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Coincidence points and common fixed points for hybrid pair of mappings in b-metric spaces endowed with a graph

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Abstract. In this paper, we introduce the notion of strictly $(\alpha, \psi, \xi) - G$-contractive mappings in b-metric spaces endowed with a graph G. We establish a sufficient condition for existence and uniqueness of points of coincidence and common fixed points for such mappings. Our results extend and unify many existing results in the literature. Finally, we construct some examples to analyze and support our results.

Keywords: b-metric, digraph, weakly compatible mappings, common fixed point.

2010 AMS Subject Classification: 54H25, 47H10.

1. Introduction

Fixed point theory has many applications in different branches of mathematics and applied sciences. Banach contraction principle [6] is considered to be the initial result of the study of fixed point theory in metric spaces. It is one of the most useful tool in solving existence and uniqueness problems of fixed points. In 1969, Nadler [27] extended the famous Banach contraction principle to set valued form. Later on, a number of articles have been published to the development of fixed point theory of multi-valued mappings in metric spaces(see [1, 3]). Afterwards, hybrid fixed point theory for nonlinear single-valued and multi-valued mappings takes a prominent place in many aspects (see [22, 23]).

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In 1989, Bakhtin [5] introduced b-metric spaces as a generalization of metric spaces and generalized the famous Banach contraction principle in metric spaces to b-metric spaces.

The notion of $\alpha-\psi$-contractive mappings was introduced and studied by Samet et. al. [30]. Some results in this direction are given in [1, 3, 19, 24]. Recently, the study of fixed point theory endowed with a graph is a new development in the domain of contractive type multi-valued theory. Many important results of [1–4, 8–10, 14–17, 21, 23, 26, 28, 30, 31] have become the source of motivation for many researchers that do research in fixed point theory. The main aim of this paper is to introduce the concept of strictly $(\alpha,\, \psi,\, \xi) - G$-contractive mappings of a hybrid pair of single-valued and multi-valued mappings in the framework of b-metric spaces and to derive some coincidence point and common fixed point results for such mappings. As some consequences of this study, we obtain several related results in the setting of b-metric spaces.

2. Some Basic Concepts

First we recall some basic notations and definitions in b-metric spaces.

Definition 2.1 [12] Let X be a nonempty set and $s \geqslant 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leqslant s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

It is to be noted that the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. The following example illustrates the above fact.

***Example* 2.2** [25] Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3,\ d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

***Example* 2.3** [29] Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

Definition 2.4 [11] Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.
(ii) (x_n) is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Remark 1 *[11] In a b-metric space (X, d), the following assertions hold:*

(i) A convergent sequence has a unique limit.
(ii) Each convergent sequence is Cauchy.

(iii) In general, a b-metric is not continuous.

Definition 2.5 [20] Let (X, d) be a b-metric space. A subset $A \subseteq X$ is said to be open if and only if for any $a \in A$, there exists $\epsilon > 0$ such that the open ball $B(a, \epsilon) \subseteq A$. The family of all open subsets of X will be denoted by τ.

Theorem 2.6 [20] τ defines a topology on (X, d).

Theorem 2.7 [20] Let (X, d) be a b-metric space and τ be the topology defined above. Then for any nonempty subset $A \subseteq X$ we have

(i) A is closed if and only if for any sequence (x_n) in A which converges to x, we have $x \in A$;

(ii) if we define $\overline{A}$ to be the intersection of all closed subsets of X which contains A, then for any $x \in \overline{A}$ and for any $\epsilon > 0$, we have $B(x, \epsilon) \cap A \neq \emptyset$.

Definition 2.8 Let (X, d) be a b-metric space and A be a nonempty subset of X. The diameter of A, denoted by $\delta(A)$, is defined by $\delta(A) = sup\{d(x, y) : x,\ y \in A\}$. The subset A is said to be bounded if $\delta(A)$ is finite.

For a b-metric space (X, d), we let $CB(X)$ be the set of all nonempty closed bounded subsets of X. A point $x \in X$ is called a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$. For every $A,\ B \in CB(X)$, let

$$H(A, B) = max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

where $d(x, B) = inf\{d(x, y) : y \in B\}$. Such a map H is called the Hausdorff b-metric on $CB(X)$ induced by the b-metric d.

It is easy to verify that $H(A, B) \geqslant 0$ for all $A,\ B \in CB(X)$ and $H(A, B) = 0$ if and only if $A = B$ and $H(A, B) = H(B, A)$ for all $A,\ B \in CB(X)$. We now verify that

$$H(A, B) \leqslant s(H(A, C) + H(C, B)),\ \forall\, A,\ B,\ C \in CB(X).$$

We first note that,

$$d(x, P) \leqslant s(d(x, y) + d(y, P)),\ \forall\, x,\ y \in X \textit{ and any subset } P \subseteq X.$$

Now, for all $A,\ B,\ C \in CB(X)$ and $x \in A$, we have

$$d(x, B) \leqslant s(d(x, z) + d(z, B)) \leqslant s(d(x, z) + \sup_{z \in C} d(z, B))$$

for all $z \in C$, which implies that

$$d(x, B) \leqslant s(d(x, C) + \sup_{z \in C} d(z, B))$$

for all $x \in A$. Therefore,

$$\sup_{x \in A} d(x, B) \leqslant s(\sup_{x \in A} d(x, C) + \sup_{z \in C} d(z, B)).$$

By an argument similar to that used above, we have

$$\sup_{y\in B} d(y,A) \leqslant s(\sup_{y\in B} d(y,C) + \sup_{z\in C} d(z,A)).$$

Again,

$$\begin{aligned}
H(A,B) &= max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}\\
&\leqslant max\{s(\sup_{x\in A} d(x,C) + \sup_{z\in C} d(z,B)), s(\sup_{y\in B} d(y,C) + \sup_{z\in C} d(z,A))\}\\
&\leqslant max\{s \sup_{x\in A} d(x,C), s \sup_{z\in C} d(z,A)\} + max\{s \sup_{z\in C} d(z,B), s \sup_{y\in B} d(y,C)\}\\
&= s\,(H(A,C) + H(C,B)).
\end{aligned}$$

Thus, H is a b-metric on $CB(X)$ with the coefficient $s \geqslant 1$.

Let Ψ be a class of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:
(ψ_1) ψ is a nondecreasing function;
(ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Remark 2 *[23] For each $\psi \in \Psi$, we see that the following assertions hold:*

(i) $\lim_{n\to\infty} \psi^n(t) = 0$*, for all* $t > 0$*;*
(ii) $\psi(t) < t$ *for each* $t > 0$*;*
(iii) $\psi(0) = 0$.

Definition 2.9 [30] Let T be a self-mapping on a nonempty set X and $\alpha : X \times X \to [0,\infty)$ be another mapping. We say that T is α-admissible if the following condition holds:

$$x,\, y \in X,\ \ \alpha(x,y) \geqslant 1 \Rightarrow \alpha(Tx,Ty) \geqslant 1.$$

Definition 2.10 Let (X,d) be a b-metric space and $\alpha : X \times X \to [0,\infty)$ be a mapping. A mapping $T : X \to CB(X)$ is called α_*-admissible if

$$x,\, y \in X,\ \ \alpha(x,y) \geqslant 1 \Rightarrow \alpha_*(Tx,Ty) \geqslant 1,$$

where $\alpha_*(Tx,Ty) = inf\{\alpha(a,b) : a \in Tx,\, b \in Ty\}$.

In 2014, Ali et. al.[1] introduced a family Ξ of functions $\xi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ξ is continuous;
(ii) ξ is nondecreasing on $[0,\infty)$;
(iii) $\xi(0) = 0$ and $\xi(t) > 0$ for all $t \in (0,\infty)$;
(iv) ξ is subadditive.

***Example* 2.11** [1] Suppose that $\phi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,\infty)$, for each $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$ and for each a, $b > 0$, we have

$$\int_0^{a+b} \phi(t)dt \leqslant \int_0^a \phi(t)dt + \int_0^b \phi(t)dt.$$

Define $\xi : [0, \infty) \to [0, \infty)$ by $\xi(t) = \int_0^t \phi(w)dw$ for each $t \in [0, \infty)$. Then, $\xi \in \Xi$.

