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An introduction to fixed-circle problem on soft metric spaces

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Abstract. Recently, soft set theory has been extensively studied both theoretically and practically with different approaches. On the other hand, fixed-circle problem has been investigated as a geometric generalization of fixed-point theory and this problem can be applied to some applicable areas. With these two perspectives, in this paper, we obtain some soft fixed-circle results using different auxiliary functions on a soft metric space. To do this, we are inspired various contractive conditions. The obtained results can be considered as an existence or uniqueness theorem. The proved theorems are supported by some illustrative examples. Finally, we give a list of geometric consequences of these results.

Keywords: Soft circle, soft fixed circle, soft metric space.

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1. Introduction and motivation

"*Soft set theory"* was introduced as a general mathematical tool for coping with encountered difficulties and uncertain problems in different fields such as engineering, medical science etc. [21]. After then, this theory has been extensively studied for both theoretic and applicable studies with various approaches (for example, see [8, 9, 13, 14, 16, 18, 23, 33–35] and the references therein). For example, some basic operations related to soft set theory were defined in [19]. Also, the notion of a soft topological space was introduce[d b](#page-14-0)y Shabir and Naz [29]. After then, some basic topological notions such as soft interior, soft continuity, soft compactness etc. were given in [38].

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If we want to give an example for the application area, in [14], an application to food engineering was obtained and the obtained results were optimized the results by using appropriate parameters and degree of membership functions. On the other hand, recently various decision making problems have been studied using the notion of soft set (for example, see [7, 27, 28] and the references therein). After thes[e e](#page-14-1)xamples, the advantages of studying soft set theory can be briefly listed as follows:

- Using a parametrization point of view, soft set theory presented a new mathematical model.
- Thanks to soft set theory, the constructed model can allow to deal with uncertainties.
- Using a soft set, complicated objects can become more understandable by means of parametrization.
- Soft set theory is related to soft computing models such as fuzzy set theory, rough set theory etc.
- Soft set theory has been used in the various applications such as topology, decision making problem, algebraic structure etc.

Recently, "*fixed-point theory*" has been extensively studied with different aspects. One of these aspects is to investigate the geometric properties of the fixed point set $Fix(T)$ of a self-mapping $T : X \to X$ when *T* has more than one fixed point. For this purpose, "*fixed-circle problem*" was introduced on a metric space by Ozgür and Tas [26]. This problem can be considered as a geometric generalization to the fixed-point theory. In many studies, some solutions was investigated using diverse approaches and contractions (see, for example $[20, 22, 25, 30, 31]$ and the references therein). For example, in $[22]$, some fixed-disc results were obtained using the set of simulation functions [on](#page-14-2) metric spaces. In [30], bilateral-type solutions to the fixed-circle problem were proved and an application to rectified linear units application was given. Also, this problem came to be thought of mor[e g](#page-14-3)[ene](#page-14-4)r[ally](#page-14-5) [as](#page-15-1) t[he](#page-15-2) fixed figure problem such as fixed-ellipse probl[em](#page-14-4), fixed-Cassini curve problem etc. (see, [10, 15, 24, 32] for more details). What kind of advantages [do](#page-15-1)es working with a fixed-circle or fixed-figure results give us?

- In cases where the number of fixed points is more than one, a geometric meaning can be attributed to the set of fixed poin[ts.](#page-14-6)
- Thanks to the increase in the number of fixed points, the applicability of the obtained theory increases.
- Non-unique fixed points plays an important role a real life problem such as discontinuous activation functions etc.
- The set of non-unique fixed points may form a geometric shape such as circle, ellipse, hyperbola, Apollonius circle etc.
- By means of the fixed-circle problem, some theoretical results in the literature can be generalized.

From the above motivation, the main purpose is to obtain new solution to the "*Fixedcircle problem*" using the notion of a soft set. To do this, at first, we introduce the notion of a soft circle with some nice examples. After then, we prove six existence fixed-circle theorems and three uniqueness fixed-circle theorems on a soft metric space. Also, we prove a theorem which exclude the identity soft mapping. To show the validity of our obtained results, we give one necessary examples. Finally, we construct a list of geometric consequences of these results. The obtained results are important in generalizing the known results in the literature.

To summarize, the paper is organized as follows: After we give fundamental objectives

and advantages of this work in Section 1, we recall some basic notions related to soft set theory in Section 2. We define a concept of a fixed circle and prove some existence and uniqueness theorems on a soft metric space in Section 3. In the last section, we mention the importance of this paper and propose some open problems as the future work.

2. Preliminaries

In this section, we recall some basic concepts related to soft set theory.

Definition 2.1 [21] Let *X* be an initial universe set and *E* be a set of parameters. A pair (F, E) is called a soft set over X if and only if F is a mapping from E into the set of all subsets of the universe set X, that is, $F: E \to P(X)$, where $P(X)$ is the set of all subsets of the set *X*. (1) (*F, E*) is said to be a null soft set denoted by $\widetilde{\theta}$ if $F(e) = \emptyset$ for all $e \in E$.

(1) (*F, E*) is said to be a null soft set denoted by $\widetilde{\theta}$ if $F(e) = \emptyset$ for all $e \in E$.

Definition 2.2 [[19\]](#page-14-0) Let (F, E) be a soft set over a universe set X.

-
- (1) (*F, E*) is said to be a null soft set denoted by $\widetilde{\theta}$ if $F(e) = \emptyset$ for all $e \in E$.
(2) (*F, E*) is said to be an absolute soft set denoted by \widetilde{X} if $F(e) = X$ for all $e \in E$.

Definition 2.3 [\[12](#page-14-7)] Let *A, B* \subset *E* be nonempty subsets.

- (1) For two soft sets (F, A) and (G, B) over a common universe X, (F, A) is said to be a soft subset of (G, B) if $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. Then we write (*For two soft sets*
 For two soft sets
 E a soft subset $(F, A) \tilde{\subset} (G, B)$ $(F, A) \tilde{\subset} (G, B)$ $(F, A) \tilde{\subset} (G, B)$.
- (2) Two soft sets (F, A) and (G, B) over a common universe X are said to be equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A) .

Definition 2.4 [19] Let $A, B \subset E$ be nonempty subsets. The union of two soft sets (F, A) and (G, B) over a common universe X is the soft set (H, C) , where $C = A \cup B$ and \mathbf{r}

$$
H(e) = \begin{cases} F(e) & ; e \in A - B \\ G(e) & ; e \in B - A \\ F(e) \cup G(e) & ; e \in A \cap B \end{cases}
$$

for all $e \in C$. It is denoted by $(F, A) \cup (G, B) = (H, C)$.

Definition 2.5 [11] Let $A, B \subset E$ be nonempty subsets. The intersection of two soft sets (F, A) and (G, B) over a common universe X is the soft set (H, C) , where $C = A \cap B$ for all *e* ∈ *C*. It is denoted by (F, A) $\tilde{\cup}$ $(G, B) = (H, C)$.
 Definition 2.5 [11] Let *A*, *B* ⊂ *E* be nonempty subsets. The intersection of ty sets (F, A) and (G, B) over a common universe *X* is the soft set $(H$

Definition 2.6 [\[6\]](#page-14-9) Let \mathbb{R} be the set of real numbers, $B(\mathbb{R})$ be the collection of all nonempty bounded subsets of R and *E* be a set of parameters. Then a mapping $F: E \to$ $B(\mathbb{R})$ is called a soft real set. It is denoted by (F, E) . If specifically (F, E) is a singleton soft set then identifying (F, E) with the corresponding soft element, it will be called a **Definition 2.6** [6] Let \mathbb{R} be the set of r nonempty bounded subsets of \mathbb{R} [an](#page-14-10)d E be a $B(\mathbb{R})$ is called a soft real set. It is denoted l soft set then identifying (F, E) with the cosoft real number and den

 $\overline{0}$ and $\overline{1}$ are the soft real numbers where $\overline{0}(e) = 0$ and $\overline{1}(e) = 1$ for all $e \in E$, respectively.

Definition 2.7 [6] Let (F, E) and (G, E) be two soft real numbers.

- (1) $(F, E) = (G, E)$ if $F(e) = G(e)$ for each $e \in E$.
- (2) $(F+G)(e) = \{x+y : x \in F(e), y \in G(e)\}\$ for each $e \in E$.
- (3[\)](#page-14-10) $(F G)(e) = \{x y : x \in F(e), y \in G(e)\}\$ for each $e \in E$.
- (4) $(F.G)(e) = \{x.y : x \in F(e), y \in G(e)\}$ for each $e \in E$.

(5) $(F/G)(e) = \{x/y : x \in F(e), y \in G(e) - \{0\}\}\$ for each $e \in E$. (5) $(F/G)(e) = \{x/y : x \in F(e), y \in F\}$
finition 2.8 [6] For two soft real nur
(1) $\widetilde{r} \leqslant \widetilde{s}$ if $\widetilde{r}(e) \leqslant \widetilde{s}(e)$ for all $e \in E$,

Definition 2.8 [6] For two soft real numbers

(5) $(F/G)(e) = \{x/y : x \in F(e), y \in e\}$
 finition 2.8 [6] For two soft real nur

(1) $\widetilde{r} \leqslant \widetilde{s}$ if $\widetilde{r}(e) \leqslant \widetilde{s}(e)$ for all $e \in E$,

(2) $\widetilde{r} \geqslant \widetilde{s}$ if $\widetilde{r}(e) \geqslant \widetilde{s}(e)$ for all $e \in E$, (1) $\widetilde{r} \leq \widetilde{s}$ $\widetilde{r} \leq \widetilde{s}$ $\widetilde{r} \leq \widetilde{s}$ if $\widetilde{r}(e) \leq \widetilde{s}(e)$ for all $e \in E$,

(2) $\widetilde{r} \geq \widetilde{s}$ if $\widetilde{r}(e) \geq \widetilde{s}(e)$ for all $e \in E$,

(3) $\widetilde{r} \leq \widetilde{s}$ if $\widetilde{r}(e) \leq \widetilde{s}(e)$ for all $e \in E$, (1) $\widetilde{r} \leqslant \widetilde{s}$ if $\widetilde{r}(e) \leqslant \widetilde{s}(e)$ for all $e \in E$,

(2) $\widetilde{r} \geqslant \widetilde{s}$ if $\widetilde{r}(e) \geqslant \widetilde{s}(e)$ for all $e \in E$,

(3) $\widetilde{r} \leqslant \widetilde{s}$ if $\widetilde{r}(e) < \widetilde{s}(e)$ for all $e \in E$,

(4) $\widetilde{r} \leq$

Definition 2.9 [5] A soft set (P, E) over X is said to be a soft point if there is exactly one $e \in E$ such that $P(e) = \{x\}$ for some $x \in X$ and $P(e') = \emptyset$ for all $e' \in E - \{e\}$. It will be denoted by P_e^x . one *e* ∈ *E* such that $P(e) = \{x\}$ for some $x \in X$ and
will be denoted by P_e^x .
Definition 2.10 [5] A soft point P_e^x is said to be be
and $P(e) = \{x\} \subset F(e)$. It is written by $P_e^x \in (F, E)$.

