1}) = d(x_{2n_0+1}, Sx_{2n_0+1}) = d(A, B)$. Thus x_{2n_0+1} is best proximity points of S. Since every best proximity point of S is also a best proximity point of T, x_{2n_0+1} is a best proximity point of T. Hence x_{2n_0+1} is a common best proximity point of T and S.
Case ii: Suppose $M(x_{2n}, x_{2n+1}, x_{2n+1}, x_{2n+2}) = d(x_{2n}, x_{2n+1})$ for $n = 0, 1, 2, \cdots$. From (4), we have

$$\psi(d(x_{2n+1}, x_{2n+2})) \leqslant \psi(d(x_{2n}, x_{2n+1})) - \phi(d(x_{2n}, x_{2n+1})) \leqslant \psi(d(x_{2n}, x_{2n+1})). \tag{5}$$

Since ψ is non-decreasing, the above inequality yields the following inequality:

$$d(x_{2n+1}, x_{2n+2}) \leqslant d(x_{2n}, x_{2n+1}). \tag{6}$$

Now, if we put $d_n = d(x_n, x_{n+1})$, then we get $d_{2n+1} \leqslant d_{2n}$ for all n. Also, we have $d_{2n+2} \leqslant d_{2n+1}$, which implies that the sequence $\{d_n\}$ is decreasing. So there is a $r \geqslant 0$ such that $d_n \to r$ as $n \to \infty$. We wish to show that $r = 0$. Since ϕ is lower semi continuous, we have that

$$\phi(r) \leqslant \liminf_{n\to\infty} d(x_{2n}, x_{2n+1}).$$

By taking limit superior in (5) we get

$$\limsup_{n\to\infty} \psi(d(x_{2n+1}, x_{2n+2})) \leqslant \limsup_{n\to\infty} \psi(d(x_{2n}, x_{2n+1})) - \liminf_{n\to\infty} \phi(d(x_{2n}, x_{2n+1})).$$

Consequently, we have $\psi(r) \leqslant \psi(r) - \phi(r)$ which implies $\phi(r) = 0$. So $r = 0$, i.e.

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{7}$$

Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, it is enough to prove that the subsequence $\{x_{2n}\}$ of $\{x_n\}$ is Cauchy sequence in A_0. Contrarily, let there exists an $\epsilon > 0$ for which the subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_{2n}\}$ such that $m(k)$ is the smallest integer satisfying, for all $k > 0$, $m(k) > n(k) > k$ imply that

$$d(x_{2n(k)}, x_{2m(k)}) \geqslant \epsilon, \tag{8}$$

$$d(x_{2n_k}, x_{2m_k-2}) < \epsilon. \tag{9}$$

Using triangular inequality, (8) and (9), we get that

$$\begin{aligned}\epsilon \leqslant d(x_{2n(k)}, x_{2m(k)}) &\leqslant d(x_{2n_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2m_k}) \\ &< \epsilon + d(x_{2m_k-2}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2m_k}).\end{aligned}$$

By letting $k \to \infty$ in the above inequality and using (7), we get that

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)}) = \epsilon. \tag{10}$$

Using triangular inequality again, we have

$$d(x_{2m_k}, x_{2n_k}) - d(x_{2n_k+1}, x_{2n_k}) \leqslant d(x_{2m_k}, x_{2n_k+1}) \leqslant d(x_{2m_k}, x_{2n_k}) + d(x_{2n_k}, x_{2n_k+1}).$$

Letting $k \to \infty$ in the above inequality and using (7), $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \epsilon$. Similarly, we can prove that $\lim_{k\to\infty} d(x_{2n_k+1}, x_{2m_k+1}) = \epsilon$ and $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+2}) = \epsilon$. Now, we have

$$d(x_{2m_k+1}, Tx_{2m_k}) = d(A, B)\ ,\ d(x_{2n_k+2}, Sx_{2n_k+1}) = d(A, B). \tag{11}$$