Lemma 2.12 Let (X, d) be a b-metric space with the coefficient $s \geqslant 1$ and let $\xi \in \Xi$ be such that $\xi(st) = s\xi(t)$ for each $t \in [0, \infty)$. Then $(X, \xi \circ d)$ is a b-metric space with the coefficient $s \geqslant 1$.

Proof. Proof follows from the fact that for all x, y, $z \in X$, we have

$$\begin{aligned}
(\xi \circ d)(x, y) &= \xi(d(x, y)) \\
&\leqslant \xi\left(sd(x, z) + sd(z, y)\right) \\
&\leqslant \xi(sd(x, z)) + \xi(sd(z, y)) \\
&= s\xi(d(x, z)) + s\xi(d(z, y)) \\
&= s[(\xi \circ d)(x, z) + (\xi \circ d)(z, y)].
\end{aligned}$$

■

Lemma 2.13 Let (X, d) be a b-metric space with the coefficient $s \geqslant 1$, let $\xi \in \Xi$ be such that $\xi(st) = s\xi(t)$ for each $t \in [0, \infty)$ and let $B \in CB(X)$. Assume that there exists $x \in X$ such that $\xi(d(x, B)) > 0$. Then there exists $y \in B$ such that

$$\xi(d(x, y)) < q\xi(d(x, B)), \ \text{where } q > 1.$$

Proof. Proof is similar to that of Lemma 2.3[1]. ■

Definition 2.14 Let (X, d) be a b-metric space with the coefficient $s \geqslant 1$ and $\alpha : X \times X \to [0, \infty)$ be a mapping. A mapping $T : X \to CB(X)$ is called α_*-admissible with respect to f(a self-mapping on X) if the following condition holds:

$$x,\, y \in X,\ \alpha(fx, fy) \geqslant 1 \Rightarrow \alpha_*(Tx, Ty) \geqslant 1,$$

where $\alpha_*(Tx, Ty) = inf\{\alpha(a, b) : a \in Tx,\, b \in Ty\}$.

Definition 2.15 Let (X, d) be a b-metric space with the coefficient $s \geqslant 1$. Then the mappings $T : X \to CB(X)$ and $f : X \to X$ are called (α, ψ, ξ)-contractive mappings if there exist three functions $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0, \infty)$ such that

$$x,\, y \in X,\ \alpha(fx, fy) \geqslant 1 \Rightarrow \xi(sH(Tx, Ty)) \leqslant \psi(\xi(M_s(fx, fy))),$$

where $M_s(fx, fy) = max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2s}\}$. If $\psi \in \Psi$ is strictly increasing, then T and f are called strictly (α, ψ, ξ)-contractive mappings.

Definition 2.16 Let (X, d) be a b-metric space and $T : X \to CB(X)$ and $f : X \to X$ be two mappings. If $y = fx \in Tx$ for some x in X, then x is called a coincidence point of T and f and y is called a point of coincidence of T and f.

Definition 2.17 Let (X,d) be a b-metric space. The mappings $T : X \to CB(X)$ and $f : X \to X$ are called compatible iff $fTx \in CB(X)$ for all $x \in X$ and $H(Tfx_n, fTx_n) \to 0$ whenever (x_n) is a sequence in X such that $Tx_n \to M \in CB(X)$ and $fx_n \to t \in M$.

Definition 2.18 Let (X,d) be a b-metric space. The mappings $T : X \to CB(X)$ and $f : X \to X$ are called weakly compatible if they commute at their coincidence points, i.e., if $Tfx = fTx$ whenever $fx \in Tx$.

Proposition 2.19 Let (X,d) be a b-metric space and $T : X \to CB(X)$ and $f : X \to X$ be weakly compatible. If T and f have a unique point of coincidence $y = fx \in Tx$, then y is the unique common fixed point of T and f in X.

Proof. Let $y = fx \in Tx$ for some x in X. Since f and T are weakly compatible, $Tfx = fTx$. This implies that $fy \in Ty$ and hence fy is a point of coincidence of f and T. As y is the unique point of coincidence of f and T, it follows that $y = fy \in Ty$. This shows that y is a common fixed point of f and T.

Let z be another common fixed point of f and T in X i.e., $z = fz \in Tz$. Since f and T have a unique point of coincidence in X, it follows that $fy = fz$ and hence $y = z$. This proves that y is the unique common fixed point of f and T in X. ■

We next review some basic notions in graph theory.

Let (X,d) be a b-metric space. We assume that G is a digraph with the set of vertices $V(G) = X$ and the set $E(G)$ of its edges contains all the loops, i.e., $\Delta \subseteq E(G)$ where $\Delta = \{(x,x) : x \in X\}$. We also assume that G has no parallel edges. So we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention, $E(\tilde{G}) = E(G) \cup E(G^{-1})$.

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [7, 13, 18]. If x, y are vertices of the digraph G, then a path in G from x to y of length n ($n \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

Definition 2.20 Let (X,d) be a b-metric space with the coefficient $s \geqslant 1$ and let $G = (V(G), E(G))$ be a graph. Then the mappings $T : X \to CB(X)$ and $f : X \to X$ are called $(\alpha, \psi, \xi) - G$-contractive mappings if there exist three functions $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0, \infty)$ such that x, $y \in X$ with $(fx, fy) \in E(\tilde{G})$,

$$\alpha(fx, fy) \geqslant 1 \Rightarrow \xi(sH(Tx, Ty)) \leqslant \psi(\xi(M_s(fx, fy))),$$

where $M_s(fx, fy) = max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2s}\}$. If $\psi \in \Psi$ is strictly increasing, then T and f are called strictly $(\alpha, \psi, \xi) - G$-contractive mappings.

It is valuable to note that strictly (α, ψ, ξ)-contractive mappings are strictly $(\alpha, \psi, \xi) - G_0$-contractive. But strictly $(\alpha, \psi, \xi) - G$-contractive mappings need not be strictly (α, ψ, ξ)-contractive mappings (see Remark 6).

Definition 2.21 Let (X,d) be a b-metric space with the coefficient $s \geqslant 1$ and let $f : X \to X$ be a given mapping. We say that f is continuous at $x_0 \in X$ if for every sequence (x_n) in X, we have $x_n \to x_0$ as $n \to \infty \Longrightarrow fx_n \to fx_0$ as $n \to \infty$. If f is continuous at each point $x_0 \in X$, then we say that f is continuous on X.

Definition 2.22 Let (X, d) be a b-metric space endowed with a graph G. A mapping $f : X \to X$ is called G-continuous if given $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$,

$$x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for } n \in \mathbb{N} \text{ imply } fx_n \to fx.$$

Similarly, a mapping $T : X \to CB(X)$ is called G-continuous if given $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$,

$$x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for } n \in \mathbb{N} \text{ imply } Tx_n \to Tx.$$

It is easy to observe that continuity $\Rightarrow$ G-continuity.

3. Main results

Let (X, d) be a b-metric space with the coefficient $s \geqslant 1$.

Definition 3.1 The point p belongs to the lower limit $Li_{n\to\infty}A_n$ of a sequence (A_n) of subsets of (X, d), if every open ball with center p intersects all the A_n from a sufficiently great index n onward.

Definition 3.2 The point p belongs to the upper limit $Ls_{n\to\infty}A_n$ of a sequence (A_n) of subsets of (X, d), if every open ball with center p intersects an infinite set of the terms A_n.

Definition 3.3 The sequence (A_n) of subsets of (X, d) is said to be convergent to A, if $Li_{n\to\infty}A_n = A = Ls_{n\to\infty}A_n$. We then write $A = Lim_{n\to\infty}A_n$.

Theorem 3.4 The point $p \in Ls_{n\to\infty}A_n$ is equivalent to the existence of a sequence of points (p_{k_n}) such that $k_1 < k_2 < \cdots$, $p = \lim_{n\to\infty} p_{k_n}$ and $p_{k_n} \in A_{k_n}$.