Definition 2.10 [\[](#page-14-11)5] A soft point P_e^x is said to be belongs to a soft set (F, E) if $e \in E$

Definition 2.11 [5] Two soft points P_e^x , $P_{e'}^y$ e' are said to be equal if $e = e'$ and $P(e) =$ $P(e')$, that is, $x = y$ $x = y$. Thus,

$$
P_e^x \neq P_{e'}^y \Longleftrightarrow x \neq y \text{ or } e \neq e'.
$$

Proposition 2.12 [5] The union of any collection of soft point can be considered as a soft set and every soft set can be expressed as union of all soft points belonging to it, that is, (*F, E*) = $\left(\overline{F,E}\right)$

$$
(F, E) = \bigcup_{P_e^x \widetilde{\in} (F, E)} P_e^x.
$$

Remark 1 [1, 36] Let A, B ⊂ E be nonempty subsets. If f is a soft mapping from a **Remark** 1 [1, 36] Let $A, B \subset E$ be nonempty subsets. If f is a soft mapping from a soft set (F, A) to a soft set (G, B), which is denoted by $f : (F, A) \longrightarrow (G, B)$, then for **Remark 1** [1, 36] Let $A, B \subset E$ be nonempty subsets. If f is a soft mapping from a soft set (F, A) to a soft set (G, B) , which is denoted by $f : (F, A) \longrightarrow (G, B)$, then for each soft point $P_{e_1}^x \nightharpoonup (F, A)$, there exists o $f(P_{e_1}^x) = P_{e_2}^y$. ft set (F, A) to a soft set (G, B) , which is denoted by $f : (F, A)$
ch soft point $P_{e_1}^x \tilde{\in} (F, A)$, there exists only one soft point $P_{e_1}^x = P_{e_2}^y$.
Let $SP(\tilde{X})$ be the collection of all soft points of \tilde{X} a

Let $SP(\widetilde{X})$ be the collection of all soft points of \widetilde{X} and $\mathbb{R}(E)^*$ be the set of all nonnegative soft real numbers. Let $SP(\tilde{X})$ be the collection of all soft points of \tilde{X} and $\mathbb{R}(E)^*$ be the set of all non-
negative soft real numbers.
Definition 2.13 [5] A mapping $\tilde{d}: SP(\tilde{X}) \times SP(\tilde{X}) \to \mathbb{R}(E)^*$ is said to be a soft me

Let $SP(\tilde{X})$ be the collection of all soft points of \tilde{X}
negative soft real numbers.
Definition 2.13 [5] A mapping $\tilde{d}: SP(\tilde{X}) \times SP(\tilde{X})$ -
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the soft set \tilde{X} if \tilde{d} satisfies the following con
 $(\tilde{d}1)$ $\tilde{d}(P_{e_1}^x, P_{e_2}^y) \geqslant \overline{0}$ for all $P_{e_1}^x, P_{e_2}^y \in SP(\til$ efinition
the soft
 $(\tilde{d}1) \tilde{d}(P)$
 $(\tilde{d}2) \tilde{d}(P)$ the soft set \tilde{X} if \tilde{d} satisfies the following conditions:
 $(\tilde{d}1) \tilde{d}(P_{e_1}^x, P_{e_2}^y) \geq 0$ for all $P_{e_1}^x, P_{e_2}^y \in SP(\tilde{X})$.
 $(\tilde{d}2) \tilde{d}(P_{e_1}^x, P_{e_2}^y) = 0$ if and only if $P_{e_1}^x = P_{e_2}^y$

 $P_{e_1}^x, P_{e_2}^y$ $\geqslant \overline{0}$ for all $P_{e_1}^x, P_{e_2}^y$

 $(P_{e_1}^x, P_{e_2}^y) = \overline{0}$ $(P_{e_1}^x, P_{e_2}^y) = \overline{0}$ $(P_{e_1}^x, P_{e_2}^y) = \overline{0}$ if and only if $P_{e_1}^x = P_{e_2}^y$.

(d) $\tilde{d}(P_{e_1}^x, P_{e_2}^y) \geq \overline{0}$ for all $P_{e_1}^x, P_{e_2}^y \in SP(\tilde{X})$.

(d) $\tilde{d}(P_{e_1}^x, P_{e_2}^y) = \overline{0}$ if and only if $P_{e_1}^x = P_{e_2}^y$.

(d) $\tilde{d}(P_{e_1}^x, P_{e_2}^y) = \tilde{d}(P_{e_2}^y, P_{e_1}^x)$ for all $P_{$

($\tilde{d}2$) $\tilde{d}(P_{e_1}^x, P_{e_2}^y) = \overline{0}$ if and only if $P_{e_1}^x = P_{e_2}^y$.

($\tilde{d}3$) $\tilde{d}(P_{e_1}^x, P_{e_2}^y) = \tilde{d}(P_{e_2}^y, P_{e_1}^x)$ for all $P_{e_1}^x, P_{e_2}^y \in SP(\tilde{X})$.

($\tilde{d}4$) $\tilde{d}(P_{e_1}^x, P_{e_2}^$ $\begin{array}{c} \n(\widetilde{d}3) \, \hat{d} \\
(\widetilde{d}4) \, \hat{d} \\
\end{array}$

The so
 $\widetilde{X}, \widetilde{d}, E$ $\left(\widetilde{X}, \widetilde{d}, E\right)$.

Example **2.14** [5] Let *X ⊂* R be a nonempty set and *E ⊂* R be the nonempty set of The soft set *X* with a soft metric *a* on *X* is called a soft metric space and denoted by $(\tilde{X}, \tilde{d}, E)$.
 Example 2.14 [5] Let $X \subset \mathbb{R}$ be a nonempty set and $E \subset \mathbb{R}$ be the nonempty set of parameters. Let $\$ *<i><i>x kmple* **2.14** [5] Let $X \subset \mathbb{R}$ be a nonempty set and $E \subset \mathbb{R}$ be the nonempty sparameters. Let \widetilde{X} be the absolute soft set and \overline{x} denotes the soft real number such $\overline{x}(e) = x$ for all $e \in E$. T $\frac{d}{d}$
d(*P*

$$
\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = |\overline{x} - \overline{y}| + |\overline{e}_1 - \overline{e}_2|,
$$

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for all $P_{e_1}^x, P_{e_2}^y \in SP(\tilde{X})$, where " $|.|$ " denotes the modulus of soft real numbers, is a soft *N.*
for all $P_{e_1}^x, P_e^y$
metric on \widetilde{X} . for all $P_{e_1}^x, P_{e_2}^y \in SP(X)$, where " $|.|$ " denotes the modulus of soft real nu
metric on \tilde{X} .
Example **2.15** [5] Let $X \subset \mathbb{R}$ be a nonempty set and $E \subset \mathbb{R}$ be the n
parameters. Then the function $\tilde{d}: SP$

Example 2.15 [5] Let $X \subset \mathbb{R}$ be a nonempty set and $E \subset \mathbb{R}$ be the nonempty set of **R** be a nonem:
 on $\widetilde{d}: SP(\widetilde{X}) \times$
 $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = \begin{cases}$

$$
\widetilde{d}(P_{e_{1}}^{x},P_{e_{2}}^{y})=\left\{\begin{matrix} \overline{0}\\ \overline{1}\\ \overline{1}\end{matrix};\begin{matrix} P_{e_{1}}^{x}=P_{e_{2}}^{y}\\ P_{e_{1}}^{x}\neq P_{e_{2}}^{y} \end{matrix}\right.,
$$

 $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = \begin{cases} \overline{0} \; ; P_{e_1}^x = P_{e_2}^y \\ \overline{1} \; ; P_{e_1}^x \neq P_{e_2}^y \end{cases}$ $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = \begin{cases} \overline{0} \; ; P_{e_1}^x = P_{e_2}^y \\ \overline{1} \; ; P_{e_1}^x \neq P_{e_2}^y \end{cases}$ $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = \begin{cases} \overline{0} \; ; P_{e_1}^x = P_{e_2}^y \\ \overline{1} \; ; P_{e_1}^x \neq P_{e_2}^y \end{cases}$,
for all $P_{e_1}^x, P_{e_2}^y \in SP(\widetilde{X})$, is a soft metric on \widetilde{X} . \widetilde{d} is called the discrete soft metric on the
soft is said to be the discrete soft metric space.

Example 2.16 [37] Let $E = \mathbb{R}$ be a parameter set and $X = \mathbb{R}^2$. Let us consider a metric **dot** and \overline{X} and $(\overline{X}, \overline{d}, E)$ is said to be the discrete soft metric soft set \overline{X} and $(\overline{X}, \overline{d}, E)$ is said to be the discrete soft metric *Example* **2.16** [37] Let $E = \mathbb{R}$ be a parameter set and $X = \$ *∗* defined by $=\mathbb{R}$
ctio
 $\widetilde{d}(P)$

$$
\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = d(x, y) + |\overline{e}_1 - \overline{e}_2|,
$$

 $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = d(x, y) + |\overline{e}_1 - \overline{e}_2|$,
for all $P_{e_1}^x, P_{e_2}^y \in SP(\widetilde{X})$, is a soft metric on \widetilde{X} . \widetilde{d} is a soft metric on the soft set \widetilde{X} and $\begin{aligned} \text{or all } P, \\ \widetilde{X}, \widetilde{d}, E \end{aligned}$ $(\widetilde{X}, \widetilde{d}, E)$ is a soft metric space.

