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Connected and hyperconnected generalized topological spaces

I. Basdouri[a*], R. Messaoud[a], A. Missaoui[a]

[a] *Département de Mathématiques, Faculté des Sciences de Gafsa, Zarroug 2112 Gafsa, Tunisie.*

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Abstract. Á. Császár introduced and extensively studied the notion of generalized open sets. Following Csázar, we introduce a new notion hyperconnected. We study some specific properties about connected and hyperconnected in generalized topological spaces. Finally, we characterize the connected component in generalized topological spaces.

Keywords: Generalized topology, m-structure, weak structure, connected, g-closed, hyperconnected.

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1. Introduction

The properties of structures defined by a given set X and a relation, respectively relations defined on a class of subsets of X and satisfying some conditions are often studied. Such structures are given for example in [1, 3, 5, 6, 11, 12]. The best known structures of such type are topological spaces defined by a closure operation. Generalized topological space is an important generalization of topological spaces. In the past decade, Csázar [4–10] and others have been considering generalized topological spaces, and developing a theory for them. more precisely, for the last years, different forms of open sets are being studied. Recently, a significant contribution to the theory of generalized open sets has been presented by A. Csázar [5–10]. Especially, the author defined some basic operators on generalized topological spaces. It is observed that a large number of papers are devoted

*Corresponding author.
E-mail address: basdourimed@yahoo.fr (I. Basdouri).

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to the study of generalized open sets like open sets of a topological space, containing the class of open sets and possessing properties more or less similar to those of open sets.

In the present paper, we introduce the notion of hyperconnected and we studied some specific properties about connected and hyperconnected in Generalized Topological Spaces. Finally, we characterize the connected component in Generalized Topological Spaces.

2. Preliminaires

Let X be a nonempty set and g be a collection of subsets of X. Then g is called a *generalized topology* (GT for short) on X if $\emptyset \in g$ and $G_i \in g$ for i $\in$ I $\neq \emptyset$ implies $G = \bigsqcup_{i\in I} G_i \in g$. The pair (X, g) is called a generalized topological space (GTS for short) on X. The elements of g are called $g-$open sets and their complements are called $g-$closed sets. We denote the family of all $g-$closed sets in X by $gc(X)$. The generalized closure of a subset S of X, denoted by $c_g(S)$, is the intersection of all generalized closed sets including S. And the generalized interior of S, denoted by $i_g(S)$, is the union of generalized open sets contained in S.

Definition 2.1 2Let (X, g_X) be a generalized topological space and $A \subseteq X$. Then A is said to be
$g-$semi$-$open if $A \subseteq c_g(i_g(A))$,
$g-$preopen if $A \subseteq i_g(c_g(A))$,
$g-\alpha-$ open if $A \subseteq i_g(c_g(i_g(A)))$,
$g-\beta-$ open if $A \subseteq c_g(i_g(c_g(A)))$,

The complement of $g-$semi$-$open (resp., $g-$preopen, $g-$ α-open, $g-$ β-open) is said to be $g-$semi-closed (resp., $g-$preclosed, $g-$ α-closed, $g-$ β-closed).

Let us denote the class of all g-semi-open sets, $g-$preopen sets, $g-\alpha$-open sets and g-β-open sets on X by $\sigma(g_X)$ (σ for short), $\pi(g_X)$ (π for short), $\alpha(g_X)$ (α for short),and $\beta(g_X)$ (β for short) respectively.

Denote by $c_\sigma(X)$, $c_\pi(X)$, $c_\alpha(X)$ and $c_\beta(X)$, the closures of g-semi-closed sets, $g-$preclosed stes, $g-$ α-closed sets and $g-\beta$-closed sets on X.

Definition 2.2 1Let g_X and g_Y be generalized topologies on X and Y, respectively. Then a function $f : X \to Y$ is said to be (g_X, g_Y)-continuous if $G' \in g_Y$ implies that $f^{-1}(G') \in g_X$

Definition 2.3 3Let g_X and g_Y be generalized topologies on X and Y, respectively. Then a function $f : X \to Y$ is said to be
(α, g_Y)-continuous if for each $g-$open set U in Y, $f^{-1}(U)$ is $g-\alpha$-open in X,
(σ, g_Y)-continuous if for each $g-$open set U in Y, $f^{-1}(U)$ is $g-$semi-open in X
(π, g_Y)-continuous if for each $g-$open set Uin Y, $f^{-1}(U)$ is $g-$preopen in X
(β, g_Y)-continuous if for each $g-$open set U in Y, $f^{-1}(U)$ is $g-\beta$-open in X

Definition 2.4 4Let g_X and g_Y be generalized topologies on X and Y, respectively.