Proof. If $p \in Ls_{n\to\infty}A_n$ and if S_m is the open ball with center p and radius $\frac{1}{m}$, then there exists an infinite set $\{A_{k_1}, A_{k_2}, \cdots, A_{k_n}, \cdots\}$ of the terms A_n, where $k_1 < k_2 < \cdots$ such that $S_m \cap A_{k_n} \neq \emptyset$ for each k_n. Let $p_{k_n} \in S_m \cap A_{k_n}$. Then $d(p_{k_n}, p) < \frac{1}{m}$ for all n and consequently it follows that $p = \lim_{n\to\infty} p_{k_n}$. The converse implication is obvious. ■

Theorem 3.5 Let (X, d) be a b-metric space and $A_1, A_2, A_3, \cdots$ be a sequence of subsets of (X, d). Then,

$$Ls_{n\to\infty}A_n = \bigcap_n \overline{A_n \cup A_{n+1} \cup \cdots}.$$

Proof. Let $p \in Ls_{n\to\infty}A_n$. Then $p = \lim_{n\to\infty} p_{k_n}$ and $p_{k_n} \in A_{k_n}$. Since $k_n \geqslant n$, we have $p_{k_n} \in (A_i \cup A_{i+1} \cup \cdots)$ for $n > i$. Consequently, $p \in \overline{A_i \cup A_{i+1} \cup \cdots}$ for each i. Conversely, if p is not in $Ls_{n\to\infty}A_n$, then there exists an open ball $B_r(p)$ with center p and an index m such that $B_r(p) \cap A_n = \emptyset$ for $n \geqslant m$. Therefore, $p \notin \overline{A_m \cup A_{m+1} \cup \cdots}$. ■

Definition 3.6 Let (X, d) be a b-metric space and A be a nonempty subset of X. A generalized open ball with radius $r(> 0)$ and center A is defined to be the set of all points $x \in X$ such that $d(x, A) < r$ where $d(x, A) = inf\{d(x, a) : a \in A\}$. The set of all points

$x \in X$ such that $d(x, A) \leqslant r$ is called a generalized closed ball with radius r and center A.

Theorem 3.7 If (X, d) is a complete b-metric space with the coefficient $s \geqslant 1$, then the space $(CB(X), H)$ is a complete b-metric space with the coefficient $s \geqslant 1$.

Proof. Let (A_n) be a Cauchy sequence in $(CB(X), H)$. Then, for any $\epsilon > 0$, there exists an $n(\frac{\epsilon}{s^3}) \in \mathbb{N}$ such that

$$H(A_n, A_{n(\frac{\epsilon}{s^3})}) < \frac{\epsilon}{s^3} \ for\ all\ n > n(\frac{\epsilon}{s^3}). \tag{1}$$

Let $L = Ls_{n\to\infty}A_n$. We shall prove that

$$\lim_{n\to\infty} H(L, A_n) = 0. \tag{2}$$

It is sufficient to prove that $H(L, A_{n(\frac{\epsilon}{s^3})}) \leqslant \frac{2\epsilon}{s}$, because then condition (1) will imply that $H(L, A_n) < (2 + \frac{1}{s^2})\epsilon$ for $n > n(\frac{\epsilon}{s^3})$. We have to show the following propositions.

(i) If $p \in L$, then $d(p, A_{n(\frac{\epsilon}{s^3})}) \leqslant \frac{\epsilon}{s}$.
(ii) If $q \in A_{n(\frac{\epsilon}{s^3})}$, then $d(q, L) \leqslant \frac{2\epsilon}{s}$.

Let B be the generalized closed ball with center $A_{n(\frac{\epsilon}{s^3})}$ and radius $\frac{\epsilon}{s^3}$. By condition (1), $A_n \subseteq B$ for $n > n(\frac{\epsilon}{s^3})$. Since by applying Theorem (3.5), $L \subseteq \overline{A_n \cup A_{n+1} \cup \cdots}$, it follows that $L \subseteq B$. Therefore, if $p \in L$, then $d(p, A_{n(\frac{\epsilon}{s^3})}) \leqslant \frac{\epsilon}{s^3} \leqslant \frac{\epsilon}{s}$ which proves proposition (i).

To prove proposition (ii), let $n(\frac{\epsilon}{s^2 s^{k+1} 2^k}) = n_k$. We can assume that $n_k > n_{k-1}$. We consider the sequence $\{q_{n_0}, q_{n_1}, \cdots, q_{n_k}, \cdots\}$ defined as follows. Choose a point q_{n_k} in A_{n_k} so that $q_{n_0} = q$ and $d(q_{n_{k-1}}, q_{n_k}) < \frac{\epsilon}{s^2 s^k 2^{k-1}}$, which is possible by condition (1). For $m > k$, we have

$$\begin{aligned}
d(q_{n_k}, q_{n_m}) &\leqslant sd(q_{n_k}, q_{n_{k+1}}) + s^2 d(q_{n_{k+1}}, q_{n_{k+2}}) + \cdots \\
&\quad + s^{m-k-1} d(q_{n_{m-2}}, q_{n_{m-1}}) + s^{m-k-1} d(q_{n_{m-1}}, q_{n_m}) \\
&< \frac{\epsilon}{s^2}[\frac{s}{s^{k+1}2^k} + \frac{s^2}{s^{k+2}2^{k+1}} + \frac{s^3}{s^{k+3}2^{k+2}} + \cdots + \frac{s^{m-k-1}}{s^{m-1}2^{m-2}} + \frac{s^{m-k}}{s^m 2^{m-1}}] \\
&= \frac{\epsilon}{s^2}[\frac{1}{s^k 2^k} + \frac{1}{s^k 2^{k+1}} + \frac{1}{s^k 2^{k+2}} + \cdots + \frac{1}{s^k 2^{m-2}} + \frac{1}{s^k 2^{m-1}}] \\
&< \frac{\epsilon}{s^2 (2s)^k}[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots] \\
&= \frac{2\epsilon}{s^2 (2s)^k} \to 0 \ as\ m,\ k \to \infty.
\end{aligned} \tag{3}$$

This implies that the sequence $\{q_{n_0}, q_{n_1}, \cdots, q_{n_k}, \cdots\}$ is Cauchy in (X, d). Since X is complete, the sequence converges to a point u of X. By Theorem (3.4), it follows that $u \in L$. Now, by using condition (3), we obtain

$$d(q, u) \leqslant s[d(q, q_{n_k}) + d(q_{n_k}, u)] = s[d(q_{n_0}, q_{n_k}) + d(q_{n_k}, u)] < s[\frac{2\epsilon}{s^2} + d(q_{n_k}, u)].$$

Taking limit as $k \to \infty$, it follows that $d(q, u) \leqslant \frac{2\epsilon}{s}$. Therefore, if $q \in A_{n(\frac{\epsilon}{s^3})}$, then

$d(q, L) \leqslant d(q, u) \leqslant \frac{2\epsilon}{s}$ and proposition (ii) follows. ■

We now assume that (X, d) is a b-metric space endowed with a reflexive digraph G such that $V(G) = X$ and G has no parallel edges. Let $f : X \to X$ and $T : X \to CB(X)$ be such that $T(X) \subseteq f(X)$. Let $x_0 \in X$ be arbitrary. Since $T(X) \subseteq f(X)$, there exists an element $x_1 \in X$ such that $fx_1 \in Tx_0$. Continuing in this way, we can construct a sequence (fx_n) such that $fx_n \in Tx_{n-1},\ n = 1, 2, 3, \cdots$.

Definition 3.8 Let (X, d) be a b-metric space endowed with a graph G and $f : X \to X$ and $T : X \to CB(X)$ be such that $T(X) \subseteq f(X)$. We define C^{α}_{fT} the set of all elements x_0 of X such that $(fx_n, fx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$ and $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$, for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$.