3. Main results

In this section, at first, we present the notions of a soft circle and a soft fixed circle with some illustrative examples. We prove some soft fixed-circle theorems on soft metric spaces with different aspects. The importance of working with a fixed-circle problem with the help of soft set is to increase the number of fixed points of a mapping by means of parametrization. In this way, these theorems can be used in various fields of application such as neural networks, activation functions etc. spaces with different aspects.

the help of soft set is to incre

parametrization. In this way,

such as neural networks, activ
 Definition 3.1 Let $(\tilde{X}, \tilde{d}, E)$ be the number of fixed points of a mapping by means of ese theorems can be used in various fields of application functions etc.
be a soft metric space and \tilde{r} be a soft real number with

parametrization. In this way, these
such as neural networks, activation
Definition 3.1 Let $(\tilde{X}, \tilde{d}, E)$ be
 $\tilde{r} \geq 0$. The soft circle is defined by $(\widetilde{K}, \widetilde{d}, E)$ be a soft metric space and \widetilde{r}
s defined by
 $(P_{e_1}^{x_0}, \widetilde{r}) = \widetilde{\bigcup} \{ P_{e_2}^{x} \in \widetilde{X} : \widetilde{d} (P_{e_2}^{x}, P_{e_1}^{x_0}) \}$ $\frac{E}{r}$, \widetilde{r} $\frac{1}{2}$ be a
 $=\widetilde{r}$

$$
C\left(P_{e_1}^{x_0}, \widetilde{r}\right) = \widetilde{\bigcup} \left\{P_{e_2}^x \widetilde{\in} \widetilde{X} : \widetilde{d}\left(P_{e_2}^x, P_{e_1}^{x_0}\right) = \widetilde{r}\right\},\
$$
\nwith the center $P_{e_1}^{x_0}$ and the radius \widetilde{r} .
\n**Example 3.2** Let us consider the discrete soft metric space $\left(\widetilde{X}, \widetilde{d}, E\right)$

sider the discrete soft metric space $(\widetilde{X}, \widetilde{d}, E)$ as in Example
 $(SP(\widetilde{X}) - \{P_{e_1}^{x_0}\}; \widetilde{r} = 1)$ 2.15. Then we get space
 $\label{eq:1} \begin{split} &\text{space}\\ \vdots \widetilde{r}=1 \end{split}$

$$
C(P_{e_1}^{x_0}, \widetilde{r}) = \begin{cases} SP(\widetilde{X}) - \{P_{e_1}^{x_0}\} : \widetilde{r} = 1 \\ \widetilde{\emptyset} & ; \widetilde{r} \neq 1 \end{cases}.
$$

C (*B*
Example **3.3** Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space as in Example 2.14. Suppose that *X* $\subset \mathbb{R}$ is a nonempty set and $E \subset \mathbb{R}$ is the set of parameters. Then we get the circle

 $C(P_1^0, \bar{1})$ with the center P_1^0 and the radius $\bar{1}$ as follows: r.
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$$
P_1^0
$$
 and the radius $\overline{1}$ as follows:
\n
$$
C(P_1^0, \overline{1}) = \widetilde{\bigcup} \left\{ P_{\lambda}^x : \widetilde{d}(P_1^0, P_{\lambda}^x) = \overline{1} \right\}
$$

and

$$
\widetilde{d}(P_1^0, P_\lambda^x) = |\overline{x} - \overline{0}| + |\overline{\lambda} - \overline{1}| \Longrightarrow |x| + |\lambda - 1| = 1.
$$

To determine the point x and λ , we investigate the following cases: **Case 1**: $x \ge 0$ and $\lambda \ge 1 \Longrightarrow x + \lambda - 1 = 1 \Longrightarrow x + \lambda = 2$. **Case 2**: $x \ge 0$ and $\lambda < 1 \implies x + 1 - \lambda = 1 \implies x = \lambda$. **Case 3**: $x < 0$ and $\lambda \geq 1 \implies -x + \lambda - 1 = 1 \implies \lambda - x = 2$. **Case 4**: $x < 0$ and $\lambda < 1 \implies -x + 1 - \lambda = 1 \implies x = -\lambda$. Then the points x and λ can be seen in the following figure.

Figure 1. The set of the points x and λ .

Therefore, using the points x and λ lies on the above figure. We write infinite number of soft points. Consequently, the circle $C(P_1^0, \bar{1})$ has infinite number of soft points. Therefore, using the point
of soft points. Consequently,
Example **3.4** Let $(\tilde{X}, \tilde{d}, E)$

be a soft metric space as in Example 2.16. Let $X = \mathbb{R}$, $E = \{1, 2, 3\}$ and the metric *d* defined as $d(x, y) = |e^x - e^y|$ for all $x, y \in \mathbb{R}$. Then we get for soft points. Conseque
 Example 3.4 Let $(X, E) = \{1, 2, 3\}$ and the metric \tilde{d} as $\tilde{d}(P)$ $\begin{aligned} \n\mathcal{L}_{e_1}, P_{e_2}^y &= |e^x - e^y| + |\overline{e}_1 - \overline{e}_2|. \text{ Then we get the circle } C\left(P_2^1, \overline{1}\right), \text{ the radius } \overline{1} \text{ as follows:} \n\mathcal{L}_{e_1}^y &= \overline{1} \left\{ \begin{array}{l} \text{and } \widetilde{d}\left(P_2^1, P_\lambda^x\right) = |e^x - e| + |\lambda - 2| = 1. \end{array} \right. \n\end{aligned}$ with the center P_2^1 and the radius $\overline{1}$ as follows: $\begin{align} \mathbf{e} \ \mathbf{a} \ \mathbf{a} \end{align}$

$$
C(P_2^1, \overline{1}) = \widetilde{\bigcup} \left\{ P_\lambda^x : \widetilde{d}(P_2^1, P_\lambda^x) = \overline{1} \right\} \text{ and } \widetilde{d}(P_2^1, P_\lambda^x) = |e^x - e| + |\lambda - 2| = 1.
$$

To determine the point x and λ , we investigate the following cases:

Case 1: If $\lambda - 2 \ge 0$ and $e^x - e \ge 0$, then we get $P_{\lambda}^x = P_3^1$. **Case 2**: If $\lambda - 2 \geq 0$ and $e^x - e < 0$, then we get $P_{\lambda}^x = P_{\lambda}^1$ and $P_{\lambda}^x = P_{\lambda}^{\ln(e-1)}$. **Case 3**: If $\lambda - 2 < 0$ and $e^x - e \ge 0$, then we get $P_{\lambda}^x = P_1^{\ln(e-2)}$. **Case 4**: If $\lambda - 2 < 0$ and $e^x - e < 0$, then we get $P_{\lambda}^{\hat{x}} = P_1^{\hat{1}}$. Consequently, we have nd $e^x - e \geq 0$, then we get $P_1^x = P_1^{\text{ln}(e)}$

$$
C\left(P_2^1,\overline{1}\right) = \left\{P_3^1,P_2^{\text{ln}(e-1)},P_1^{\text{ln}(e-2)},P_1^1\right\}.
$$

N. Taş and O. B. Özbak
 Definition 3.5 Let $(\widetilde{X}, \widetilde{d}, E)$ *o I. Linear. Topological. Algebra.* 12(04) (2023) 243-258. 249
be a soft metric space, $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping *N. Taş and O. B. Özbakır / J. Linear. Topological. Algebra.* 12(0)
 Definition 3.5 Let $(\widetilde{X}, \widetilde{d}, E)$ be a soft metric space, $T : \widetilde{X}$

and $C(P_{e_1}^{x_0}, \widetilde{r})$ be a soft circle. If $T(P_{e_2}^{x}) = P_{e_2}^{x}$ for a *r*
<i>s</sup>, \widetilde{r} $P_{e_1}^{x_0}, \widetilde{r}$, then $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ $(\frac{\pi}{r})$
 $(\frac{\pi}{r})$ $\frac{24}{\text{ing}}$
*i*ng is called as a soft fixed circle of *T*.

In the following subsection, we give some existence soft fixed-circle theorems.

3.1 *Existence theorems for a soft fixed-circle*

In this subsection, we prove new existence theorems for a soft fixed circle and give some necessary illustrative examples. To do this, we inspire the Caristi type contraction [3]. **For the induce the form a soft fixed-circle**
In this subsection, we prove new existence theorems for a soft fixed circle a
necessary illustrative examples. To do this, we inspire the Caristi type co
For this purpose, let *∗* defined as ence the
this,
 blowin
 $) = \tilde{d}$ heorems fe ng $\varphi:\widetilde{X}$

$$
\varphi(P_e^x) = \tilde{d}\left(P_e^x, P_{e_1}^{x_0}\right),\tag{1}
$$
\n
$$
\text{center of a soft circle } C\left(P_{e_1}^{x_0}, \tilde{r}\right).
$$

for all $P_e^x \in \tilde{X}$, where $P_{e_1}^{x_0}$ is the center of a soft circle $C(P_{e_1}^{x_0}, \tilde{r})$.

Now we obtain the existence theorem for a soft fixed circle.

 $\varphi(P_e^x) = d\left(P_e^x, P_{e_1}^{x_0}\right),$ (1)

for all $P_e^x \in \tilde{X}$, where $P_{e_1}^{x_0}$ is the center of a soft circle $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$.

Now we obtain the existence theorem for a soft fixed circle.
 Theorem 3.6 Let m 3.6 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ $C(P_{e_1}^{x_0}, \tilde{r})$ be a soft circle and a soft mapping $\varphi : \tilde{X} \to \mathbb{R}(E)^*$ be defined as in (1). If *T* or all $P_e^x \n\t\tilde{\in} \n\t\tilde{X}$, where $P_{e_1}^{x_0}$ is the center of a soft circle $C\left(P_{e_1}^{x_0}\right)$
Now we obtain the existence theorem for a soft fixed circle
heorem **3.6** Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space, $P_e^x \overline{w}$
en \widetilde{r} satisfies the following conditions **heorem 3.6** Let $(\widetilde{X}, \widetilde{d}, \widetilde{d}, \widetilde{P}_{e_1}, \widetilde{r})$ be a soft circle tisfies the following cond (*i*) $\widetilde{d}(P_e^x, T(P_e^x)) \le \varphi(P_e)$ $(P_{e_1}^{x_0}, \tilde{r})$ be a soft circle and a soft map
tisfies the following conditions
(*i*) $\tilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x) - \varphi(T(P_e^x))$
(*ii*) $\tilde{d}(T(P_e^x), P_{e_1}^{x_0}) \geq \tilde{r}$, for $\left(I_{e_1}, f\right)$ be a *i*
satisfies the follo
 (i) $\tilde{d}\left(P_e^x, T(P_e)\right)$
 (ii) $\tilde{d}\left(T(P_e^x)\right)$
for each $P_e^x \in C$