Suppose for any $l \geqslant 1$, there exist $2m_{k_l}$ and $2n_{k_l}$ such that $2m_{k_l} > 2n_{k_l}$ and $d(x_{2m_{k_l}}, x_{2n_{k_l}+1}) < d(x_{2m_{k_l}}, x_{2m_{k_l}+1})$. As $l \to \infty$, from the above inequality we get

$$\epsilon = \lim_{l\to\infty} d(x_{2m_{k_l}}, x_{2n_{k_l}+1}) \leqslant \lim_{l\to\infty} d(x_{2m_{k_l}}, x_{2m_{k_l}+1}) = 0,$$

which is not true. Therefore there exist $l \in N$ such that for any $k \geqslant l$

$$\frac{1}{2}\min\{d(x_{2m_k}, x_{2m_k+1}), d(x_{2n_k+1}, x_{2n_k+2})\} \leqslant d(x_{2m_k}, x_{2n_k+1}). \tag{12}$$

From (11) and (12), for any $k \geqslant l$, we have

$$\begin{aligned}\psi(d(x_{2m_k+1}, x_{2n_k+2})) \leqslant{}& \psi(M(x_{2m_k}, x_{2n_k+1}, x_{2m_k+1}, x_{2n_k+2})) \\ &- \phi(M(x_{2m_k}, x_{2n_k+1}, x_{2m_k+1}, x_{2n_k+2})),\end{aligned} \tag{13}$$

where

$$\begin{aligned}M(x_{2m_k}, x_{2n_k+1}, x_{2m_k+1}, x_{2n_k+2}) = \max\{&d(x_{2m_k}, x_{2n_k+1}), d(x_{2m_k}, x_{2m_k+1}), \\ &d(x_{2n_k+1}, x_{2n_k+2}), \frac{d(x_{2m_k}, x_{2n_k+2}) + d(x_{2n_k+1}, x_{2m_k+1})}{2}\}.\end{aligned} \tag{14}$$

Letting $k \to \infty$ in (14) and using (13), the continuity of ψ, and the lower semi continuity of ϕ, we get that $\psi(\epsilon) \leqslant \psi(\epsilon) - \phi(\epsilon)$, which implies that $\phi(\epsilon) = 0$. From the property ϕ, $\epsilon = 0$. This contradict the fact that $\epsilon > 0$. So, $\{x_{2n}\}$ is Cauchy sequence. Consequently $\{x_n\}$ is Cauchy sequence in A_0. Since (X, d) is a complete metric space and A is a closed subset of X, there exist $v \in A$ such that $x_n \to v$. With out loss of generality we may assume that T is continuous. From (3), we have

$$d(A, B) = \lim_{n\to\infty} d(x_{2n+1}, Tx_{2n}) = d(v, Tv).$$

Therefore, v is a best proximity point of T. Since we have proved that every best proximity point of T is also a best proximity point of S, v is also a best proximity point of S. Hence v is a common best proximity of the pair T and S.

Now, we show that this v is unique. Let $u \in P_{T,S}(A) = P_T(A) = P_S(A)$. Then we have

$$d(u, Tu) = d(A, B), d(v, Tv) = d(A, B).$$

Furthermore, we have $\frac{1}{2}\min\{d(u,v), d(v,v)\} \leqslant d(u,v)$. Thus, inequality (2) implies that

$$\psi(d(u,v)) \leqslant \psi(M(u,v,u,v)) - \phi(M(u,v,u,v)),$$

where $M(u,v,u,v) = \max\{d(u,v), d(u,u), d(v,v), \frac{d(u,v)+d(u,v)}{2}\} = d(u,v)$. Therefore, $\psi(d(u,v)) \leqslant \psi(d(u,v)) - \phi(d(u,v))$. Hence, $\phi(d(u,z)) = 0$. So $d(u,v) = 0$. Consequently, $u = v$. Hence T and S have a unique common best proximity point. ■

Now, we draw some corollaries to our theorem. If we relax condition (i) in Theorem 3.3, we obtain the following corollary.