Taking $f = I$, the identity map on X, C^{α}_{fT} becomes C^{α}_{T} which is the collection of all elements x_0 of X such that $(x_n, x_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$ and $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$, for every sequence (x_n) such that $x_n \in Tx_{n-1}$.

Theorem 3.9 Let (X, d) be a b-metric space with the coefficient $s \geqslant 1$ and let $G = (V(G), E(G))$ be a graph. Let $T : X \to CB(X)$ and $f : X \to X$ be strictly $(\alpha,\ \psi,\ \xi) - G$-contractive mappings with $\xi(st) = s\xi(t)$ for each $t \in [0, \infty)$ and $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t > 0$. Suppose that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X with the following property:

$(*)$ If (fx_n) is a sequence in X such that $fx_n \to x$ and $(fx_n, fx_{n+1}) \in E(\tilde{G})$ for all $n \geqslant 1$ and $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$, then there exists a subsequence (fx_{n_i}) of (fx_n) such that $(fx_{n_i}, x) \in E(\tilde{G})$ and $\alpha(fx_{n_i}, x) \geqslant 1$ for all $i \geqslant 1$.

Then f and T have a point of coincidence in X if $C^{\alpha}_{fT} \neq \emptyset$. Moreover, f and T have a unique point of coincidence in X if the graph G has the following property:

$(**)$ If x, y are points of coincidence of f and T in X, then $(x, y) \in E(\tilde{G})$ and $\alpha(x, y) \geqslant 1$. Furthermore, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. Suppose that $C^{\alpha}_{fT} \neq \emptyset$. We choose an $x_0 \in C^{\alpha}_{fT}$ and keep it fixed. Since $Tx_0 \subseteq f(X)$, there exists $x_1 \in X$ such that $fx_1 \in Tx_0$ and $(fx_0, fx_1) \in E(\tilde{G})$ with $\alpha(fx_0, fx_1) \geqslant 1$. If $x_1 = x_0$, then f and T have a point of coincidence in X. So, we assume that $x_1 \neq x_0$. If $fx_1 \in Tx_1$, then we have nothing to prove. Therefore, let $fx_1 \notin Tx_1$. Since T and f are strictly $(\alpha,\ \psi,\ \xi) - G$-contractive, we have

$$\begin{aligned}
\xi(sH(Tx_0, Tx_1)) &\leqslant \psi\left(\xi\left(max\left\{\begin{matrix} d(fx_0, fx_1), d(fx_0, Tx_0), d(fx_1, Tx_1), \\ \frac{d(fx_0, Tx_1) + d(fx_1, Tx_0)}{2s} \end{matrix}\right\}\right)\right) \\
&\leqslant \psi(\xi(max\{d(fx_0, fx_1), sd(fx_0, fx_1), d(fx_1, Tx_1), \frac{d(fx_0, Tx_1)}{2s}\})) \\
&\leqslant \psi\left(\xi\left(max\left\{\begin{matrix} d(fx_0, fx_1), sd(fx_0, fx_1), d(fx_1, Tx_1), \\ \frac{d(fx_0, fx_1) + d(fx_1, Tx_1)}{2} \end{matrix}\right\}\right)\right) \\
&\leqslant \psi(\xi(max\{sd(fx_0, fx_1), sd(fx_1, Tx_1)\})) \\
&= \psi(\xi(s\, max\{d(fx_0, fx_1), d(fx_1, Tx_1)\})). \qquad (4)
\end{aligned}$$

If $max\{d(fx_0, fx_1), d(fx_1, Tx_1)\} = d(fx_1, Tx_1)$, then it follows from condition (4) that

$$0 < \xi(sd(fx_1, Tx_1)) \leqslant \xi(sH(Tx_0, Tx_1)) \leqslant \psi(\xi(sd(fx_1, Tx_1))), \tag{5}$$

which is a contradiction, since $\psi(r) < r$ for each $r > 0$. Therefore,

$$max\{d(fx_0, fx_1), d(fx_1, Tx_1)\} = d(fx_0, fx_1).$$

Thus, from condition (4), we obtain

$$0 < \xi(sd(fx_1, Tx_1)) \leqslant \xi(sH(Tx_0, Tx_1)) \leqslant \psi(\xi(sd(fx_0, fx_1))). \tag{6}$$

Since $\xi(st) = s\xi(t)$, by Lemma 2.13, for $q > 1$, there exists $fx_2 \in Tx_1$ such that

$$0 < \xi(d(fx_1, fx_2)) < q\xi(d(fx_1, Tx_1)). \tag{7}$$

From conditions (6) and (7), we get

$$0 < \xi(sd(fx_1, fx_2)) < q\psi(\xi(sd(fx_0, fx_1))).$$

Since ψ is strictly increasing, we have

$$0 < \psi(\xi(sd(fx_1, fx_2))) < \psi(q\psi(\xi(sd(fx_0, fx_1)))).$$

Put $q_1 = \frac{\psi(q\psi(\xi(sd(fx_0, fx_1))))}{\psi(\xi(sd(fx_1, fx_2)))}$. Then, $q_1 > 1$. If $x_1 = x_2$ or $fx_2 \in Tx_2$, then we have nothing to prove. Therefore, we assume that $x_1 \neq x_2$ and $fx_2 \notin Tx_2$. Since $x_0 \in C^{\alpha}_{fT}$, $fx_1 \in Tx_0$, $fx_2 \in Tx_1$, it follows that $(fx_1, fx_2) \in E(\tilde{G})$ and $\alpha(fx_1, fx_2) \geqslant 1$. Applying strictly $(\alpha,\, \psi,\, \xi) - G$-contractive condition, we get

$$\xi(sH(Tx_1, Tx_2)) \leqslant \psi(\xi(s\, max\{d(fx_1, fx_2), d(fx_2, Tx_2)\})). \tag{8}$$

If $max\{d(fx_1, fx_2), d(fx_2, Tx_2)\} = d(fx_2, Tx_2)$, then it follows from condition (8) that

$$0 < \xi(sd(fx_2, Tx_2)) \leqslant \xi(sH(Tx_1, Tx_2)) \leqslant \psi(\xi(sd(fx_2, Tx_2))),$$

which is a contradiction. Therefore, $max\{d(fx_1, fx_2), d(fx_2, Tx_2)\} = d(fx_1, fx_2)$. Now, by using condition (8), we obtain

$$0 < \xi(sd(fx_2, Tx_2)) \leqslant \xi(sH(Tx_1, Tx_2)) \leqslant \psi(\xi(sd(fx_1, fx_2))). \tag{9}$$

By Lemma 2.13, for $q_1 > 1$, there exists $fx_3 \in Tx_2$ such that

$$0 < \xi(d(fx_2, fx_3)) < q_1\xi(d(fx_2, Tx_2)). \tag{10}$$

From conditions (9) and (10), we get

$$0 < \xi(sd(fx_2, fx_3)) < q_1\psi(\xi(sd(fx_1, fx_2))) = \psi(q\psi(\xi(sd(fx_0, fx_1)))).$$

ψ being strictly increasing, it follows that

$$0 < \psi(\xi(sd(fx_2, fx_3))) < \psi^2(q\psi(\xi(sd(fx_0, fx_1)))).$$

Since $x_0 \in C_{fT}^{\alpha}$, $fx_1 \in Tx_0$, $fx_2 \in Tx_1$, $fx_3 \in Tx_2$, it follows that $(fx_n, fx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, 3$ and $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for $n = 0, 1, 2$. Continuing this process, we can construct a sequence (fx_n) in X such that $fx_n \in Tx_{n-1}$, $(fx_n, fx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$ and $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and

$$0 < \xi(sd(fx_{n+1}, fx_{n+2})) < \psi^n(q\psi(\xi(sd(fx_0, fx_1)))) = \psi^n(k), \ \forall n \in \mathbb{N} \cup \{0\},$$

where $k = q\psi(\xi(sd(fx_0, fx_1)))$. We now show that (fx_n) is a Cauchy sequence in $f(X)$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(fx_n, fx_m) &\leqslant sd(fx_n, fx_{n+1}) + s^2 d(fx_{n+1}, fx_{n+2}) + \cdots \\ &\quad + s^{m-n-1} d(fx_{m-2}, fx_{m-1}) + s^{m-n-1} d(fx_{m-1}, fx_m). \end{aligned}$$