- $\varphi_e^x, T(P_e^x)$) $\leq \varphi(P_e^x) \varphi(T(P_e^x)),$ rcc rcc
 $\approx \text{rcc}$
 $\approx \text{rcc}$
 $\approx \text{rcc}$ $\left(\begin{matrix} x \ x \z \end{matrix}\right)$
 $\left(\begin{matrix} x \ \widetilde{r} \end{matrix}\right)$ $\begin{array}{c} (P_e^x)) \in C \ (P_e^x) \ C \left(P_e^x \right) \in C \ (P_e^x) \in C \end{array}$ $\mathcal{P}_e^x, \ \mathcal{P}_f^x, \ \mathcal{P}_f^x$
- $T(P_e^x), P_{e_1}^{x_0}$

 $P_{e_1}^{x_0}$, \tilde{r}), then $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ is a soft fixed circle of *T*. med circle of

Proof. Let $P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})$ be an arbitrary soft point. Now we show $T(P_e^x) = P_e^x$ for \tilde{d} $(T(P_e^x), P_{e_1}^{x_0})$
 \tilde{d} $(T(P_e^x), P_{e_1}^{x_0})$
 \tilde{d} $P_e^x \tilde{\in} C$ $(P_{e_1}^{x_0}, P_{e_1}^{x_0})$
 \tilde{e} \tilde{e} C $(P_{e_1}^{x_0}, P_{e_1}^{x_0})$ $\overset{x_0}{P^x_{e_1}}$, $\overset{x_1}{C}, \widetilde{r}$ et $P_e^x \n\tilde{\in} C(P_{e_1}^{x_0}, \tilde{r})$ be an arbitrary soft point. Now we
 $\tilde{E} C(P_{e_1}^{x_0}, \tilde{r})$. By the condition *(i)* and the definition of
 $\tilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x) - \varphi(T(P_e^x)) = \tilde{d}(P_e^x, P_{e_1}^{x_0}) - \tilde{d}$

each
$$
P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})
$$
. By the condition (i) and the definition of φ , we get
\n
$$
\tilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x) - \varphi(T(P_e^x)) = \tilde{d}(P_e^x, P_{e_1}^{x_0}) - \tilde{d}(T(P_e^x), P_{e_1}^{x_0})
$$
\n
$$
= \tilde{r} - \tilde{d}(T(P_e^x), P_{e_1}^{x_0}). \qquad (2)
$$
\nFrom the condition (ii), we get the following cases:
\nCase 1: If $\tilde{d}(T(P_e^x), P_{e_1}^{x_0}) \leq \tilde{r}$, then we get a contradiction with the inequality (2).

From the condition (*ii*), we get the following cases:

om the cond
Case 1: If \tilde{d} $T(P_e^x), P_{e_1}^{x_0}$ om the con
Case 1: If \tilde{d}
Case2: If \tilde{d} $T(P_e^x), P_{e_1}^{x_0}$ get the following cases:
 $\tilde{\gt} \tilde{r}$, then we get a contradiction with the in
 $= \tilde{r}$, then using the inequality (2), we obtain $\left(\begin{array}{c} \overline{\mathcal{E}}(k) \\ \overline{\mathcal{E}}(k) \end{array}\right)$, $\left(\begin{array}{c} P_{e_1}^x \\ \overline{P}_e \end{array}\right)$ $\leq \widetilde{r}$, then we get a contradiction with the *p*_{e_1} $\overline{\mathcal{E}}(P_{e_1}^x) = \widetilde{r}$, then using the inequality (2), we ob e inequ

$$
\widetilde{d}(P_e^x, T(P_e^x)) \leq \widetilde{r} - \widetilde{d}(T(P_e^x), P_{e_1}^{x_0}) = \widetilde{r} - \widetilde{r} = \overline{0}
$$
\n
$$
= P_e^x. \text{ Consequently, } C(P_{e_1}^{x_0}, \widetilde{r}) \text{ is a soft fixed}
$$

and so we get $T(P_e^x) = P_e^x$. Consequently, $C(P_{e_1}^{x_0}, \tilde{r})$ is a s[of](#page-6-0)t fixed circle of *T*.

Remark 2 Theorem 3.6 is a generalization of Theorem 2*.*1 *given in [26]. If we take the parameter set E with only one element, then Theorem 3.6 coincide Theorem* 2*.*1 *given in [26]. Ex[am](#page-14-2)ple* 3.7 Let $\left(\tilde{X}, \tilde{d}, E\right)$
 Example 3.7 Let $\left(\tilde{X}, \tilde{d}, E\right)$ *, r*e

We give a following [illu](#page-6-1)strative example.

be a soft metric space and $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ be a soft circle. Let We give a following illustrative example 3.7 Let $(\tilde{X}, \tilde{d}, E)$ be a soft us define a soft mapping $T : \tilde{X} \to \tilde{X}$ as *e* a soft metric space
 $\rightarrow \widetilde{X}$ as
 $\rangle = \begin{cases} P_e^x ; P_e^x \in C \\ P_e^{\beta} \end{cases}$ *i c*
i, \tilde{r} \tilde{r} \tilde{r}

$$
T(P_e^x) = \begin{cases} P_e^x ; P_e^x \tilde{\in} C\left(P_{e_1}^{x_0}, \tilde{r}\right) \\ P_{e_2}^{\beta} ; \qquad \text{otherwise} \end{cases}
$$

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for all $P_e^x \n\widetilde{\in} \n\widetilde{X}$, where *P* $\ddot{O}zbakr / J$. *Linear*. *Topological. Algebra.* 12(04)
 $\theta_{e_2}^{\beta}$ is a constant soft point such that \widetilde{d} $P^{\beta}_{e_2}, P^{x_0}_{e_1}$ *xkir / J. Linear. Topological. Algebra.* 12(04) (2023) 243-258.
 i a constant soft point such that $\widetilde{d}\left(P_{e_2}^{\beta}, P_{e_1}^{x_0}\right) \lesssim \widetilde{r}$, that is, $P_{e_2}^{\beta}$ is exterior of the soft circle $C(P_{e_1}^{x_0}, \tilde{r})$. Then *T* satisfies the conditions of Theorem *ar.*
 $\frac{1}{r}$ 3.6. Consequently, $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ is a soft fixed circle of T . $\frac{1}{2}$ i
cin
 \widetilde{r} , \widetilde{r} For all $P_{e_2}^{\beta} \in \Lambda$, where $P_{e_2}^{\beta}$ is a constant sort point such that $d(P_{e_2}^{\beta}, P_{e_1}^{\gamma}) > r$, that is,
 $P_{e_2}^{\beta}$ is exterior of the soft circle $C(P_{e_1}^{x_0}, \tilde{r})$. Then *T* satisfies the conditions of T 2 10 0100

 $C(P_{e_1}^{x_0}, \tilde{r})$ $C(P_{e_1}^{x_0}, \tilde{r})$ be a soft circle and a soft mapping $\varphi : \tilde{X} \to \mathbb{R}(E)^*$ be defined as in (1). If *T* $\frac{1}{r}$ en quently, $C(P_{e_1}^{x_0}, \tilde{r})$ is a soft fixed circle of *T*.
 3.8 Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space, $T : \tilde{X}$

be a soft circle and a soft mapping $\varphi : \tilde{X} \to \mathbb{R}(E)$ satisfies the following conditions **heorem 3.8** Let $(\widetilde{X}, \widetilde{d}, E)$ be a soft metric $(P_{e_1}^{x_0}, \widetilde{r})$ be a soft circle and a soft mapping tisfies the following conditions $(i) \widetilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x) + \varphi(T(P_e^x)) - \overline{2}\widetilde{r},$ $(P_{e_1}^{x_0},$
tisfies
(*i*) \widetilde{d}
(*ii*) \widetilde{d} r cle
 conc
 φ (*F*)
 $\leqslant \widetilde{r}$, $(C(F_{e_1}, f)$ be a *i*
satisfies the follo
 (i) $\tilde{d}(P_e^x, T(P_e^x))$
 (ii) $\tilde{d}(T(P_e^x))$
for each $P_e^x \in C$ vec $\text{vec$ $\left(\begin{matrix} x \ x \z \end{matrix}\right)$
 $\left(\begin{matrix} x \ \widetilde{r} \end{matrix}\right)$

 $\varphi_e^x, T(P_e^x)$) $\leqslant \varphi(P_e^x) + \varphi(T(P_e^x))$ $\leqslant \varphi(P_e^x)$

 $T(P_e^x), P_{e_1}^{x_0}$

 $P_{e_1}^{x_0}$, \tilde{r}), then $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ is a soft fixed circle of *T*. $\widetilde{\widetilde{r}}, \ \widetilde{t}$
 $,\widetilde{\widetilde{r}}$

Proof. Let $P_e^x \n\tilde{\in} C(P_{e_1}^{x_0}, \tilde{r})$ be an arbitrary soft point. Using the condition (i) , we obtain $\begin{array}{l} e^{\alpha} \in \mathcal{E} \ F_e^x, \ F_e^x \in \mathcal{C} \end{array}$
 $\begin{array}{l} e^{\alpha} \in \mathcal{E} \ e^{\alpha} \in \mathcal{E} \end{array}$

$$
x_0^{r_0}, \widetilde{r})
$$
 be an arbitrary soft point. Using the

$$
\widetilde{d}(P_e^x, T(P_e^x)) \widetilde{\leq} \varphi(P_e^x) + \varphi(T(P_e^x)) - \overline{2}\widetilde{r}
$$

and by the definition of φ , we get

the definition of
$$
\varphi
$$
, we get
\n
$$
\tilde{d}(P_e^x, T(P_e^x)) \leq \tilde{d}(P_e^x, P_{e_1}^{x_0}) + \tilde{d}(T(P_e^x), P_{e_1}^{x_0}) - \overline{2}\tilde{r} = \tilde{d}(P_e^x, P_{e_1}^{x_0}) - \tilde{r}.
$$
\n(a)
\ne condition (*ii*), we get the following cases:
\n
$$
\text{If } \tilde{d}(T(P_e^x), P_{e_1}^{x_0}) \leq \tilde{r}, \text{ then we get a contradiction with the inequality (3).}
$$

From the condition (*ii*), we get the following cases:

 $\widetilde{d}\left(P_e^x\right)$ om the cond
Case 1: If \widetilde{d} $T(P_e^x), P_{e_1}^{x_0}$ om the condition (*ii*), we get the following cases:

Case 1: If $\tilde{d}(T(P_e^{x}), P_{e_1}^{x_0}) \leq \tilde{r}$, then we get a contradiction with the in

Case2: If $\tilde{d}(T(P_e^{x}), P_{e_1}^{x_0}) = \tilde{r}$, then using the inequality (3), we o $T(P_e^x), P_{e_1}^{x_0}$ $P_{e_1}^{x_0}$ $\leq \tilde{r}$, then we get a contradiction with
 $P_{e_1}^{x_0}$ $= \tilde{r}$, then using the inequality (3), we c
 $\tilde{d}(P_e^x, T(P_e^x)) \leq \tilde{d}(P_e^x, P_{e_1}^{x_0}) - \tilde{r} = \tilde{r} - \tilde{r} = \overline{0}$, ം
ലോക്ക

$$
\widetilde{d}(P_e^x, T(P_e^x)) \leq \widetilde{d}(P_e^x, P_{e_1}^{x_0}) - \widetilde{r} = \widetilde{r} - \widetilde{r} = \overline{0},
$$

Consequently, $C(P_{e_1}^{x_0}, \widetilde{r})$ is a soft fixed circle

that is, $T(P_e^x) = P_e^x$. Consequently, $C(P_{e_1}^{x_0}, \tilde{r})$ is a soft fixed circle of *T*.