Then a function $f : X \to Y$ is said to be
contra (g_X, g_Y)-continuous if for each $g-$open set U in Y, $f^{-1}(U)$ is $g-$closed in X,
contra (α, g_Y)-continuous if for each $g-$open set U in Y, $f^{-1}(U)$ is $g-\alpha$-closed in X,
contra (σ, g_Y)-continuous if for each $g-$open set U in Y, $f^{-1}(U)$ is $g-$semi-closed in X,
contra (π, g_Y)-continuous if for each $g-$open set U in Y, $f^{-1}(U)$ is $g-$preclosed in X,
contra (β, g_Y)-continuous if for each $g-$open set U in Y, $f^{-1}(U)$ is $g-\beta$-closed in X.

3. On generalized connected spaces

Definition 3.1 Let (X, g_X) be a GTS. X is called:
$g-$connected if there are no nonempty disjoint $g-$open subsets U, V of X such that $U \cup V = X$
$g-\alpha$-connected if there are no nonempty disjoint $g-\alpha$-open subsets U, V of X such that $U \cup V = X$
$g-\sigma$-connected if there are no nonempty disjoint $g-$semi-open subsets U, V of X such that $U \cup V = X$
$g-\pi$-connected if there are no nonempty disjoint $g-$preopen subsets U, V of X such that $U \cup V = X$
g- β-connected if there are no nonempty disjoint $g-\beta$-open subsets U, V of X such that $U \cup V = X$

Theorem 3.2 Let $f : (X, g_X) \to (Y, g_Y)$ be a contra (α, g_Y)-continuous surjection and let X be $g-\alpha$-connected. Then Y is $g-$connected.

Proof. Let $f : (X, g_X) \to (Y, g_Y)$ be a contra (α, g_Y)-continuous surjection and let X be g-α-connected. Suppose that Y is not $g-$connected. Then there are nonempty disjoint $g-$open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. Thus V_1, $V_2 \in gC(Y)$. Since f is contra (α, g_Y)-continuous, then $f^{-1}(V_1)$, $f^{-1}(V_2)$ are g-α-open . Note that $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. Then X is not $g-\alpha$-connected, contradiction. Thus Y is $g-$connected. ■

Theorem 3.3 Let $f : (X, g_X) \to (Y, g_Y)$ be a contra (σ, g_Y)-continuous surjection and let X be $g-\sigma$-connected. Then Y is $g-$connected.

Proof. Let $f : (X, g_X) \to (Y, g_Y)$ be a contra (σ, g_Y)-continuous surjection and let X be $g-\sigma$-connected. Suppose that Y is not $g-$connected. Then there are nonempty disjoint $g-$open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. Thus V_1, $V_2 \in gC(Y)$. Since f is contra (σ, g_Y)-continuous, then $f^{-1}(V_1)$, $f^{-1}(V_2)$ are $g-$semi-open . Note that $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. Then X is not $g-\sigma$-connected, contradiction. Thus Y is $g-$connected. ■

Theorem 3.4 Let $f : (X, g_X) \to (Y, g_Y)$ be a contra (π, g_Y)-continuous surjection and let X be $g-\pi$-connected. Then Y is $g-$connected.

Proof. Let $f : (X, g_X) \to (Y, g_Y)$ be a contra (π, g_Y)-continuous surjection and let X be $g-\pi$-connected. Suppose that Y is not $g-$connected. Then there are nonempty disjoint $g-$open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. Thus V_1, $V_2 \in gC(Y)$. Since f is contra (π, g_Y)-continuous, then $f^{-1}(V_1)$, $f^{-1}(V_2)$ are $g-$preopen . Note that $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. Then X is not $g-\pi$-connected, contradiction. Thus Y is $g-$connected. ■

Theorem 3.5 Let $f : (X, g_X) \to (Y, g_Y)$ be a contra (β, g_Y)-continuous surjection and let X be $g-\beta$-connected. Then Y is $g-$connected.

Proof. Let $f : (X, g_X) \to (Y, g_Y)$ be a contra (β, g_Y)-continuous surjection and let X be $g-\beta$-connected. Suppose that Y is not $g-$connected. Then there are nonempty disjoint $g-$open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. Thus V_1, $V_2 \in gC(Y)$. Since f is contra (β, g_Y)-continuous, then $f^{-1}(V_1)$, $f^{-1}(V_2)$ are $g-\beta$-open . Note that $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. Then X is not $g-\beta$-connected, contradiction. Thus Y is $g-$connected. ■

Proposition 3.6 Let (X, g_X) be a GTS and $A \subseteq X$. Then we have the following implcations.

