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On the square root of quadratic matrices

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Abstract. Here we present a new approach to calculating the square root of a quadratic matrix. Actually, the purpose of this article is to show how the Cayley-Hamilton theorem may be used to determine an explicit formula for all the square roots of 2×2 matrices.

Keywords: Square root of matrix, matrix equation, eigenvalue.

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1. Introduction

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. In mathematics, the square root of a matrix extends the notion of square root from numbers to matrices. Matrix B is said to be a square root of A if the matrix product BB is equal to A.

Now, what is the square root of a matrix such as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$? It is not, in general, $\begin{pmatrix} \sqrt{a} & \sqrt{b} \\ \sqrt{c} & \sqrt{d} \end{pmatrix}$. This is easy to see since the upper left entry of its square is $a + \sqrt{bc}$ and not a.

In recent years, several articles have been written about the root of a matrix, and one can refer to [4–6]. A number of methods have been proposed for computing the square root of a matrix, and these are usually based upon Newton's method, either directly or via the sign function (see, e.g., [1–3]).

This short paper intends to show how the Cayley-Hamilton theorem may be used to determine an explicit formula for all the square roots of 2×2 matrices.

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2. Our Method

The set of all matrices which their square is A, denotes by $\sqrt{A}$, i.e.,

$$\sqrt{A} = \left\{X : \; X \in M_n\left(\mathbb{C}\right), X^2 = A\right\}.$$

This set can be very large. For example, we will see that $\sqrt{I}$ has infinite members. We can define the n-th root of a matrix A as follows

$$\sqrt[n]{A} = \{X : \; X \in M_n\left(\mathbb{C}\right), X^n = A\}.$$

It is well known to all, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the eigenvalues, are the roots of the polynomial $\lambda^2 - (trA)\,\lambda + \det A$. From the Cayley-Hamilton theorem, we know that

$$A^2 - (trA)\,A + (\det A)\,I = 0.$$

Thus, we have $A^2 = (trA)\,A - (\det A)\,I$. Therefore, the equation $A^2 = B$ can be written as follows $(trA)\,A - (\det A)\,I = B$. Consequently,

$$A = \frac{1}{trA}\left(B + (\det A)\right)I. \tag{1}$$

We try to write trA and $\det A$ as functions of trB and $\det B$. To do this end, we need the following three lemmas.

Lemma 2.1 Let A be a 2×2 matrix. Then $trA^2 = (trA)^2 - 2\det A$.

Proof. Assume that λ_1 and λ_2 are the eigenvalues of the matrix A. Then we can easily see that λ_1^2, λ_2^2 are the eigenvalues of A^2. Moreover, $trA = \lambda_1 + \lambda_2$ and $\det A = \lambda_1\lambda_2$. So $trA^2 = \lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2$. In other words, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$. Therefore

$$\begin{aligned} tr\left(A^2\right) = \left(a^2 + bc\right) + \left(bc + d^2\right) &= a^2 + 2bc + d^2 \\ &= (a + d)^2 - 2ad + 2bc \\ &= (a + d)^2 - 2\,(ad - bc) \\ &= (trA)^2 - 2\,(\det A)\,. \end{aligned}$$

■

We can state the following remark, whose proof is omitted being similar to the proof of Lemma 2.1.

Remark 1 *Let $A, B \in M_2\left(\mathbb{C}\right)$ and $A^2 = B$. Then the following statements are hold:*

(1) $\det A = \sqrt{\det B}$.
(2) $trA = \sqrt{trB + 2\sqrt{\det B}}$.

Lemma 2.2 Let $A \in M_2\left(\mathbb{C}\right)$. If $trA = 0$ then $A^2 \in \langle I \rangle$.

Proof. We will prove Lemma 2.2 in two ways. In general, we have

$$A^2 - (trA)\,A + (\det A)\,I = 0.$$

Therefore, if $trA = 0$ then we obtain $A^2 = -(\det A)\, I$ and $A^2 \in \langle I \rangle$. We can state another proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $a + d = 0$. Then $A^2 = \begin{pmatrix} a^2 + bc & 0 \\ 0 & bc + d^2 \end{pmatrix}$. Hence, $A^2 = \left(a^2 + bc\right) I$, since $a^2 = d^2$. ■

Now, we can consider two cases:

I. If $B \notin \langle I \rangle$, by Lemma 2.2, $trA \neq 0$. Thus, we can compute A by $A = \frac{1}{trA}\left(B + (\det A)\, I\right)$. Note that trA and $\det A$ are given by Lemma 1.