Remark 3 Theorem 3.8 is a generalization of Theorem 2*.*2 *given in [26]. If we take the parameter set E with only one element, then Theorem 3.8 coincide Theorem* 2*.*2 *given in [26].*

Example **3.9** Let us [con](#page-7-0)sider Example 3.7. Then *T* satisfies the con[diti](#page-14-2)ons of Theorem **EXAMPLE 3.5** Let us consider Example 3.1. Then 1 satisfacture of *T*. $\frac{1}{\tilde{r}}$
 $\frac{1}{\tilde{r}}$ *Example* 3.9 Let us consider
3.8. Consequently, $C(P_{e_1}^{x_0}, \tilde{r})$
Theorem 3.10 Let $(\tilde{X}, \tilde{d}, E)$ be a soft fixed circle of *T*.
be a sof[t m](#page-6-2)etric space, $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping,

 $C(P_{e_1}^{x_0}, \tilde{r})$ $C(P_{e_1}^{x_0}, \tilde{r})$ be a soft circle and a soft mapping $\varphi : \tilde{X} \to \mathbb{R}(E)^*$ be defined as in (1). If *T* $\frac{1}{\widetilde{r}}$
en puently, $C(P_{e_1}^{x_0}, \tilde{r})$ is a soft fixed circle of T .
 3.10 Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space, $T : \tilde{X}$

be a soft circle and a soft mapping $\varphi : \tilde{X} \to \mathbb{R}(E)$ satisfies the following conditions **heorem 3.10** Let $(\tilde{X}, \tilde{d}, E)$ be a so
 $(P_{e_1}^{x_0}, \tilde{r})$ be a soft circle and a soft m

tisfies the following conditions

(*i*) $\tilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x) - \varphi(T(P_e^x))$ $(P_{e_1}^{x_0}, \tilde{r})$ be a soft circle
tisfies the following cone
(*i*) $\tilde{d}(P_e^x, T(P_e^x)) \le \varphi(L$
(*ii*) $\overline{h}d(P_e^x, T(P_e^x)) + \tilde{d}$ appin

(),

≤ \widetilde{r} , (f_{e_1}, f) be a *f*
satisfies the follo
 (i) $\tilde{d}(P_e^x, T(P_e^x))$
 (ii) $\tilde{h}\tilde{d}(P_e^x, T)$
for each $P_e^x \in C$ cc φ
 θ + \tilde{r} P_e^x and a soft mapping φ : φ

ditions
 P_e^x ρ φ $(T(P_e^x))$,
 $\widetilde{I}(T(P_e^x), P_{e_1}^{x_0}) \leq \widetilde{r}$,
 \widetilde{I} , where $\overline{0} \leqslant \overline{h} \leqslant \overline{1}$, then C $\frac{1}{r}$, $\frac{1}{r}$

 $\varphi_e^x, T(P_e^x)$) $\leqslant \varphi(P_e^x) - \varphi(T(P_e^x)),$

 $T(P_e^x), P_{e_1}^{x_0}$ $\begin{array}{l} \mathcal{C}(P_e^x),\ P(P_e^x),P_{e_1}^{x_0})\leqslant \ \mathcal{C}(P_e^x), P_{e_1}^{x_0})\leqslant \ \mathcal{C}(P_{e_1}^{x_0},\widetilde{r})\ \mathcal{C}(P_{e_1}^{x_0},\widetilde{r})\ \mathcal{C}(P_{e_1}^{x_0},\widetilde{r})\ \mathcal{C}(P_{e_1}^{x_0},\widetilde{r})\ \mathcal{C}(P_{e_1}^{x_0},\widetilde{r})\ \mathcal{C}(P_{e_1}^{x_0},\widetilde{r})\$ $\begin{array}{c} \n\sqrt{2} \leqslant \\ \n\sqrt{2} \leqslant \\ \n\sqrt{r} \end{array}$

 $P_{e_1}^{x_0}, \widetilde{r}$ $P_{e_1}^{x_0}$, \tilde{r}) is a soft fixed circle [o](#page-6-3)f *T*.

Proof. We suppose that $P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})$ such that $T(P_e^x) \neq P_e^x$. Using the conditions

 (i) , (ii) and the definition of φ , we have

\n- (i), (ii) and the definition of
$$
\varphi
$$
, we have
\n- $$
\tilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x) - \varphi(T(P_e^x))
$$
\n
$$
= \tilde{d}(P_e^x, P_{e_1}^{x_0}) - \tilde{d}(T(P_e^x), P_{e_1}^{x_0}) = \tilde{r} - \tilde{d}(T(P_e^x), P_{e_1}^{x_0})
$$
\n
$$
\leq \overline{h}\tilde{d}(P_e^x, T(P_e^x)) - \tilde{d}(T(P_e^x), P_{e_1}^{x_0}) = \tilde{r} - \tilde{d}(T(P_e^x), P_{e_1}^{x_0})
$$
\n
$$
\leq \overline{h}\tilde{d}(P_e^x, T(P_e^x)) + \tilde{d}(T(P_e^x), P_{e_1}^{x_0}) - \tilde{d}(T(P_e^x), P_{e_1}^{x_0})
$$
\n
$$
= \overline{h}\tilde{d}(P_e^x, T(P_e^x)),
$$
\na contradiction with $\overline{0} \leq \overline{h} \leq \overline{1}$. So it should be $T(P_e^x) = P_e^x$. Consequently, $C(P_{e_1}^{x_0})$.

 $, \widetilde{r}$ is a soft fixed circle of T.

Remark 4 Theorem 3.10 is a generalization of Theorem 2*.*3 *given in [26]. If we take the parameter set E with only one element, then Theorem 3.10 coincide Theorem* 2*.*3 *given in [26]. on*
 sid , \widetilde{r}

Example **3.11** Let u[s con](#page-7-1)sider Example 3.7. Then *T* satisfies the con[diti](#page-14-2)ons of Theorem 3[.](#page-7-1)10. Consequently, $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ is a soft fixed circle of T . *in [26].*
 Example **3.11** Let us conside

3.10. Consequently, $C(P_{e_1}^{x_0}, \tilde{r})$
 Theorem 3.12 Let $(\tilde{X}, \tilde{d}, E)$ Example 3.7. Then *T* satisfies the conditions of Theorem
is a soft fixed circle of *T*.
be a soft metric space, $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping,

 $C(P_{e_1}^{x_0}, \tilde{r})$ $C(P_{e_1}^{x_0}, \tilde{r})$ $C(P_{e_1}^{x_0}, \tilde{r})$ $C(P_{e_1}^{x_0}, \tilde{r})$ be a soft circle and a soft mapping $\varphi_{\tilde{r}} : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ be defined by
 $\varphi_{\tilde{r}}(\tilde{u}) = \begin{cases} \tilde{r} & \text{if } \tilde{u} = \tilde{r} \\ \tilde{u} + \tilde{r} & \text{if } \tilde{u} \neq \tilde{r} \end{cases}$ $\frac{1}{2}$
en
 \widetilde{r} *be* a soft α
*φ*_{*r*}</sub> $(\tilde{u}) = \begin{cases} 1 & \text{if } i \neq j. \end{cases}$ re *ric* space,
g $\varphi_{\tilde{r}} : \mathbb{R}(E)$
 \tilde{r} ; $\tilde{u} = \tilde{r}$

$$
\varphi_{\widetilde{r}}\left(\widetilde{u}\right) = \left\{\begin{matrix} \widetilde{r} & ; \widetilde{u} = \widetilde{r} \\ \widetilde{u} + \widetilde{r} & ; \widetilde{u} \neq \widetilde{r} \end{matrix}\right.,
$$

 $\varphi_{\widetilde{r}}\left(\widetilde{u}\right)=\bigg\{$ for all
 $\widetilde{u}\in\mathbb{R}(E)^{*}$ and $\widetilde{r}\stackrel{<}{>} \overline{0}.$ Assume that \sim

 $:$ all \tilde{i}
(*i*) \tilde{d} $T(P_e^x), P_{e_1}^{x_0}$ l \widetilde{r} > ζ
ξ φ_{\widetilde{r}} $\widetilde{d}\left(P_e^x,P_{e_1}^{x_0}\right)\right)$ $\frac{d}{dt}$ $\left\{ \begin{array}{l} \tilde{u} + \tilde{r} \; ; \; \tilde{u} \neq \tilde{r} \\ t \\ t \\ + \overline{L} \tilde{d} \left(P_e^x, T(P_e^x) \right) \; \text{for some} \; \overline{L} \; \leqslant \; \overline{0} \; \text{and each} \; P_e^x \end{array} \right\}$ for al
 (i)
 $\widetilde{\in} \widetilde{X}$, $\begin{align*} \n\text{all } \widetilde{u} \in \mathbb{R} \\ \n(i) \widetilde{d} \left(T(P, \widetilde{X}, \widetilde{X}) \right) \\ \n(ii) \widetilde{r} \leqslant \widetilde{d} \n\end{align*}$ *E*)^{*} and $\widetilde{r} \leq \overline{0}$. Assume that
 j, $P_{e_1}^{x_0}$) $\leq \varphi_{\widetilde{r}} \left(\widetilde{d} \left(P_e^x, P_{e_1}^{x_0} \right) \right) +$
 T(P_e^x), $P_{e_1}^{x_0}$) for each $P_e^x \in C$ P_e^x
 $,\widetilde{r}$ $(i) \widetilde{d}(T(P_e^x), P_{e_1}^{x_0}) \widetilde{\leq} \varphi_{\widetilde{r}}(\widetilde{d}(P_e^x, P_{e_1}^{x_0})) + \overline{L}\widetilde{d}(P_e^x, \widetilde{X},$
 $(ii) \widetilde{r} \widetilde{\leq} \widetilde{d}(T(P_e^x), P_{e_1}^{x_0}) \text{ for each } P_e^x \widetilde{\in} C(P_{e_1}^{x_0}, \widetilde{r})$
 $(iii) \widetilde{d}(T(P_e^x), T(P_{e_2}^y)) \widetilde{\geq} \widetilde{2r}$ $\mathcal{L} = \mathcal{L} \mathcal{L}$ *, r*e $\widetilde{X},$