Corollary 3.4 Let A and B be two nonempty, closed subsets of a complete metric space (X,d). Let $T, S : A \to B$ be mappings. Suppose the following conditions are satisfied:

(i) for every $x, y, u, v \in A$

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Sy) = d(A,B) \end{array}\right\} \implies \psi(d(u,v)) \leqslant \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)), \tag{15}$$

where $M(x,y,u,v) = \max\{d(x,y), d(x,u), d(y,v), \frac{d(x,v)+d(y,u)}{2}\}$, $\psi \in \Psi$ and $\phi \in \Phi$;

(ii) $A_0 \neq \emptyset$, $T(A_0) \subseteq B_0$,and $S(A_0) \subseteq B_0$;

(iii) either T or S is continuous.

Then T and S have a unique common best proximity point.

If we take $\psi(t) = t$ and $\phi(t) = (1-k)t$, where $0 \leqslant k < 1$ in Theorem 3.3 we get the following corollary.

Corollary 3.5 Let A and B be two nonempty closed subsets of a complete metric space (X,d). Let $T, S : A \to B$ be mappings. Suppose that for all $x, y, u, v \in A$ the following conditions are satisfied:

(i) for every $x, y, u, v \in A$

$$\left.\begin{array}{l} \frac{1}{2}\min\{d(x,u), d(y,v)\} \leqslant d(x,y) \\ d(u,Tx) = d(A,B) \\ d(v,Sy) = d(A,B) \end{array}\right\} \implies d(u,v) \leqslant kM(x,y,u,v), \tag{16}$$

where $M(x,y,u,v) = \max\{d(x,y), d(x,u), d(y,v), \frac{d(x,v)+d(y,u)}{2}\}$;

(ii) $A_0 \neq \emptyset$, $T(A_0) \subseteq B_0$,and $S(A_0) \subseteq B_0$;

(iii) either T or S is continuous.

Then T and S have a unique common best proximity point.

The following example shows that Theorem 3.3 is a generalization of Corollary 3.4. Further, it is interesting to note that the maps T and S of Example 3.6 does not satisfy the hypotheses of the Corollary 3.4.

***Example* 3.6** Let $X = R^3$ and $d : X \times X \to R$ defined by

$$d((x_1,x_2,x_3),(y_1,y_2,y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|.$$

Also, let

$$A = \{(0,0,0),(0,4,0),(4,0,0),(4,5,0),(5,4,0)\},$$
$$B = \{(0,0,1),(0,4,1),(4,0,1),(4,5,1),(5,4,1)\}.$$

We define $T, S : A \to B$ by

$$T(x_1,x_2,0) = \begin{cases} (x_1,0,1), & \text{if } x_1 \leqslant x_2; \\ (0,x_2,1), & \text{if } x_1 > x_2. \end{cases} \quad \text{and} \quad S(x_1,x_2,0) = \begin{cases} (x_1,0,1), & \text{if } x_1 \leqslant x_2; \\ (0,0,1), & \text{if } x_1 > x_2. \end{cases}$$

Clearly $d(A,B) = 1$, $A_0 = A$, $B_0 = B$, $T(A_0) \subseteq B_0$, $S(A_0) \subseteq B_0$ and T is continuous. Now we define functions $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t^2$ and $\phi(t) = \frac{t}{2}$.

Now, we show that T and S satisfies (i). For this regard, let $x = (x_1,x_2,0)$, $y = (y_1,y_2,0)$, $u = (u_1,u_2,0)$ and $v = (v_1,v_2,0) \in A$ such that

$$\tfrac{1}{2}\min\{d(x,u), d(y,v)\} \leqslant d(x,y), d(u,Tx) = d(A,B), d(v,Sy) = d(A,B). \tag{17}$$

Now, from (17) we get that

$$v = (v_1,v_2,0) = \begin{cases} (y_1,0,0), & \text{if } y_1 \leqslant y_2, \\ (0,0,0), & \text{if } y_1 > y_2. \end{cases}$$

Case i) Let $x = (0,0,0)$ and $y = (y_1,y_2,0)$. From $d((u_1,u_2,0),T(0,0,0)) = 1$ we get $u_1 = 0$ and $u_2 = 0$. Thus, $u = (0,0,0)$.