(1) A is $g-$open set $\Rightarrow$ (2) $g-\alpha$-open set $\Rightarrow$ (3) $g-$semi-open set $\Rightarrow$ (4) $g-\beta$-open set

and

(5) A is $g-\alpha$-open set $\Rightarrow$ (6) $g-$preopen set $\Rightarrow$ (7) $g-\beta$-open set

Proof. (1) $\Rightarrow$ (2). A is $g-$open set ie $A = i_g(A)$. Since $A \subseteq c_g(A) = c_g(i_g(A))$. Thus $i_g(A) = A \subseteq i_g(c_g(i_g(A)))$. Then A is $g-\alpha$-open set.

(2) $\Rightarrow$ (3). A is $g-\alpha$-open set ie $A \subseteq i_g(c_g(i_g(A))) \subseteq c_g(i_g(A)))$. Thus A is $g-$semi-open set

(3) $\Rightarrow$ (4). We have $A \subseteq c_g(A)$ and $i_g(A) \subseteq i_g(c_g(A))$. Thus $c_g(i_g(A)) \subseteq c_g(i_g(c_g(A)))$. Since A is $g-$semi-open, we have $A \subseteq c_g(i_g(A))$. Then A is $g-\beta$-open.

(5) $\Rightarrow$ (6). We have $i_g(A) \subseteq A$ and $c_g(i_g(A)) \subseteq c_g(A)$. Thus $i_g(c_g(i_g(A))) \subseteq i_g(c_g(A))$. Since A is $g-\alpha$-open we have $A \subseteq i_g(c_g(A))$. Then A is $g-$preopen.

(6) $\Rightarrow$ (7). We have $i_g(c_g(A)) \subseteq c_g(i_g(c_g(A)))$. Since A is $g-$preopen, we have $A \subseteq i_g(c_g(A))$. Thus $A \subseteq c_g(i_g(c_g(A)))$. Then A is $g-\beta$-open. ■

Corollary 3.7 Let (X, g_X) be a GTS. Then we have the following implcations.

(1) X is $g-\beta$-connected $\Rightarrow$ (2) $g-\sigma$-connected $\Rightarrow$ (3) $g-\alpha$-connected $\Rightarrow$ (4) $g-$connected

and

(5) X is $g-\beta$-connected $\Rightarrow$ (6) $g-\pi$-connected $\Rightarrow$ (7) $g-\alpha$-connected

Proof. (1) $\Rightarrow$ (2). Let X be $g-\beta$-connected. Suppose that X is not $g-\sigma$-connected Then there are nonempty disjoint g-semi-open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. By Proposition 3.6, $g-$semi-open implies $g-\beta$-open. Thus there are nonempty disjoint $g-\beta$-open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. Then X is not $g-\beta$-connected, contradiction. Then X is $g-\sigma$-connected.

(2) $\Rightarrow$ (3). Let X be $g-\sigma$-connected. Suppose that X is not $g-\alpha$-connected Then there are nonempty disjoint $g-$semi-open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. By Proposition 3.6, $g-\alpha$-open implies $g-$semi-open. Thus there are nonempty disjoint $g-$semi-open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. Then X is not $g-\sigma$-connected, contradiction. Then X is $g-\alpha$-connected.

(3) $\Rightarrow$ (4). Let X be g-α-connected. Suppose that X is not $g-$connected Then there are nonempty disjoint $g-$open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. By Proposition 3.6, $g-$open implies $g-\alpha$-open. Thus there are nonempty disjoint $g-\alpha$-open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. Then X is not $g-\alpha$-connected, contradiction. Then X is $g-$connected.

(5) $\Rightarrow$ (6). Let X be $g-\beta$-connected. Suppose that X is not $g-\pi$-connected Then there are nonempty disjoint $g-\pi$-open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. By Proposition 3.6, $g-$preopen implies $g-\beta$-open. Thus there are nonempty disjoint $g-\beta$-open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. Then X is not $g-\beta$-connected, contradiction. Then X is $g-\pi$-connected.

(6) $\Rightarrow$ (7). Let X be $g-\pi$-connected. Suppose that X is not $g-\alpha$-connected Then there are nonempty disjoint $g-\alpha$-open subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. By Proposition 3.6, $g-$semi-open implies $g-\pi$-open. Thus there are nonempty disjoint $g-$preopen subsets V_1, V_2 of Y such that $V_1 \cup V_2 = Y$. Then X is not $g-\pi$-connected, contradiction. Then X is $g-\alpha$-connected. ■

At the end of this section, we give some examples.

***Example* 3.8** Let

$$X = Y = \{a, b, c, d\} \quad \text{and} \quad g_X = g_Y = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}\}.$$

Let $f : (X, g_X) \to (X, g_X)$ be the identity function. f is contra (σ, g_X)-continuous.Then X is $g-$ connected but not $g - \sigma$-connected.