(*ii*) $\widetilde{r} \leq \widetilde{d} (T(P_e^x), P_{e_1}^{x_0})$ for each $P_e^x \in C(P_{e_1}^{x_0}, \widetilde{r}),$

(*iii*) $\widetilde{d} (T(P_e^x), T(P_{e_2}^y)) \geqslant \overline{2\widetilde{r}}$ for each $P_e^x, P_{e_2}^y \in C(P_{e_1}^{x_0}, \widetilde{r})$ and

(*iv*) $\widetilde{d} (T(P_e^x), T(P$ $\frac{1}{2}$ $\frac{1}{r}$ *F*

- $P_{e_1}^{x_0}, \widetilde{r}),$
- $P_{e_1}^{x_0}, \widetilde{r}$ and $P_e^x \neq P_{e_2}^y$,

 $P_{e_1}^{x_0}, \tilde{r}$ and $P_e^x \neq P_{e_2}^y$. Then *T* fixes the soft circle *C* $P_{e_1}^{x_0}$ P_e^2
 a *r*_{a}, \widetilde{r} . $\begin{array}{l} \left(\mathcal{F}_{e}^{x}\right),T(P_{e_{2}}^{y})\right)\tilde{\geq}\ \left(\mathcal{F}_{e}^{x}\right),T(P_{e_{2}}^{y})\right)\tilde{<}\ \tilde{r}^{x},\ \text{the\ soft\ circle}\ \left(\mathcal{F}_{e_{1}}^{x_{0}},\tilde{r}\right). \end{array}$ $\begin{array}{c}\n\geq i,\ \geq \infty,\ \tilde{r}\n\end{array}$ The solution of (I_{e_1}, I) .

Proof. Let $P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})$ be an arbitrary soft point. Using the conditions (*i*) and (*ii*),
 $\tilde{d}(T(P_e^x), P_{e_1}^{x_0}) \leq \varphi_{\tilde{r}}(\tilde{d}(P_e^x, P_{e_1}^{x_0})) + \overline{L}\tilde{d}(P_e^x, T(P_e^x)) = \tilde{r} + \overline{L}\tilde{d}(P_e^x, T(P_e^x))$ we get $\frac{d}{d}$ be
 \tilde{d}

$$
\widetilde{d}\left(T(P_e^{x}), P_{e_1}^{x_0}\right) \leqslant \varphi_{\widetilde{r}}\left(\widetilde{d}\left(P_e^{x}, P_{e_1}^{x_0}\right)\right) + \overline{L}\widetilde{d}\left(P_e^{x}, T(P_e^{x})\right) = \widetilde{r} + \overline{L}\widetilde{d}\left(P_e^{x}, T(P_e^{x})\right)
$$

and so

$$
\tilde{r} \leq \tilde{d} \left(T(P_e^x), P_{e_1}^{x_0} \right) + \tilde{L} \tilde{d} \left(P_e^x, T(P_e^x) \right) - \tilde{r} + \tilde{L} \tilde{d} \left(P_e, T(P_e^x) \right)
$$
\n
$$
\tilde{r} \leq \tilde{d} \left(T(P_e^x), P_{e_1}^{x_0} \right) \leq \tilde{r} + \overline{L} \tilde{d} \left(P_e^x, T(P_e^x) \right). \tag{4}
$$
\nhave $\tilde{d} \left(T(P_e^x), P_{e_1}^{x_0} \right) = \tilde{r}$ by the inequality (4), that is, $T(P_e^x)$

If *^L* ⁼ 0, then we have *^d*^e $T(P_e^x), P_{e_1}^{x_0}$ *∈*e *C* $P_{e_1}^{x_0}, \tilde{r}$. Suppose that $\tilde{d}(P_e^x, T(P_e^x)) \neq \overline{0}$, that is, $T(P_e^x) \neq P_e^x$ for each $P_e^x \in \tilde{P}_e^{(0)}, \tilde{r}$. By the condition (*iii*), we get
 $\tilde{d}(T(P_e^x), T(T(P_e^x))) \geq \overline{2}\tilde{r}$ (5) $\frac{\overline{0}}{\widetilde{r}}$ $\widetilde{r} \leq \widetilde{d}(T(P_e^x), P_{e_1}^{x_0}) \leq \widetilde{r} + \overline{L}\widetilde{d}(P_e^x, T(P_e^x)).$ (4)

then we have $\widetilde{d}(T(P_e^x), P_{e_1}^{x_0}) = \widetilde{r}$ by the inequality (4), that is, $T(P_e^x)$.

Suppose that $\widetilde{d}(P_e^x, T(P_e^x)) \neq \overline{0}$, that is, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. By the condition (*iii*), we get $=\frac{1}{\sum_{i=1}^{n} x_i}$

$$
\widetilde{d}(T(P_e^x), T(T(P_e^x))) \widetilde{\geq} \widetilde{2r}
$$
\n⁽⁵⁾

and using the condition (*iv*), we obtain

\n252 *N. Tag and O. B. Özbak T. Linear. Topological. Algebra.* 12(04) (2023) 243-258.\n

\n\n253 and using the condition
$$
(iv)
$$
, we obtain\n

\n
$$
\widetilde{d}(T(P_e^x), T(T(P_e^x))) \leq \widetilde{r} + \widetilde{d}(T(P_e^x), T(P_e^x)) = \widetilde{r},
$$
\n\n\n254 *W W*

ich contradicts when \overline{H} \overline{L} \leq $\overline{0}$ and \overline{d} (*P*

 $\binom{x}{e}$, $T(P_e^x)$ \neq $\overline{0}$, then we a get a contradiction with the inequality (4). which contradicts with $\text{If } \overline{L} \leq \overline{0} \text{ and } \tilde{d}(P_{e}^{x}, T)$
Hence it should be $\tilde{d}(F)$ $P_e^x, T(P_e^x)$ = $\overline{0}$; that is, $T(P_e^x) = P_e^x$. Consequently, *T* fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ $(P_{\text{p}}$
 α
 \widetilde{r} . ■

Remark 5 Theorem 3.12 is a generalization of Theorem 2*.*1 *given in [20]. If we take [th](#page-8-0)e parameter set E with only one element, then Theorem 3.12 coincide Theorem* 2*.*1 *given in [20].* **Example 3.13** Let $(\tilde{X}, \tilde{d}, E)$
 Example 3.13 Let $(\tilde{X}, \tilde{d}, E)$

 be a soft metric space defined as in Exa[mp](#page-14-3)le 2.14. Assume that $X \subset \mathbb{R}$ is a nonempty set, $E = \{1, 2\}$ is the set [of pa](#page-8-1)rameters and $C(P_1^0, 1)$ is a cir[cle w](#page-14-3)ith the center P_1^0 \widetilde{A}, E be a soft metric space defined as in Example 2.14. Assume
bty set, $E = \{1, 2\}$ is the set of parameters and $C(P_1^0, 1)$ is a
and the radius $\overline{1}$. Let us define the soft mapping $T : \widetilde{X} \to \widetilde{X}$ as a soft metric space do
 $E = \{1, 2\}$ is the set

radius $\overline{1}$. Let us defin
 $E = \begin{cases} P_e^x & ; P_e^x \in C(F_e) \\ \frac{e}{\overline{C}} & ; \text{otherwise} \end{cases}$

$$
T(P_e^x) = \begin{cases} P_e^x ; P_e^x \tilde{\in} C(P_1^0, 1) \\ \overline{0} & \text{otherwise} \end{cases}
$$

 $T(P_e^x) = \begin{cases} P_e^x \, ; \, P_e^x \in C(P_1^0, 1) \\ 0 \qquad \text{otherwise} \end{cases}$,
for all $P_e^x \in \tilde{X}$. Then *T* satisfies the conditions of Theorem 3.12 with $\overline{L} = \overline{0}$. Consequently, $T(P_e^x) = \begin{cases} P_e^x; P_e^x \in C \\ \overline{0} \end{cases}$

for all $P_e^x \in \widetilde{X}$. Then *T* satisfies the conditions of The
 $C(P_1^0, 1) = \{P_1^{-1}, P_1^1, P_2^0\}$ is a soft fixed circle of *T*. for all $P_e^x \in \tilde{X}$. Then *T* satisfies the conditions of Theorem 3.12 with $\overline{L} = \overline{0}$. Consequently,
 $C(P_1^0, 1) = \{P_1^{-1}, P_1^1, P_2^0\}$ is a soft fixed circle of *T*.
 Theorem 3.14 Let $(\tilde{X}, \tilde{d}, E)$ be a so r all P^x $\stackrel{\frown}{}$ -11 , $11, 12$ is a solutined effect of 1.