Sub-case i) if $y_1 \leqslant y_2$ then $v = (v_1,v_2,0) = (y_1,0,0)$. Here for all $y = (y_1,y_2,0) \in A$ and $y_1 \leqslant y_2$ we get $d(u,v) = y_1$, $d(x,y) = y_1 + y_2$, $d(x,u) = 0$, $d(y,v) = y_2$ and $\frac{d(x,v)+d(y,u)}{2} = \frac{2y_1+y_2}{2}$. In this sub-case $M(x,y,u,v) = y_1 + y_2$. Moreover, $(y_1)^2 \leqslant (y_1 + y_2)^2 - \frac{y_1+y_2}{2}$. Thus, $\psi(d(u,v)) \leqslant \psi(M(x,y,u,v)) - \phi(M(x,y,u,v))$ is true.

Sub-case ii) if $y_1 > y_2$ then $v = (v_1,v_2,0) = (0,0,0)$. Here for all $y = (y_1,y_2,0) \in A$ and $y_1 > y_2$ we get $d(u,v) = 0$, $d(x,y) = y_1 + y_2$, $d(x,u) = 0$, $d(y,v) = y_1 + y_2$ and $\frac{d(x,v)+d(y,u)}{2} = \frac{y_1+y_2}{2}$. In this sub-case, $M(x,y,u,v) = y_1 + y_2$. Moreover, $0 \leqslant (y_1+y_2)^2 - \frac{y_1+y_2}{2}$. Therefore, $\psi(d(u,v)) \leqslant \psi(M(x,y,u,v)) - \phi(M(x,y,u,v))$ is true.

Case ii) Let $x = (0,4,0)$ and $y = (y_1,y_2,0)$. From $d((u_1,u_2,0),T(0,4,0)) = 1$, we get $u_1 = 0$ and $u_2 = 0$. Thus, $u = (0,0,0)$.

Sub-case i) if $y_1 \leqslant y_2$ then $v = (y_1,0,0)$. Now $d(u,v) = y_1$, $d(x,y) = |y_2 - 4|$, $d(x,u) = 4$, $d(y,v) = y_2$ and $\frac{d(x,v)+d(y,u)}{2} = \frac{2y_1+y_2+4}{2}$. Here for all $y = (y_1,y_2,0) \in A$ and $y_1 \leqslant y_2$ we observe that $y_1^2 \leqslant 4^2 - \frac{4}{2}$. Moreover $f(x) = x^2 - \frac{x}{2}$ is an increasing function on the interval $[\frac{1}{4},\infty)$ and $M(x,y,u,v) \geqslant 4$ imply that $[d(u,v)]^2 = y_1^2 \leqslant [M(x,y,u,v)]^2 - \frac{M(x,y,u,v)}{2}$. Therefore, $\psi(d(u,v)) \leqslant \psi(M(x,y,u,v)) - \phi(M(x,y,u,v))$.

Sub-case ii) if $y_1 > y_2$ then $(v_1,v_2,0) = (0,0,0)$. Now $d(u,v) = 0$, $d(x,u) = 4$, $d(y,v) = y_1 + y_2$. Since $0 \leqslant 4^2 - \frac{4}{2}$ by similar reason as in sub-case (i) we get the inequality, $\psi(d(u,v)) \leqslant \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)))$.

Case iii) Let $x = (4, 0, 0)$ and $y = (y_1, y_2, 0)$. From $d((u_1, u_2, 0), T(4, 0, 0)) = 1$ we get $u_1 = 0$ and $u_2 = 0$. Thus, $u = (0, 0, 0)$.

Sub-case i) if $y_1 \leqslant y_2$ then $v = (y_1, 0, 0)$. Now, $d(u, v) = y_1$ and $M(x, y, u, v) = max\{|y_1 - 4| + y_2, 4, y_2, \frac{y_1+|y_1-4|+y_2}{2}\}$. Here for all $y = (y_1, y_2, 0) \in A$ and $y_1 \leqslant y_2$ we can easily observe that $y_1^2 \leqslant y_2^2 - \frac{y_2}{2}$. Since $M(x, y, u, v) \geqslant 4$ we conclude that $y_1^2 = \psi(d(u, v)) \leqslant \psi(M(x, y, u, v)) - \phi(M(x, y, u, v))$.