***Example* 3.9** Let

$$X = Y = \{a, b, c, d\} \quad \text{and} \quad g_X = g_Y = \{\emptyset, X, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}\}.$$

Define the contra (β, g_X)-continuous identity function $f : (X, g_X) \to (X, g_X)$. Then X is $g-$ connected but not $g - \beta$-connected.

4. On generalized hyperconnected spaces

Definition 4.1 Let (X, g_X) be a GTS. X is called
$g-$hyperconnected, if every nonempty $g-$open subset U of X is $g-$dense (i.e, $c_g(U) = X$).
$g-\alpha$-hyperconnected, if every nonempty $g - \alpha$-open subset U of X is $g-$dense.
$g-\sigma$-hyperconnected, if every nonempty $g-$semi-open subset U of X is $g-$dense.
$g-\pi$-hyperconnected, if every nonempty $g-$preopen subset U of X is $g-$dense.
$g-\beta$-hyperconnected, if every nonempty $g - \beta$-open subset U of X is $g-$dense.

Corollary 4.2 Let (X, g_X) be a GTS. Then we have the following implcations.
(1) X is $g-\beta$-hyperconnected $\Rightarrow$ (2)$g-\sigma$-hyperconnected $\Rightarrow$ (3) $g-\alpha$-hyperconnected $\Rightarrow$ (4) $g-$hyperconnected
and
(5) X is $g-\beta$-hyperconnected $\Rightarrow$ (6) $g-\pi$-hyperconnected $\Rightarrow$ (7) $g-\alpha$-hyperconnected

Proof. (1) $\Rightarrow$ (2). Let X be $g - \beta$-hypeconnected and let A be nonempty $g-$semi-open subset of X. By proposition 3.6, A is nonempty g-β-open subset. Since X is $g - \beta$-hypeconnected, we have $c_g(A) = X$. Then X is $g - \sigma$-hyperconnected.

(2) $\Rightarrow$ (3).Let X be $g-\sigma$-hypeconnected and let A be nonempty $g-\alpha$-open subset of X. By proposition 3.6, A is nonempty $g-$semi-open subset. Since X is $g-\sigma$-hypeconnected, we have $c_g(A) = X$. Then X is $g - \alpha$-hyperconnected.

(3) $\Rightarrow$ (4). Let X be $g-\alpha$-hypeconnected and let A be nonempty $g-$open subset of X. By proposition 3.6, A is nonempty $g - \alpha$-open subset. Since X is $g - \alpha$-hypeconnected, we have $c_g(A) = X$. Then X is $g-$hyperconnected.

(5) $\Rightarrow$ (6). Let X be $g-\beta$-hypeconnected and let A be nonempty $g-$preopen subset of X. By proposition 3.6, A is nonempty $g-\beta$-open subset. Since X is $g-\beta$-hypeconnected, we have $c_g(A) = X$. Then X is g-π-hyperconnected.

(6) $\Rightarrow$ (7). Let X be $g-\pi$-hypeconnected and let A be nonempty $g-\alpha$-open subset of X. By proposition 3.6, A is nonempty $g-$preopen subset. Since X is $g-\pi$-hypeconnected, we have $c_g(A) = X$. Then X is $g - \alpha$-hyperconnected. ■

5. Caracterisation of composent connected on generalization topological spaces

Definition 5.1 Let (X, g_X) be a general topological space and let $a \in X$. The $g-$connected component in X containing a is the greater $g-$connected subset of X

containing a.

Lemma 5.2 If A a $g-$connected subset of a general topological space X, then all subset B such that $A \subseteq B \subseteq c_g(A)$ is $g-$connected.

Theorem 5.3 Any general topological space (X, g_X) is written as the disjoint union of its $g-$connected components. Each g-connected component is g-closed.

Proof. Let (X, g_X) be a general topological space and $a \in X$. We define g- $C(a) = \bigcup_{a\in A_i}$ with A_i connected. g-$C(a)$ is connected because is the union of all g-connected A_i containing a. $g - C(a)$ is the largest $g-$connected part container a, since $g - C(a)$ is the union of all $g-$connected parts containing a. Easily verified:

(1) $b \notin g - C(a) \Longrightarrow g - C(a) \cap g - C(b) = \emptyset$.

(2) $b \in g - C(a) \Longrightarrow g - C(b) = g - C(a)$.

(1) and (2) show that the relationship $a\mathcal{R}b \Longleftrightarrow g - C(a) = g - C(b)$ is an equivalence relation whose equivalence classes are related composentes X. In particular:

$$X = \bigsqcup_{\bar{a}\in X/\mathcal{R}} g - C(a)$$

As $g - C(a)$ is $g-$connected, to after Lemma 5.2, adhesion $c_g(g - C(a))$ is also connected. Or $g - C(a)$ is the largest $g-$connected part containing a, we get $g - C(a) = c_g(g - C(a))$, ie, $g - C(a)$ is g-closed. ■

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