Theorem 3.14 Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space, $T : \tilde{X} \to \tilde{X}$ be a soft mapping, $C(P_{e_1}^{x_0}, \tilde{r})$ $C(P_{e_1}^{x_0}, \tilde{r})$ be a soft circle and a soft mapping $\varphi_{\tilde{r}} : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ be defined as in $\left(\begin{matrix} e\ 1 \end{matrix}\right)$
en
, \widetilde{r} Theorem 3.12. Assume that $(\overline{P^{x_0}_{e_1}}, \hat{i}^{x_0}_{\text{neorem}})$ et (
 ft et
 um e
 $-\tilde{d}$ soft
ft ma
 $\widetilde{\leqslant} \varphi_{\widetilde{r}}$ $\frac{1}{e}$ pi $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping,
 $E)^* \to \mathbb{R}(E)^*$ be defined as in
 $+ \overline{L}\widetilde{d}(P_e^x, T(P_e^x))$ for some $\overline{L} \leq$ *e* a soft
2. Assu:
 $P_{e_1}^{x_0}$ –
 $P_e^{x_0} \in \widetilde{X}$,

 $P_e^x, P_{e_1}^{x_0}$ $T(P_e^x), P_{e_1}^{x_0}$ $\widetilde{d}\left(P_e^x,P_{e_1}^{x_0}\right)\right)$ (*i*) $2\tilde{d}$ $(P_e^x, P_{e_1}^{x_0}) - \tilde{d}$ $(T(P_e^x), P_{e_1}^{x_0}) \leq \varphi_{\tilde{r}}$ $(\tilde{d} (P_e^x, P_{e_1}^{x_0}) + \overline{L}$
and each $P_e^x \in \widetilde{X}$,
(*ii*) $\tilde{d} (T(P_e^x), P_{e_1}^{x_0}) \leq \tilde{r}$ for each $P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})$, $\frac{r}{\widetilde{r}}, \vec{I}$ (*i*) $\overline{2d}$ $(P_e^x, P_{e_1}^{x_0}) - \overline{d}$ $(T(P_e^x), P_{e_1}^{x_0}) \leq \varphi_{\tilde{r}}$ $(\overline{d} (P_e^x, P_{e_1}^{x_0}))$
and each $P_e^x \in \widetilde{X}$,
(*ii*) $\overline{d} (T(P_e^x), P_{e_1}^{x_0}) \leq \widetilde{r}$ for each $P_e^x \in C (P_{e_1}^{x_0}, \widetilde{r})$,
(*iii*) \over +.
 \overline{r} , \overline{r}

 $\overline{0}$ and each P_e^x

- $T(P_e^x), P_{e_1}^{x_0}$ $T(P_e^x), P_{e_1}^{x_0}$ $T(P_e^x), P_{e_1}^{x_0}$ $\leqslant \widetilde{r}$ for each $P_e^x \in C(P_{e_1}^{x_0}, \widetilde{r}),$
- $P_{e_1}^{x_0}, \tilde{r}$ and $P_e^x \neq P_{e_2}^y$, $\frac{1}{r}$ *F* $y \rightarrow z \overline{z}$

and each $P_e^x \n\widetilde{\in} \n\widetilde{X}$,

(*ii*) $\widetilde{d}(T(P_e^x), P_{e_1}^{x_0}) \leq \widetilde{r}$ for each $P_e^x \in C(P_{e_1}^{x_0}, \widetilde{r})$,

(*iii*) $\widetilde{d}(T(P_e^x), T(P_{e_2}^y)) \geq \overline{2\widetilde{r}}$ for each $P_e^x, P_{e_2}^y \in C(P_{e_1}^{x_0}, \widetilde{r})$ and

($(x_e^x), T(P_{e_2}^y)) \leq \widetilde{r} + \widetilde{d}(P_{e_2}^y, T(P_e^x))$ for each $P_e^x, P_{e_2}^y \in C(P_{e_1}^{x_0}, \widetilde{r})$ and $P_e^x \neq P_{e_2}^y$. Then *T* fixes the soft circle $C(P_{e_1}^{x_0}, \tilde{r})$. P_e^2
 a_2 , \widetilde{r} , \widetilde{r} $\begin{aligned} \n\mathcal{F}^x(z) &, T(I) \\ \n\text{the soft} \\ \n\mathcal{F}^x & \tilde{\in} \ C \n\end{aligned}$ \geqslant \geqslant \widehat{r}
le
 $, \widetilde{r}$

Proof. Let $P_e^x \n\tilde{\in} C(P_{e_1}^{x_0}, \tilde{r})$ be an arbitrary soft point. By the conditions (*i*) and (*ii*), we obtain $\overline{e}^{\overline{c}} \in C(P_{e_1}^{x_0}, \tilde{r})$ be an arbitrary soft point. By the con
 $\overline{2} \tilde{d}(P_{e}^x, P_{e_1}^{x_0}) - \tilde{d}(T(P_{e}^x), P_{e_1}^{x_0}) \leq \tilde{d}(P_{e}^x, P_{e_1}^{x_0}) + \overline{L} \tilde{d}(P_{e}^x, P_{e_1}^{x_0})$ $(P_{e_1}^{x_0},r)$ be are

$$
\overline{2}\tilde{d}\left(P_e^x, P_{e_1}^{x_0}\right) - \tilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \leq \tilde{d}\left(P_e^x, P_{e_1}^{x_0}\right) + \overline{L}\tilde{d}\left(P_e^x, T(P_e^x)\right)
$$
\n
$$
\implies \overline{2}\tilde{r} - \tilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \leq \tilde{r} + \overline{L}\tilde{d}\left(P_e^x, T(P_e^x)\right)
$$
\n
$$
\implies \tilde{r} \leq \tilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) + \overline{L}\tilde{d}\left(P_e^x, T(P_e^x)\right) \leq \tilde{r} + \overline{L}\tilde{d}\left(P_e^x, T(P_e^x)\right).
$$
\n(6)

\nequality (6) and the similar arguments used in the proof of Theorem 3.12, it is is seen that T fixes the soft circle $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$.

By the inequality (6) and the similar arguments used in the proof of Theorem 3.12, it can be easily seen that *T* fixes the soft circle $C(P_{e_1}^{x_0}, \tilde{r})$. **■**

Remark 6 Theorem 3.14 is a generalization of Theorem 2*.*6 *given in [20]. If we take the parameter set E wi[th](#page-9-0) only one element, then Theorem 3.14 coincide Theorem* 2*.*6 *[giv](#page-8-1)en in [20].* **Remark 6** Theorem 3.14 is

parameter set *E* with only of
 in [20].
 Example 3.15 Let $(\tilde{X}, \tilde{d}, E)$

be a soft metric space defined as in Exa[mp](#page-14-3)le 2.14. Assume

that $X \subset \mathbb{R}$ is a nonempty set, $E = \{1, 2\}$ is the set of parameters and $C(P_1^0, 1)$ is a circle with the center P_1^0 $\ddot{O}zbak\pi / J$. Linear. Topological. Algebra. 12(04) (2023) 243-258. 253

oty set, $E = \{1, 2\}$ is the set of parameters and $C(P_1^0, 1)$ is a

and the radius $\overline{1}$. Let us define the soft mapping $T : \widetilde{X} \to \widetilde{X}$ $E = \{1, 2\}$ is the set
radius $\overline{1}$. Let us defin
 $= \begin{cases} P_e^x ; P_e^x \in C(P, \\ P_{ex}^{2x} \end{cases}$

$$
T(P_e^x) = \begin{cases} P_e^x ; P_e^x \tilde{\in} C(P_1^0, 1) \\ P_e^{2x} \qquad \text{otherwise} \end{cases},
$$

 $T(P_e^x) = \begin{cases} P_e^x \, \, ; \, P_e^x \, \, \widetilde{\in} \, \, C(P_1^0,1) \\ P_e^{2x} \qquad \qquad \text{otherwise} \end{cases} \, ,$ for all $P_e^x \, \widetilde{\in} \, \, \widetilde{X}.$ Then T satisfies the conditions of Theorem 3.14 with $\overline{L} = \overline{0}.$ Consequently, $T(P_e^x) = \begin{cases} P_e^x ; P_e^x \in C \\ P_e^{2x} \end{cases}$ other
for all $P_e^x \in \widetilde{X}$. Then *T* satisfies the conditions of The
 $C(P_1^0, 1) = \{P_1^{-1}, P_1^1, P_2^0\}$ is a soft fixed circle of *T*. for all $P_e^x \in \tilde{X}$. Then *T* satisfies the conditions of Theorem 3.14 with $\overline{L} = \overline{0}$. Consequently,
 $C(P_1^0, 1) = \{P_1^{-1}, P_1^1, P_2^0\}$ is a soft fixed circle of *T*.
 Theorem 3.16 Let $(\tilde{X}, \tilde{d}, E)$ be a so r all P^x \hat{e}

 $C(P_{e_1}^{x_0}, \tilde{r})$ be a soft circle and a soft mapping $\varphi_{\tilde{r}}^* : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ be defined by $\left(\begin{matrix} e\ 1 \end{matrix}\right)$
en
, \widetilde{r} e a soft metric space,

oft mapping $\varphi_{\tilde{r}}^* : \mathbb{R}(E)$
 $(\tilde{u}) = \begin{cases} 0; \quad \tilde{u} = 0 \\ 0; \quad \tilde{u} = \tilde{u} \end{cases}$

a soft mapping
$$
\varphi_{\widetilde{r}}^* : \mathbb{R}(E)^*
$$

\n
$$
\varphi_{\widetilde{r}}^*(\widetilde{u}) = \begin{cases} \overline{0} & ; \widetilde{u} = \overline{0} \\ \widetilde{u} - \widetilde{r} & ; \widetilde{u} \le \overline{0} \end{cases},
$$
\n*i* a soft mapping *T* satisfies
\n
$$
\operatorname{ch} P_e^x \in C(P_{e_0}^x, \widetilde{r}),
$$

 $\varphi^*_{\tilde{r}}(\tilde{u}) = \left\{ \begin{matrix} \overline{0} & ; \, \widetilde{u} = \overline{0} \\ \widetilde{u} - \widetilde{r} & ; \, \widetilde{u} \stackrel{\frown}{<} \overline{0} \end{matrix} \right. ,$ for all $\widetilde{u} \in \mathbb{R}(E)^*$ and $\widetilde{r} \stackrel{\frown}{>} \overline{0}$. If a soft mapping *T* satisfies the following conditions $:$ all \tilde{i}
(*i*) \tilde{d} $\varphi_{\tilde{r}}^* (\tilde{u}) = \begin{cases} \n\frac{1}{r} \leq \overline{0}.\n\end{cases}$ If a soft may
 $= \tilde{r}$ for each $P_e^x \in C$ $(i) \widetilde{d}(T(P_e^x), P_{e_1}^{x_0}) = \widetilde{r}$ for each $P_e^x \in C(P_{e_1}^{x_0}, \widetilde{r})$
 $(ii) \widetilde{d}(T(P_e^x), T(P_{e_2}^y)) \leq \widetilde{r}$ for each $P_e^x, P_{e_2}^y \in C$ fies
 $, \widetilde{r}$ $(i) \widetilde{d}(T(P_e^x), P_{e_1}^{x_0}) = \widetilde{r}$ for each $P_e^x \in C(I_i^i)$
 $(\widetilde{d}(T(P_e^x), P_{e_1}^{x_0}) = \widetilde{r}$ for each $P_e^x \in C(I_i^i)$
 $(\widetilde{d}(T(P_e^x), T(P_{e_2}^y)) \leq \widetilde{d}(P_e^x, P_{e_2}^y) - \varphi_{\widetilde{r}}^*\left(\widetilde{d}(P_{e_1}^x, P_{e_2}^y) - \varphi_{\widetilde{r$ $P_{e_1}^{x_0}, \tilde{r}$,
 $P_z^{x_0}, \tilde{r}$,
 $\tilde{e} \in C(P_{e_1}^{x_0}, \tilde{r})$ and $P_e^x \neq P_{e_2}^y$,
 $\tilde{d}(P_e^x, T(P_e^x))$ for each $P_e^x, P_{e_2}^y$ *e*₂ $\widetilde{\in}$ *C* ($\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