Sub-case ii) if $y_1 > y_2$ then $v = (v_1, v_2, 0) = (0, 0, 0)$. Now, $d(u, v) = 0$ and $M(x, y, u, v) = max\{|y_1 - 4| + y_2, 4, y_1 + y_2, \frac{y_1+y_2+4}{2}\}$. For all $y = (y_1, y_2, 0) \in A$ and $y_1 > y_2$, we observe that $0 \leqslant 4^2 - \frac{4}{2}$. Thus, $\psi(d(u, v)) = 0 \leqslant 14 \leqslant \psi(M(x, y, u, v)) - \phi(M(x, y, u, v))$.

Case iv). Let $x = (4, 5, 0)$ and $y = (y_1, y_2, 0)$. From $d((u_1, u_2, 0), T(4, 5, 0)) = 1$, we get $u_1 = 4$ and $u_2 = 0$. Thus, $u = (4, 0, 0)$.

Sub-case i) if $y_1 \leqslant y_2$ then $v = (y_1, 0, 0)$. Now, $d(u, v) = |y_1 - 4|$ and $M(x, y, u, v) = max\{|y_1 - 4| + |y_2 - 5|, 5, y_2, \frac{2|y_1-4|+y_2+5}{2}\}$. Here for all $y = (y_1, y_2, 0) \in A$ and $y_1 \leqslant y_2$, we observe that $(|y_1 - 4|)^2 \leqslant 5^2 - \frac{5}{2}$ and $M(x, y, u, v) \geqslant 5$. These imply that $\psi(d(u, v)) = (|y_1 - 4|)^2 \leqslant \psi(M(x, y, u, v)) - \phi(M(x, y, u, v))$.

Sub-case ii) if $y_1 > y_2$ then $v = (v_1, v_2, 0) = (0, 0, 0)$. Now, $d(u, v) = 4$ and $M(x, y, u, v) = max\{|y_1 - 4| + |y_2 - 5|, 5, y_1 + y_2, \frac{9+|y_1-4|+y_2}{2}\}$. We can easily observe that $4^2 \leqslant 5^2 - \frac{5}{2}$ and $M(x, y, u, v) \geqslant 5$. Thus, $\psi(d(u, v)) = 4^2 \leqslant 5^2 - \frac{5}{2} \leqslant [M(x, y, u, v)]^2 - \frac{M(x,y,u,v)}{2} = \psi(M(x, y, u, v)) - \phi(M(x, y, u, v))$.

Case v). Let $x = (5, 4, 0)$ and $y = (y_1, y_2, 0)$. From $d((u_1, u_2, 0), T(5, 4, 0)) = 1$ we get $u_1 = 0$ and $u_2 = 4$. Thus, $u = (0, 4, 0)$.

Sub-case i) if $y_1 \leqslant y_2$ then $v = (y_1, 0, 0)$, $d(u, v) = y_1 + 4$, $d(x, u) = 5$, $d(x, y) = |y_1 - 5| + |y_2 - 4|$, $d(y, v) = y_2$ and $\frac{d(x,v)+d(y,u)}{2} = \frac{|y_1-5|+4+y_1+|y_2-4|}{2}$. Now, $\frac{1}{2}\min\{d(x, u), d(y, v)\} \leqslant d(x, y)$ implies that $\frac{1}{2}\min\{5, y_2\} \leqslant d(x, y)$ i.e., $\frac{y_2}{2} \leqslant |y_1 - 5| + |y_2 - 4|$. Here $M(x, y, u, v) = max\{|y_1 - 5| + |y_2 - 4|, 5, y_2, \frac{|y_1-5|+|y_2-4|+y_1+4}{2}\}$. For $y = (4, 5, 0)$, former inequality is not satisfied, so we don't need to verify (2) for $y = (4, 5, 0)$. Here for all $y = (y_1, y_2, 0) \in A$, $y \neq (4, 5, 0)$ and $y_1 \leqslant y_2$ we can easily observe that $(y_1 + 4)^2 \leqslant 5^2 - \frac{5}{2}$. Thus, $\psi(d(u, v)) = (y_1 + 4)^2 \leqslant \psi(M(x, y, u, v)) - \phi(M(x, y, u, v))$.