Therefore,

$$\begin{aligned} \xi(d(fx_n, fx_m)) &\leqslant \xi(sd(fx_n, fx_{n+1})) + s\xi(sd(fx_{n+1}, fx_{n+2})) + \cdots \\ &\quad + s^{m-n-2}\xi(sd(fx_{m-2}, fx_{m-1})) + s^{m-n-1}\xi(sd(fx_{m-1}, fx_m)) \\ &< \psi^{n-1}(k) + s\psi^n(k) + \cdots + s^{m-n-2}\psi^{m-3}(k) + s^{m-n-1}\psi^{m-2}(k) \\ &= \frac{1}{s^{n-1}} \sum_{i=n}^{m-1} s^{i-1}\psi^{i-1}(k). \end{aligned}$$

Since $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$, it follows that

$$\lim_{n,m\to\infty} \xi(d(fx_n, fx_m)) = 0.$$

By using (ξ_1) and (ξ_3), we get

$$\lim_{n,m\to\infty} d(fx_n, fx_m) = 0.$$

This gives that (fx_n) is a Cauchy sequence in $f(X)$. As $f(X)$ is complete, there exists an $t \in f(X)$ such that $fx_n \to t = fu$ for some $u \in X$.

As $(fx_n, fx_{n+1}) \in E(\tilde{G})$ and $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \geqslant 1$, by property $(*)$, there exists a subsequence (fx_{n_i}) of (fx_n) such that $(fx_{n_i}, fu) \in E(\tilde{G})$ and $\alpha(fx_{n_i}, fu) \geqslant 1$ for all $i \geqslant 1$. Then by applying strictly $(\alpha, \psi, \xi) - G$-contractivity, we have

$$\xi(sH(Tx_{n_i}, Tu)) \leqslant \psi\left(\xi\left(max\left\{\begin{matrix} d(fx_{n_i}, fu), d(fx_{n_i}, Tx_{n_i}), d(fu, Tu), \\ \frac{d(fx_{n_i}, Tu) + d(fu, Tx_{n_i})}{2s} \end{matrix}\right\}\right)\right). \quad (11)$$

Suppose that $d(fu,Tu) \neq 0$. Let $\epsilon = \frac{d(fu,Tu)}{2s} > 0$. Since $fx_{n_i} \to fu$, there exists $k_1 \in \mathbb{N}$ such that

$$d(fx_{n_i}, fu) < \frac{d(fu,Tu)}{2s}, \ for \ each \ i \geqslant k_1. \tag{12}$$

As $fx_n \to fu$, there exists $k_2 \in \mathbb{N}$ such that

$$d(fx_{n_i+1}, fu) < \frac{d(fu,Tu)}{2s}, \ for \ each \ i \geqslant k_2. \tag{13}$$

So, it must be the case that

$$d(fu, Tx_{n_i}) \leqslant d(fu, fx_{n_i+1}) < \frac{d(fu,Tu)}{2s}, \ for \ each \ i \geqslant k_2. \tag{14}$$

As $d(fx_{n_i}, Tu) \leqslant sd(fx_{n_i}, fu) + sd(fu, Tu)$, it follows that

$$d(fx_{n_i}, Tu) < \frac{d(fu,Tu)}{2} + sd(fu,Tu) \leqslant \frac{3s}{2} d(fu,Tu), \ for \ each \ i \geqslant k_1. \tag{15}$$

Put $k = max\{k_1, k_2\}$. Then, for $i \geqslant k$, we have

$$\begin{aligned} d(fx_{n_i}, Tx_{n_i}) &\leqslant d(fx_{n_i}, fx_{n_i+1}) \\ &\leqslant s[d(fx_{n_i}, fu) + d(fu, fx_{n_i+1})] \\ &< d(fu, Tu). \end{aligned} \tag{16}$$

Thus, for $i \geqslant k$, it follows from conditions (12), (14), (15) and (16) that

$$max \left\{ \begin{array}{l} d(fx_{n_i}, fu), d(fx_{n_i}, Tx_{n_i}), d(fu,Tu), \\ \frac{d(fx_{n_i},Tu)+d(fu,Tx_{n_i})}{2s} \end{array} \right\} = d(fu,Tu).$$

Therefore, for $i \geqslant k$, we obtain from (11) that

$$\xi(sH(Tx_{n_i}, Tu)) \leqslant \psi(\xi(d(fu,Tu))). \tag{17}$$

By using condition (17), for $i \geqslant k$, we have

$$\begin{aligned} \xi(d(fu,Tu)) &\leqslant \xi(sd(fu, fx_{n_i+1})) + \xi(sd(fx_{n_i+1}, Tu)) \\ &\leqslant \xi(sd(fu, fx_{n_i+1})) + \xi(sH(Tx_{n_i}, Tu)) \\ &\leqslant \xi(sd(fu, fx_{n_i+1})) + \psi(\xi(d(fu,Tu))). \end{aligned}$$

Taking limit as $i \to \infty$, we get $\xi(d(fu,Tu)) \leqslant \psi(\xi(d(fu,Tu)))$, which is a contradiction, since $\xi(d(fu,Tu)) > 0$. Therefore, $d(fu,Tu) = 0$ and so, $t = fu \in Tu$, i.e., t is a point of coincidence of f and T.

For uniqueness, assume that there is another point of coincidence $s(\neq t)$ in X such that $s = fv \in Tv$ for some $v \in X$. By property $(**)$, we have $(fu, fv) \in E(\tilde{G})$ and

$\alpha(fu, fv) \geqslant 1$. Then, $\xi(sH(Tu, Tv)) \leqslant \psi(\xi(M_s(fu, fv)))$, where

$$\begin{aligned} M_s(fu, fv) &= max\{d(fu, fv), d(fu, Tu), d(fv, Tv), \frac{d(fu, Tv) + d(fv, Tu)}{2s}\} \\ &= max\{d(fu, fv), \frac{d(fu, Tv) + d(fv, Tu)}{2s}\} \\ &\leqslant max\{d(fu, fv), \frac{d(fu, fv) + d(fv, fu)}{2}\} \\ &\leqslant sd(fu, fv). \end{aligned}$$

Thus,

$$0 < \xi(sd(fu, fv)) \leqslant \xi(sH(Tu, Tv)) \leqslant \psi(\xi(sd(fu, fv))),$$

which is a contradiction, since $\psi(r) < r$ for each $r > 0$.

So, it must be the case that, $d(fu, fv) = 0$ and hence, $fu = fv$. Therefore, f and T have a unique point of coincidence in X. If f and T are weakly compatible, then by Proposition 2.19, f and T have a unique common fixed point in X. ■

Corollary 3.10 Let (X, d) be a complete b-metric space endowed with a graph G and the coefficient $s \geqslant 1$. Let $T : X \to CB(X)$ be a strictly $(\alpha,\ \psi,\ \xi)-G$-contractive mapping with $\xi(st) = s\xi(t)$ for each $t \in [0, \infty)$ and $\sum_{n=1}^{\infty} s^n\psi^n(t) < \infty$ for each $t > 0$. Suppose the triple (X, d, G) has the following property:

$(*)'$ If (x_n) is a sequence in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$, $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \geqslant 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, x) \in E(\tilde{G})$ and $\alpha(x_{n_i}, x) \geqslant 1$ for all $i \geqslant 1$.

Then T has a fixed point in X if $C_T^{\alpha} \neq \emptyset$. Moreover, T has a unique fixed point in X if the graph G has the following property:

$(**)'$ If x, y are fixed points of T in X, then $(x, y) \in E(\tilde{G})$ and $\alpha(x, y) \geqslant 1$.