II. If $B \in \langle I \rangle$, we have $B = \alpha I$ for some $\alpha \in \mathbb{C}$. Hence we should calculate $\sqrt{\alpha I}$.

The next lemma says that, it refers to $\sqrt{I}$.

Lemma 2.3 For each $\alpha \in \mathbb{C}$ and any matrix A, $\sqrt{\alpha A} = \sqrt{\alpha}\sqrt{A}$.

Proof. Assume that $\alpha \neq 0$ and $X \in \sqrt{\alpha A}$. So $X^2 \in \alpha A$ hence $\frac{1}{\alpha} X^2 = A$ Therefore, $\frac{1}{\sqrt{\alpha}} X \in \sqrt{A}$, which implies that $X \in \sqrt{\alpha}\sqrt{A}$. Conversely, if $X \in \sqrt{\alpha}\sqrt{A}$, then $\frac{1}{\alpha} X^2 = A$. Hence, $X^2 = \alpha A$ and $X \in \sqrt{\alpha A}$. ■

Now, we try to compute $\sqrt{I}$. Assume that $A \in M_2(\mathbb{C})$ and $A^2 = I$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$A^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = I.$$

Hence, we have

$$a^2 + bc = 1 \tag{2}$$

$$b\,(a + d) = 0 \tag{3}$$

$$c\,(a + d) = 0 \tag{4}$$

$$bc + d^2 = 1. \tag{5}$$

We should solve this system of equations. The equation (3) says that $b = 0$ or $a + d = 0$ and the equation (4) says that $c = 0$ or $a + d = 0$. We consider two cases:

i. If $a + d = 0$. The equations (3) and (4) hold. We have $a^2 + bc = 1$ or $a = \sqrt{1 - bc}$ and since $a + d = 0$ we have $d = -a = -\sqrt{1 - bc}$. Therefore

$$A \in \left\{ \begin{pmatrix} \sqrt{1 - bc} & b \\ c & -\sqrt{1 - bc} \end{pmatrix} : \; b, c \in \mathbb{C} \right\}.$$

ii. If $a + d \neq 0$, we must have $b = 0$ and $c = 0$. Hence $a = \pm 1$ and $d = \pm 1$. Therefore, there are two solutions $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence, we can write

$$\sqrt{I} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \bigcup \left\{ \begin{pmatrix} \sqrt{1 - bc} & b \\ c & -\sqrt{1 - bc} \end{pmatrix} : \; b, c \in \mathbb{C} \right\}.$$

Finally, we give some examples to show the efficiency of the presented method.

***Example* 2.4** Let $B = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}$. So $\det B = 4$ and $trB = 9$. Therefore, if $A^2 = B$ then $\det A = \sqrt{\det B} = \pm 2$ and $trA = \pm\sqrt{5}$ or $trA = \pm\sqrt{13}$. Hence, we have

$$A = \frac{1}{\pm\sqrt{13}} \left[\begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \pm 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \text{ or } A = \frac{1}{\pm\sqrt{5}} \left[\begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \pm 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right].$$

Thus,

$$A = \pm\frac{1}{\sqrt{13}} \begin{pmatrix} 3 & 2 \\ 2 & 10 \end{pmatrix} \text{ or } A = \pm\frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 6 \end{pmatrix}.$$

We may also apply this method to matrices without real eigenvalues.

***Example* 2.5** Let $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Then $\det B = -1$ and $trB = 2$. If $A^2 = B$ then $\det A = \pm i$ and $trA = \pm\sqrt{2 \pm 2i}$. Thus, we have

$$A = \pm\frac{1}{\sqrt{2+2i}} \left[\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} + i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right].$$

The following example consider the case $B \in \langle I \rangle$.

***Example* 2.6** Let $B = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. Therefore, $B = 4I$ and $\sqrt{B} = 2\sqrt{I}$. Hence, we have

$$\sqrt{B} = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \right\} \bigcup \left\{ \begin{pmatrix} 2\sqrt{1-bc} & 2b \\ 2c & -2\sqrt{1-bc} \end{pmatrix} : \ b, c \in \mathbb{C} \right\}.$$

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