Sub-case ii) if $y_1 > y_2$ then $(v_1, v_2, 0) = (0, 0, 0)$. Now $d(u, v) = 4$ and $M(x, y, u, v) = max\{|y_1 - 5| + |y_2 - 4|, 5, y_1 + y_2, \frac{9+|y_2-4|}{2}\}$. Here for all $y = (y_1, y_2, 0) \in A$, $y_1 > y_2$ and $(y_1, y_2, 0) \neq (5, 4, 0)$, we can easily observe that $4^2 \leqslant 5^2 - \frac{5}{2}$ and $M(x, y, u, v) \geqslant 5$. Thus, $\psi(d(u, v)) = 16 \leqslant \psi(M(x, y, u, v)) - \phi(M(x, y, u, v))$.

From all the cases, Case (i) to Case (iv), we conclude that (T, S) is (ψ, ϕ)-generalized weak proximal contraction pair. Hence, all the hypotheses of Theorem 3.3 are satisfied, thus T and S have a unique common best proximity point. Here we can not apply Corollary 3.4 to show that T and S have a unique common best proximity point, since (T, S) does not satisfy the condition of Corollary 3.4 at $x = (4, 5, 0)$ and $y = (5, 4, 0)$.

4. Consequences

In this section we introduce the following definition and obtain some results of best proximity points.

Definition 4.1 Let (X, d) be a metric space, A, B be nonempty subsets of X, and $T, S : A \to B$ are non self mappings. We say that (T, S) is (ψ, ϕ)-weak proximal contraction pair if, for all $x, y, u, v \in A$,

$$\left.\begin{array}{l} \frac{1}{2}\min\{d(x,u), d(y,v)\} \leqslant d(x,y) \\ d(u,Tx) = d(A,B) \\ d(v,Sy) = d(A,B) \end{array}\right\} \implies \psi(d(u,v)) \leqslant \psi(d(x,y)) - \phi(d(x,y)), \quad (18)$$

where $\psi \in \Psi$ and $\phi \in \Phi$.

Theorem 4.2 Let A and B be two nonempty, closed subsets of a complete metric space (X, d). Let $T, S : A \to B$ be mappings. Suppose the following conditions are satisfied:

(i) (T, S) is (ψ, ϕ)-weak proximal contraction;
(ii) $A_0 \neq \emptyset$, $T(A_0) \subseteq B_0$,and $S(A_0) \subseteq B_0$;
(iii) either T or S is continuous.

Then T and S have a unique common best proximity point.

Proof. Let $x_0 \in A_0$. As in the proof of Theorem 3.3 we construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_{2n+1}, Tx_{2n}) = d(A, B),\ d(x_{2n+2}, Sx_{2n+1}) = d(A, B)$$

for all $n = 0, 1, 2, \cdots$ and converging to some $x^* \in A_0$. With out loss of generality, we assume T is a continuous mapping. Thus, we have

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(x_{2n+1}, Tx_{2n}) = \lim_{n\to\infty} d(A, B) = d(A, B).$$

Hence, T has a best proximity point. Similar to Theorem 3.3, we can show that x^* is also a best proximity point of S and we also prove that x^* is unique. ■

If we take $\psi(t) = t = \phi(t)$ in Theorem 4.2 we get the following corollary.

Corollary 4.3 Let A and B be two nonempty, closed subsets of a complete metric space (X, d). Let $T, S : A \to B$ be mappings. Suppose the following conditions are satisfied:

(i) for every $x, y, u, v \in A$,

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Sy) = d(A,B) \end{array}\right\} \implies u = v;$$

(ii) $A_0 \neq \emptyset$, $T(A_0) \subseteq B_0$,and $S(A_0) \subseteq B_0$;
(iii) either T or S is continuous.