Proof. The proof follows from Theorem 3.9 by taking $f = I$, the identity map on X. ■

Corollary 3.11 Let (X, d) be a b-metric space with the coefficient $s \geqslant 1$. Let $f : X \to X$ and $T : X \to CB(X)$ be strictly $(\alpha,\ \psi,\ \xi)$-contractive mappings with $\xi(st) = s\xi(t)$ for each $t \in [0, \infty)$ and $\sum_{n=1}^{\infty} s^n\psi^n(t) < \infty$ for each $t > 0$. Suppose that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X with the following property:

(†) If (fx_n) is a sequence in X such that $fx_n \to x$ and $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \geqslant 1$, then there exists a subsequence (fx_{n_i}) of (fx_n) such that $\alpha(fx_{n_i}, x) \geqslant 1$ for all $i \geqslant 1$.

If there exists $x_0 \in X$ such that $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$, then f and T have a point of coincidence in X. Moreover, f and T have a unique point of coincidence in X if the following property holds:

(‡) If x, y are points of coincidence of f and T in X, then $\alpha(x, y) \geqslant 1$.

Furthermore, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. The proof follows from Theorem 3.9 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. ■

Corollary 3.12 Let $(X,d,\preceq)$ be a partially ordered b-metric space with the coefficient $s \geqslant 1$. Let $f : X \to X$ and $T : X \to CB(X)$ be such that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Suppose that there exist $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0,\infty)$ with $\xi(st) = s\xi(t)$ for each $t \in [0,\infty)$ and $\sum_{n=1}^{\infty} s^n\psi^n(t) < \infty$ for each $t > 0$ satisfying

$$\alpha(fx, fy) \geqslant 1 \Rightarrow \xi(sH(Tx,Ty)) \leqslant \psi(\xi(M_s(fx,fy)))$$

for all $x, y \in X$ with $fx \preceq fy$ or, $fy \preceq fx$. Suppose the triple $(X,d,\preceq)$ has the following property:

$(\acute{\dagger})$ If (fx_n) is a sequence in X such that $fx_n \to x$ and fx_n, fx_{n+1} are comparable with $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \geqslant 1$, then there exists a subsequence (fx_{n_i}) of (fx_n) such that fx_{n_i}, x are comparable with $\alpha(fx_{n_i}, x) \geqslant 1$ for all $i \geqslant 1$.

If there exists $x_0 \in X$ such that fx_n, fx_m are comparable for $m, n = 0, 1, 2, \cdots$ and $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$, for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$, then f and T have a point of coincidence in X. Moreover, f and T have a unique point of coincidence in X if the following property holds:

$(\acute{\ddagger})$ If x, y are points of coincidence of f and T in X, then x, y are comparable and $\alpha(x,y) \geqslant 1$.

Furthermore, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. The proof can be obtained from Theorem 3.9 by taking $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x,y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. ■

As an application of Theorem 3.9, we obtain the following theorem.

Theorem 3.13 Let (X,d) be a b-metric space with the coefficient $s \geqslant 1$ and let $T : X \to CB(X)$ and $f : X \to X$ be a hybrid pair such that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Suppose that T and f are strictly (α, ψ, ξ)-contractive mappings with $\xi(st) = s\xi(t)$ for each $t \in [0,\infty)$ and $\sum_{n=1}^{\infty} s^n\psi^n(t) < \infty$ for each $t > 0$ satisfying the following conditions:

(i) T is an α_*-admissible multi-valued mapping w.r.t. f;
(ii) there exists $x_0 \in X$ such that $\alpha(fx_0, fx_1) \geqslant 1$, $\forall fx_1 \in Tx_0$;
(iii) if (fx_n) is a sequence in X with $fx_n \to x$ and $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for each $n \geqslant 1$, then there exists a subsequence (fx_{n_i}) of (fx_n) such that $\alpha(fx_{n_i}, x) \geqslant 1$ for all $i \geqslant 1$.

Then f and T have a point of coincidence in X. Moreover, f and T have a unique point of coincidence in X if the property $(\ddagger)$ holds. Furthermore, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. We take $G = G_0 = (X, X \times X)$. Then, T and f are strictly $(\alpha,\psi,\xi) - G_0$-contractive. By hypothesis (ii), there exists $x_0 \in X$ such that $\alpha(fx_0, fx_1) \geqslant 1$ for all $fx_1 \in Tx_0$. By hypothesis (i), it follows that $\alpha_*(Tx_0, Tx_1) \geqslant 1$ and hence $\alpha(fx_1, fx_2) \geqslant 1$ for all $fx_1, fx_2 \in Tx_1$. By repeated use of hypothesis (i), we get that $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$. Moreover, $(fx_n, fx_m) \in E(\widetilde{G_0})$ for $m, n = 0, 1, 2, \cdots$. This ensures that $x_0 \in C^{\alpha}_{fT}$ and hence $C^{\alpha}_{fT} \neq \emptyset$. Furthermore, hypothesis (iii) shows that property $(*)$ holds. Thus, all the conditions of Theorem 3.9 are satisfied and the conclusion of Theorem 3.13 can be obtained from Theorem 3.9. ■

Theorem 3.14 Let (X, d) be a complete b-metric space with the coefficient $s \geqslant 1$ and let $G = (V(G), E(G))$ be a graph. Let $f : X \to X$ and $T : X \to CB(X)$ be the $\tilde{G}$-continuous and compatible hybrid pair such that $T(X) \subseteq f(X)$. Suppose that T and f are strictly $(\alpha, \psi, \xi) - G$-contractive mappings with $\xi(st) = s\xi(t)$ for each $t \in [0, \infty)$ and $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t > 0$. Then f and T have a point of coincidence in X if $C_{fT}^{\alpha} \neq \emptyset$. Moreover, f and T have a unique common fixed point in X if the graph G has the property $(**)$.

Proof. As in the proof of Theorem 3.9, we can construct a Cauchy sequence (fx_n) in X such that $fx_n \in Tx_{n-1}$, $(fx_n, fx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$ and $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and

$$\xi(sH(Tx_n, Tx_{n+1})) \leqslant \psi(\xi(sd(fx_n, fx_{n+1}))) < \psi^n(k), \tag{18}$$

where $k = q\psi(\xi(sd(fx_0, fx_1)))$. (X, d) being complete, there exists $t \in X$ such that $fx_n \to t$ as $n \to \infty$. We now show that (Tx_n) is a Cauchy sequence in $(CB(X), H)$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} H(Tx_n, Tx_m) \leqslant\, & sH(Tx_n, Tx_{n+1}) + s^2 H(Tx_{n+1}, Tx_{n+2}) + \cdots \\ & + s^{m-n-1} H(Tx_{m-2}, Tx_{m-1}) + s^{m-n-1} H(Tx_{m-1}, Tx_m). \end{aligned}$$

Therefore, by using condition (18), we obtain that

$$\begin{aligned} \xi(H(Tx_n, Tx_m)) &\leqslant \xi(sH(Tx_n, Tx_{n+1})) + s\xi(sH(Tx_{n+1}, Tx_{n+2})) + \cdots \\ &\quad + s^{m-n-2}\xi(sH(Tx_{m-2}, Tx_{m-1})) + s^{m-n-1}\xi(sH(Tx_{m-1}, Tx_m)) \\ &< \psi^n(k) + s\psi^{n+1}(k) + \cdots + s^{m-n-2}\psi^{m-2}(k) + s^{m-n-1}\psi^{m-1}(k) \\ &= \frac{1}{s^n} \sum_{i=n}^{m-1} s^i \psi^i(k). \end{aligned}$$

Since $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$, it follows that

$$\lim_{n,m\to\infty} \xi(H(Tx_n, Tx_m)) = 0.$$

By using (ξ_1) and (ξ_3), we get

$$\lim_{n,m\to\infty} H(Tx_n, Tx_m) = 0.$$

This proves that (Tx_n) is a Cauchy sequence in the complete b-metric space $(CB(X), H)$. So, there exists $M \in CB(X)$ such that $Tx_n \to M$. Now,

$$\begin{aligned} d(t, M) &\leqslant s[d(t, fx_n) + d(fx_n, M)] \\ &\leqslant s[d(t, fx_n) + H(Tx_{n-1}, M)] \to 0 \ as\ n \to \infty. \end{aligned}$$