Then T and S have a unique common best proximity point.

If we relax condition (i) in Theorem 4.2 we get the following corollary.

Corollary 4.4 Let A and B be two nonempty and closed subsets of a complete metric space (X, d). Let $T, S : A \to B$ be mappings. Suppose that for all $x, y, u, v \in A$, the following conditions are satisfied:

(i) $\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Sy) = d(A, B) \end{array}\right\} \implies \psi(d(u, v)) \leqslant \psi(d(x, y)) - \phi(d(x, y))$;

(ii) $A_0 \neq \emptyset$, $T(A_0) \subseteq B_0$ and $S(A_0) \subseteq B_0$;

(iii) either T or S is continuous.

Then T and S have a unique common best proximity point.

If we take $\psi(t) = t$ and $\phi(t) = (1-k)t$, where $0 \leqslant k < 1$, in Corollary 4.4 we get the following corollary.

Corollary 4.5 Let A and B be two nonempty and closed subsets of a complete metric space (X, d). Let $T, S : A \to B$ be mappings. Suppose that for all $x, y, u, v \in A$, the following conditions are satisfied:

(i) $\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Sy) = d(A, B) \end{array}\right\} \implies d(u, v) \leqslant kd(x, y)$;

(ii) $A_0 \neq \emptyset$, $T(A_0) \subseteq B_0$ and $S(A_0) \subseteq B_0$,

(iii) either T or S is continuous.

Then T and S have a unique common best proximity point.

The following example shows that the main result (Theorem 3.3) of this paper is a generalization of Theorem 4.2.

***Example* 4.6** Let $X = R^3$, $d : X \times X \to R$ defined by

$$d((x_1, x_2, x_3), (y_1, y_2, y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$$

and $A = \{(1,1,0), (1,4,0), (4,1,0)\}$, $B = \{(1,1,1), (1,4,1), (4,1,1)\}$. We define $T, S : A \to B$ by $T(x_1, x_2, 0) = \begin{cases} (1,1,1), & \text{if } x_1 \leqslant x_2; \\ (1,4,1), & \text{if } x_1 > x_2. \end{cases}$ and $S(x_1, x_2, 0) = (1,1,1)$.

We can easily see that $d(A, B) = 1$ and for $x = (4,1,0)$, $y = (1,1,0)$, $u = (u_1, u_2, 0)$ and $v = (v_1, v_2, 0)$ from $d(u, Tx) = 1$ and $d(v, Ty) = 1$. Hence it follows that $u = (1,4,0)$ and $v = (1,1,0)$. Since there is no $\psi \in \Psi$ and $\phi \in \Phi$ such that $\frac{1}{2}\{d(x, u), d(y, v)\} = \frac{1}{2}\{d((4,1,0), (1,4,0)) = 6$ and $d((1,1,0), (1,1,0)) = 0\} \leqslant d((4,1,0), (1,1,0)) = 3$ implies $\psi(d(u, v)) = \psi(3) \leqslant \psi(3) - \phi(3) = \psi(d(x, y)) - \phi(d(x, y))$. So we can not apply Theorem 4.2 to T and S regarding best proximity point. However, it can be easily verified all the hypotheses of Theorem 3.3 of this paper are satisfied for the maps T and S and conclude that T and S have a unique best proximity point.

5. Application in Fixed point theory

As an application of our results, by adding additional condition that either T or S is continuous in the conjecture Theorem 2.3 of Singh et al. [10], we prove the following fixed point theorem.

Theorem 5.1 Let X be a complete metric space. Let $T, S : X \to X$ be self maps such that

(i) either T or S is continuous;

(ii) for every $x, y \in X$, $\frac{1}{2}\min\{d(x,Tx), d(y,Sy)\} \leqslant d(x,y)$ implies that $\psi(d(Tx,Sy)) \leqslant \psi(m(x,y)) - \phi(m(x,y))$, where $\psi \in \Psi$ and $\phi \in \Phi$ are defined as in Theorem 3.3 and $m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Ty)+d(y,Sx)}{2}\}$.