Since M is closed, $t \in M$. The compatibility of f and T gives that $H(Tfx_n, fTx_n) \to 0$ as $n \to \infty$. By $\tilde{G}$-continuity of f and T, we have

$$\begin{aligned} d(ft, Tt) &\leqslant s[d(ft, ffx_{n+1}) + d(ffx_{n+1}, Tt)] \\ &\leqslant s[d(ft, ffx_{n+1}) + H(fTx_n, Tt)] \\ &\leqslant sd(ft, ffx_{n+1}) + s^2H(fTx_n, Tfx_n) + s^2H(Tfx_n, Tt) \\ &\to 0 \ as \ n \to \infty, \end{aligned}$$

which implies that $ft \in Tt$, since Tt is closed. Taking $u = ft$, it follows that u is a point of coincidence of f and T in X. By an argument similar to that used in Theorem 3.9, it follows that u is the unique point of coincidence of f and T in X. Since compatibility implies weak compatibility, by Proposition 2.19, it follows that f and T have a unique common fixed point in X. ∎

Corollary 3.15 Let (X, d) be a complete b-metric space with the coefficient $s \geqslant 1$ and let $G = (V(G), E(G))$ be a graph. Let $T : X \to CB(X)$ be a $\tilde{G}$-continuous strictly $(\alpha, \psi, \xi) - G$-contractive mapping with $\xi(st) = s\xi(t)$ for each $t \in [0, \infty)$ and $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t > 0$. Then T has a fixed point in X if $C_T^\alpha \neq \emptyset$. Moreover, T has a unique fixed point in X if the graph G has the property $(**^{\prime})$.

Proof. The proof follows from Theorem 3.14 by taking $f = I$. ∎

Corollary 3.16 Let (X, d) be a complete b-metric space with the coefficient $s \geqslant 1$ and let $f : X \to X$ and $T : X \to CB(X)$ be the continuous and compatible hybrid pair such that $T(X) \subseteq f(X)$. Suppose that T and f are strictly (α, ψ, ξ)-contractive mappings with $\xi(st) = s\xi(t)$ for each $t \in [0, \infty)$ and $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t > 0$. If there exists $x_0 \in X$ such that $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$, then f and T have a point of coincidence in X. Moreover, f and T have a unique common fixed point in X if the property (‡) holds.

Proof. The proof follows from Theorem 3.14 by taking $G = G_0$. ∎

Corollary 3.17 Let $(X, d, \preceq)$ be a partially ordered complete b-metric space with the coefficient $s \geqslant 1$. Let $f : X \to X$ and $T : X \to CB(X)$ be the continuous and compatible hybrid pair such that $T(X) \subseteq f(X)$. Suppose that there exist $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0, \infty)$ with $\xi(st) = s\xi(t)$ for each $t \in [0, \infty)$ and $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t > 0$ such that

$$\alpha(fx, fy) \geqslant 1 \Rightarrow \xi(sH(Tx, Ty)) \leqslant \psi(\xi(M_s(fx, fy)))$$

for all $x, y \in X$ with $fx \preceq fy$ or, $fy \preceq fx$. If there exists $x_0 \in X$ such that fx_n, fx_m are comparable for $m, n = 0, 1, 2, \cdots$ and $\alpha(fx_n, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$, for every sequence (fx_n) such that $fx_n \in Tx_{n-1}$, then f and T have a point of coincidence in X. Moreover, f and T have a unique common fixed point in X if the property $(\ddagger^{\prime})$ holds.

Proof. The proof can be obtained from Theorem 3.14 by taking $G = G_2$. ∎

The following corollary is indeed a generalization of Theorem 2 of Kaneko and Sessa [22].

Corollary 3.18 Let (X, d) be a complete b-metric space with the coefficient $s \geqslant 1$, $f : X \to X$ and $T : X \to CB(X)$ be compatible continuous mappings such that $T(X) \subseteq f(X)$ and

$$sH(Tx, Ty) \leqslant h\, max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2s}\}$$

for all $x, y \in X$, where $0 < h < \frac{1}{s}$. Then there exists a point $t \in X$ such that $ft \in Tt$.

Proof. The proof can be obtained from Theorem 3.14 by taking $G = G_0$, $\alpha(x, y) = 1$ for all $x, y \in X$, $\xi(t) = t$ for each $t \geqslant 0$ and $\psi(t) = ht$ for each $t \geqslant 0$, where $h \in (0, \frac{1}{s})$ is a fixed number. ■

Remark 3 *It is worth mentioning that in Corollary 3.18, f and T have a unique common fixed point in X.*

Remark 4 *Several special cases of our results can be obtained by restricting $T : X \to X$ and taking $\xi(t) = t$, $\psi(t) = ht$ for each $t \geqslant 0$, where $h \in (0, \frac{1}{s})$ is a fixed number, $\alpha(x, y) = 1$, $G = G_0$.*

As an application of Theorem 3.14, we obtain the following theorem.

Theorem 3.19 Let (X, d) be a complete b-metric space with the coefficient $s \geqslant 1$ and let $T : X \to CB(X)$ and $f : X \to X$ be the continuous and compatible hybrid pair such that $T(X) \subseteq f(X)$. Suppose that T and f are strictly (α, ψ, ξ)-contractive mappings with $\xi(st) = s\xi(t)$ for each $t \in [0, \infty)$ and $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t > 0$ satisfying the following conditions:

(i) T is an α_*-admissible multi-valued mapping w.r.t. f;
(ii) there exists $x_0 \in X$ such that $\alpha(fx_0, fx_1) \geqslant 1$, $\forall fx_1 \in Tx_0$.

Then f and T have a point of coincidence in X. Moreover, f and T have a unique common fixed point in X if the property (‡) holds.

Proof. The proof is similar to that of Theorem 3.13. ■

Finally, we furnish some examples to discuss the validity of our results.

***Example* 3.20** Let $X = [0, \infty)$ with $d(x, y) = \mid x - y \mid^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{n}) : n = 1, 2, 3, \cdots\}$. Let $T : X \to CB(X)$ be defined by $Tx = \{0, \frac{x}{3}\}$ for all $x \in X$ and $fx = 3x$ for all $x \in X$. Obviously, $T(X) \subseteq f(X) = X$. Let $\alpha : X \times X \to [0, \infty)$ be defined by $\alpha(x, y) = 1$ if $x, y \in [0, 1]$ and otherwise, $\alpha(x, y) = \frac{1}{2}$. Take $\psi(t) = \frac{t}{8}$ and $\xi(t) = \frac{t}{2}$ for each $t \geqslant 0$. If $x = 0$ and $y = \frac{1}{3n}$, then $fx = 0$ and $fy = \frac{1}{n}$ and so $(fx, fy) \in E(\tilde{G})$ and $\alpha(fx, fy) = 1$. For $x = 0$ and $y = \frac{1}{3n}$, we have $Tx = \{0\}$,

$Ty = \{0, \frac{1}{9n}\}$ and $\xi(sH(Tx,Ty)) = \xi(\frac{s}{81n^2}) = \frac{1}{81n^2}$. Moreover,

$$
\begin{aligned}
M_s(fx,fy) &= max \left\{ \begin{array}{l} d(fx,fy), d(fx,Tx), d(fy,Ty), \\ \frac{d(fx,Ty)+d(fy,Tx)}{2s} \end{array} \right\} \\
&= max \left\{ \begin{array}{l} d(0,\frac{1}{n}), d(0,\{0\}), d(\frac{1}{n},\{0,\frac{1}{9n}\}), \\ \frac{d(0,\{0,\frac{1}{9n}\})+d(\frac{1}{n},\{0\})}{2s} \end{array} \right\} \\
&= max \left\{ \frac{1}{n^2}, 0, \frac{64}{81n^2}, \frac{1}{4n^2} \right\} \\
&= \frac{1}{n^2}.
\end{aligned}
$$

So, $\psi(\xi(M_s(fx,fy))) = \psi(\xi(\frac{1}{n^2})) = \psi(\frac{1}{2n^2}) = \frac{1}{16n^2}$. Thus, $\xi(sH(Tx,Ty)) \leqslant \psi(\xi(M_s(fx,fy)))$ for all $x,y \in X$ with $(fx,fy) \in E(\tilde{G})$ and $\alpha(fx,fy) = 1$. Therefore, T and f are strictly $(\alpha,\psi,\xi) - G$-contractive mappings.