Then T and S have a unique common fixed point.

Proof. Let $A = B = X$ in Theorem 3.3. Clearly $A_0 = X = B_0$. Thus, $T(A_0) \subseteq B_0, S(A_0) \subseteq B_0$. Now, we prove that (T,S) is (ψ,ϕ)-generalized weak proximal contraction. Let $x, y, u, v \in X$ satisfying the following conditions:

$\frac{1}{2}\min\{d(x,u), d(y,v)\} \leqslant d(x,y)$, $d(u,Tx) = d(A,B)$ and $d(v,Sy) = d(A,B)$.

Since $d(A,B) = 0$, we have $u = Tx$ and $v = Sy$. By hypothesis of theorem, we have $\psi(d(u,v)) = \psi(d(Tx,Sy)) \leqslant \psi(m(x,y)) - \phi(m(x,y))$, where

$$\begin{aligned} m(x,y) &= \max\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Ty)+d(y,Sx)}{2}\} \\ &= \max\{d(x,y), d(x,u), d(y,v), \frac{d(x,v)+d(y,u)}{2}\} \\ &= M(x,y,u,v). \end{aligned}$$

Therefore, $\psi(d(u,v)) \leqslant \psi(M(x,y,u,v)) - \phi(M(x,y,u,v))$, which implies that (T,S) is a $(\psi-\phi)$-generalized weak proximal contraction pair. Since $A_0 = A \neq \emptyset$, condition (iii) of Theorem 3.3 is satisfied.

Therefore all the conditions of Theorem 3.3 are satisfied. Consequently there exists a unique common best proximity point $x^* \in X$ of T and S. That is there is a unique $x^* \in X$ such that $d(x^*, Tx^*) = 0 = d(A,B)$ and $d(x^*, Sx^*) = 0 = d(A,B)$. This implies $x^* = Tx^*$ and $x^* = Sx^*$. i.e., $Tx^* = x^* = Sx^*$. So T and S have a common fixed point x^* in X. The uniqueness of the common fixed point of T and S follows from the condition (ii) of the hypothesis. Hence, T and S have a unique common fixed point. ∎

References

[1] S. Basha Sadiq, N. Shahzad, R. Jeyaraj, Common best proximity points, Global optimization of multi-objective functions, Appl. Math. Lett. 24 (6) (2011), 883-886.

[2] K. Fan, Extensions of two fixed point theorems of F.E. Browder, Mathematische zeitschrift 112 (1969), 234-240.

[3] M. Gabeleh, Best proximity point theorems via proximal non-self mappings, J. Optim. Theory Appl. 164 (2) (2015), 565-576.

[4] M. Jleli, B. Samet, Best proximity points for $(\alpha-\psi)$-proximal contractive type mappings and applications, Bulletin des Sciences Mathematiques. 137 (8) (2013), 977-995.

[5] W. A. Kirk, S. Reich, P. Veeramani, Proximinal retracts and best proximity pair theorems, Numer. Funct. Anal. Optim. 24 (2003), 851-862.

[6] J. B. Prolla, Fixed-point theorems for set-valued mappings and existence of best approximants,Numerical Funct. Anal. Optimiz. 5 (4) (1982-1983), 449-455.

[7] S. Reich, Approximate selections, best approximations, fixed points, and invariant sets, Journal of Math. Anal. Appl. 62 (1) (1978), 104-113.

[8] V. M. Sehgal, S. P. Singh, A generalization to multi functions of Fans best approximation theorem, Proc. Amer. Math. Soc. 102 (3) (1988), 534-537.

[9] B. Samet, Some results on best proximity points, J. of Optimization Theory and Appl. 159 (1) (2013), 281-291.

[10] S. L. Singh, R. Kamal, M. De la Sen, R. Chugh, A fixed point theorem for generalized weak contractions, Filomat. 29 (7) (2015), 1481-1490.

[11] K. K. M. Sarma, G. Yohannes, Best proximity points for generalized $(\alpha-\eta-\psi)$-Geraghty proximal contraction mappings, Mathematica Moravica. 21 (2) (2017), 85-102.