We can verify that $x_0 = 0 \in C^{\alpha}_{fT}$. In fact, $fx_n \in Tx_{n-1},\ n = 1, 2, 3, \cdots$ gives that $fx_1 \in T0 = \{0\} \Rightarrow x_1 = 0$ and so $fx_2 \in Tx_1 = \{0\} \Rightarrow x_2 = 0$. Proceeding in this way, we get $fx_n = 0$ for $n = 0, 1, 2, \cdots$ and hence $(fx_n, fx_m) = (0,0) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$ and $\alpha(fx_n, fx_{n+1}) = 1$ for all $n \in \mathbb{N} \cup \{0\}$. Also, any sequence (fx_n) with the property $\alpha(fx_n, fx_{n+1}) = 1$ must be a sequence in $[0,1]$. Moreover, $(fx_n, fx_{n+1}) \in E(\tilde{G})$ must be either a constant sequence or a sequence with $fx_n = 0$ if n is odd and $fx_n = \frac{1}{n}$ if n is even, where the words 'odd' and 'even' are interchangeable. Consequently it follows that property $(*)$ holds. Furthermore, the graph G has the property $(**)$ and f and T are weakly compatible. Thus, we have all the conditions of Theorem 3.9 and 0 is the unique common fixed point of f and T in X.

We now examine the necessity of property $(**)$ in Theorem 3.9 for the unique point of coincidence.

Remark 5 *In Example 3.20, if we take*

$$
\begin{aligned}
Tx &= \{0, \frac{x}{3}\},\ if\ 0 \leqslant x < 1 \\
&= \{0\},\ if\ x = 1 \\
&= [x^2, x^2+5],\ if\ x > 1
\end{aligned}
$$

instead of $Tx = \{0, \frac{x}{3}\}$ for all $x \in X$, then all the conditions of Theorem 3.9 except property $(**)$ are satisfied. We observe that f and T have infinitely many points of coincidence in X.

The following example shows that without property $(**)$, a unique common fixed point of f and T may not exists.

***Example* 3.21** Let $X = \{1, 3, 5\} \cup [6, \infty)$ with $d(x,y) = |x-y|^3$ for all $x,y \in X$. Then (X,d) is a complete b-metric space with the coefficient $s = 4$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(1, 5)\}$. Let $T : X \to CB(X)$ be defined by

$$\begin{aligned} Tx &= \{3,\, 5\},\ if\ x = 1,\, 5 \\ &= \{3\},\ if\ x = 3 \\ &= [x^2, x^2 + 2],\ if\ x \geqslant 6 \end{aligned}$$

and

$$\begin{aligned} fx &= x,\ if\ x = 1,\, 3,\, 5 \\ &= x + 1,\ if\ x \geqslant 6. \end{aligned}$$

Obviously, $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of (X, d). Let $\alpha : X \times X \to [0, \infty)$ be defined by $\alpha(x, y) = 1$ for all $x, y \in X$. Take $\psi(t) = \frac{t}{6}$ and $\xi(t) = \frac{t}{2}$ for each $t \geqslant 0$. Then it is easy to verify that $\xi(sH(Tx, Ty)) \leqslant \psi(\xi(M_s(fx, fy)))$ for all $x,\, y \in X$ with $(fx, fy) \in E(\tilde{G})$ and $\alpha(fx, fy) = 1$. Therefore, T and f are strictly $(\alpha,\, \psi,\, \xi) - G$-contractive mappings. Moreover, $3 \in C^{\alpha}_{fT}$ and property $(*)$ holds. We find that 3 and 5 are points of coincidence of f and T in X. In fact, 3 and 5 are common fixed points of f and T in X. However, f and T are weakly compatible, there does not exist unique common fixed point of f and T due to lack of property $(**)$ of the graph G.

Remark 6 *In Example 3.21, T and f are strictly $(\alpha,\, \psi,\, \xi) - G$-contractive but not strictly $(\alpha,\, \psi,\, \xi)$-contractive. In fact, for $x = 1$, $y = 3$, we have $fx = 1,\ fy = 3, Tx = \{3,\, 5\},\ Ty = \{3\}$ and so $(fx, fy) \notin E(\tilde{G})$. Then, $\xi(sH(Tx, Ty)) = \xi(32) = 16$ and*

$$\begin{aligned} M_s(fx, fy) &= max \begin{Bmatrix} d(fx, fy), d(fx, Tx), d(fy, Ty), \\ \frac{d(fx,Ty)+d(fy,Tx)}{2s} \end{Bmatrix} \\ &= max \left\{ 8,\, 8,\, 0,\, \frac{8+0}{4} \right\} \\ &= 8, \end{aligned}$$

which implies that $\xi(sH(Tx, Ty)) > \psi(\xi(M_s(fx, fy)))$. Consequently, it follows that T and f are not strictly $(\alpha,\, \psi,\, \xi)$-contractive.

The following example supports our Theorem 3.14.

***Example* 3.22** Let $X = [1, \infty)$ with $d(x, y) =\mid x - y \mid^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with the coefficient $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(n, n+1) : n \in \mathbb{N}\}$. Let $fx = 3x^2 - 2$ and $Tx = [1, x^2]$ for each $x \geqslant 1$. Then, T and f are $\tilde{G}$-continuous and $T(X) = f(X) = X$. It is to be noted that $fTx = [1, 3x^4 - 2] \in CB(X)$ for all $x \in X$. Since $fx_n \to 1$ and $Tx_n \to \{1\}$ iff $x_n \to 1$,

$$\begin{aligned} H(fTx_n, Tfx_n) &=\mid (9x_n^4 - 12x_n^2 + 4) - (3x_n^4 - 2) \mid^2 \\ &=\mid 6x_n^4 - 12x_n^2 + 6 \mid^2 \\ &= 36 \mid x_n^4 - 2x_n^2 + 1 \mid^2 \to 0\ iff\ x_n \to 1, \end{aligned}$$

it follows that f and T are compatible. Let $\alpha : X \times X \to [0, \infty)$ be defined by $\alpha(x, y) = 1$ for all $x, y \in X$. Take $\psi(t) = \frac{t}{4}$ and $\xi(t) = \frac{t}{2}$ for each $t \geqslant 0$. If $x = \sqrt{\frac{n+2}{3}},\ y = \sqrt{\frac{n+3}{3}},\ n \in \mathbb{N}$, then $fx = n,\ fy = n + 1$ and so $(fx, fy) \in E(\tilde{G})$ and $\alpha(fx, fy) = 1$.Then

$$H(Tx, Ty) = \mid x^2 - y^2 \mid^2 = \mid \frac{n+2}{3} - \frac{n+3}{3} \mid^2 = \frac{1}{9}$$

and $d(fx, fy) = 1 \leqslant M_s(fx, fy)$ which implies that $\psi(\xi(d(fx, fy))) \leqslant \psi(\xi(M_s(fx, fy)))$. Now,

$$\xi(sH(Tx, Ty)) = \xi(\frac{s}{9}) = \frac{1}{9} < \frac{1}{8} = \psi(\xi(d(fx, fy))) \leqslant \psi(\xi(M_s(fx, fy))).$$

Thus, T and f are strictly $(\alpha,\ \psi,\ \xi) - G$-contractive.

It is easy to verify that property $(**)$ holds and $1 \in C^{\alpha}_{fT}$ i.e., $C^{\alpha}_{fT} \neq \emptyset$. Thus, we have all the conditions of Theorem 3.14 and 1 is the unique common fixed point of f and T in X.

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