- $T(P_e^x), P_{e_1}^{x_0}$ $P_{e_1}^{x_0}, \widetilde{r}),$
- $P_{e_1}^{x_0}, \widetilde{r}$ and $P_e^x \neq P_{e_2}^y$,

for all
$$
\widetilde{u} \in \mathbb{R}(E)^*
$$
 and $\widetilde{r} \leq \overline{0}$. If a soft mapping T satisfies the following conditions
\n(i) $\widetilde{d}(T(P_e^x), P_{e_1}^{x_0}) = \widetilde{r}$ for each $P_e^x \in C(P_{e_1}^{x_0}, \widetilde{r})$,
\n(ii) $\widetilde{d}(T(P_e^x), T(P_{e_2}^y)) \leq \widetilde{r}$ for each $P_e^x, P_{e_2}^y \in C(P_{e_1}^{x_0}, \widetilde{r})$ and $P_e^x \neq P_{e_2}^y$,
\n(iii) $\widetilde{d}(T(P_e^x), T(P_{e_2}^y)) \leq \widetilde{d}(P_e^x, P_{e_2}^y) - \varphi_{\widetilde{r}}^*\left(\widetilde{d}(P_e^x, T(P_e^x))\right)$ for each $P_e^x, P_{e_2}^y \in C(P_{e_1}^{x_0}, \widetilde{r})$,
\nthen T fixes the soft circle $C(P_{e_1}^{x_0}, \widetilde{r})$.
\n**Proof.** Let $P_e^x \in C(P_{e_1}^{x_0}, \widetilde{r})$ be an arbitrary soft point. By the conditions (i), we have

Proof. Let $P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})$ be an arbitrary soft point. By the conditions (i) , we have $(iii) d(T$

then *T* fixe
 Proof. Le
 $T(P_e^x) \in C$ $(P_e^u, T(P_{e_2}^s)) \le d(P_e^u, P_{e_2}^s)$
the soft circle $C(P_{e_1}^{x_0}, \tilde{r})$
 $P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})$ be an a
 $P_{e_1}^{x_0}, \tilde{r})$, for all $P_e^x \in C$ *r*_{so} $\widetilde{\epsilon}$ _{*r*} \widetilde{r} $P_{e_1}^{x_0}$, \tilde{r}). To show $T(P_e^x) = P_e^x$, we assume that $\frac{a}{\tilde{r}}, \tilde{r}$ $T(P_e^x) \neq P_e^x$. By the condition *(ii)*, we obtain $\tilde{f} \in C(P_{e_1}^{x_0}, \tilde{r})$. To show $T(P_e^x) = P_e^x$, we assume that
 $(i\tilde{i})$, we obtain
 $\tilde{d}(T(P_e^x), T(T(P_e^x))) \leq \tilde{r}$ (7)

$$
\widetilde{d}(T(P_e^x), T(T(P_e^x))) \widetilde{\gt} \widetilde{r}
$$
\n⁽⁷⁾

and using the condition (*ii*), we get

$$
\widetilde{d}(T(P_e^x), T(T(P_e^x))) \leq \widetilde{r}
$$
\nthe condition (ii), we get

\n
$$
\widetilde{d}(T(P_e^x), T(T(P_e^x))) \leq \widetilde{d}(P_e^x, T(P_e^x)) - \varphi_{\widetilde{r}}^* \left(\widetilde{d}(P_e^x, T(P_e^x)) \right)
$$
\n
$$
= \widetilde{d}(P_e^x, T(P_e^x)) - \widetilde{d}(P_e^x, T(P_e^x)) + \widetilde{r} = \widetilde{r},
$$

which a contradiction with the inequality (7). Therefore, it should be $T(P_e^x) = P_e^x$ and so *T* fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ $\frac{\text{in}}{\tilde{r}}$. ■

Remark 7 Theorem 3.16 is a generalization of Theorem 3 *given in [25]. If we take the parameter set E with only one element, th[en](#page-10-0) Theorem 3.16 coincide Theorem* 2*.*1 *given in [25].* **Remark 7** Theorem 3.16 is a generalization of Theorem 3 given in [[2](#page-10-1)5]. If we take parameter set *E* with only one element, then Theorem 3.16 coincide Theorem 2.1 g in [25].
Example 3.17 Let us consider Example 3.13. Th

Example **3.17** Let u[s con](#page-10-1)sider Example 3.13. Then *T* satisfies the con[diti](#page-14-5)ons of Theorem **Example 3.17** Let us conside
3.16 with $\overline{L} = \overline{0}$. Consequently
Now, we give the following
[The](#page-10-1)orem 3.18 Let $(\widetilde{X}, \widetilde{d}, E)$

[Now](#page-14-5), we give the following theorem to exclude the identity soft mapping.

 $\big(\widetilde{X}, \widetilde{d}, E\big)$ be a sof[t me](#page-9-2)tric space. Let us consider a soft mapping Now, we give the following theorem to
Theorem 3.18 Let $(\tilde{X}, \tilde{d}, E)$ be a soft $T : \tilde{X} \to \tilde{X}$ which has a soft fixed circle *C* $P_{e_1}^{x_0}, \widetilde{r}$ $\det \begin{cases} \frac{1}{r}, & \text{if } r \leq r. \end{cases}$ be identity soft mapping.

pace. Let us consider a soft mapping

and the soft mapping $\varphi : \widetilde{X} \to \mathbb{R}(E)^*$ defined as in (1) . Then T satisfies the condition $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\mathsf{Re } C$
 $\mathsf{con} \widetilde{\mathsf{Con}}$
 $\widetilde{\leqslant} \overline{h}$

$$
\widetilde{d}\left(P_{e_2}^y, T(P_{e_2}^y)\right) \widetilde{\leqslant} \overline{h}\left[\varphi\left(P_{e_2}^y\right) - \varphi\left(T(P_{e_2}^y)\right)\right],\tag{8}
$$

for every $P_{e_2}^y$ *e*₂ *e Z i z and O. B. Özbakır* / *J. Linear. Topological. Algebra.* 12(04) (2023) 243-258.
 *e*₂ \widetilde{E} *Z X* and $\overline{0} \leq \overline{h} \leq \overline{1}$ if and only if $T = I_{\widetilde{X}}$, where the identity soft mappi *I*^{*X*} *IX I*_{*Z*} *<i>I*_{*X*} *<i>I*_{*X*} *I*_{*X*} *I* $\rightarrow \tilde{X}$ defined by $I_{\tilde{X}}(P_e^x) = P_e^x$ for all $P_e^x \in$ $\widetilde{\in} \widetilde{X}$ and $\overline{0} \widetilde{\leq} \overline{h} \widetilde{\leq}$
lefined by $I_{\widetilde{X}}(P_{\epsilon}^y)$
 $\widetilde{e}_2 \widetilde{\leq} \widetilde{X}$ and $T(P_{\epsilon}^y)$ $\frac{1}{d}$ $I_{\widetilde{X}}$ (
nd T
 $\widetilde{\leqslant} \ \overline{h}$

Proof. Let $P_{e_2}^y \n\tilde{\in} \n\tilde{X}$ and $T(P_{e_2}^y) \neq P_{e_2}^y$. Then using the inequality (8), we get

f. Let
$$
P_{e_2}^y \tilde{\in} \tilde{X}
$$
 and $T(P_{e_2}^y) \neq P_{e_2}^y$. Then using the inequality (8), we get
\n
$$
\tilde{d}(P_{e_2}^y, T(P_{e_2}^y)) \leq \overline{h} \left[\varphi(P_{e_2}^y) - \varphi(T(P_{e_2}^y)) \right]
$$
\n
$$
= \overline{h} \left[\tilde{d}(P_{e_2}^y, P_{e_1}^{x_0}) - \tilde{d}(T(P_{e_2}^y), P_{e_1}^{x_0}) \right]
$$
\n
$$
\leq \overline{h} \left[\tilde{d}(P_{e_2}^y, T(P_{e_2}^y)) + \tilde{d}(T(P_{e_2}^y), P_{e_1}^{x_0}) - \tilde{d}(T(P_{e_2}^y), P_{e_1}^{x_0}) \right]
$$
\n
$$
= \overline{h} \tilde{d}(P_{e_2}^y, T(P_{e_2}^y)),
$$

a contradiction. Hence it should be $T(P_{e_2}^y) = P_{e_2}^y$, that is, $T = I_{\tilde{X}}$. The converse statement is clear. $=\overline{h}\widetilde{d}$ (
a contradiction. Hence it shoul
is clear.
Corollary 3.19 Let $(\widetilde{X}, \widetilde{d}, E)$ be $T(P_{e_2}^y) = P_{e_2}^y$, that is, $T = I_{\tilde{X}}$. The converse statement
be a soft metric space and $T : \tilde{X} \to \tilde{X}$ be a soft mapping.

If *T* satisfies the conditions of Theorem 3.6 (resp. Theorem 3.8, Theorem 3.10, Theorem 3.12, Theorem 3.14 and Theorem 3.16) but the condition (8) is not satisfied by *T*, then $T \neq I_{\widetilde{X}}$.

[3.2](#page-8-1) *A uniq[uen](#page-9-1)ess theore[m for](#page-10-1) a soft fixed-cir[cl](#page-10-2)e*

In this subsection, we investigate a uniqueness condition for a soft fixed-circle of a soft **3.2 A** uniqueness theorem for a soft fixed-circle
In this subsection, we investigate a uniqueness condition for a soft fixed-circle of a soft
mapping $T : \widetilde{X} \to \widetilde{X}$. At first, we give the following example to sho which has two soft fixed-circles. **Example 3.20** Let (X, \tilde{X}) be the subsection, we invest

mapping $T : \tilde{X} \to \tilde{X}$. At first which has two soft fixed-circl
 Example 3.20 Let (X, \tilde{d}, E) the a uniqueness condition for a soft fixed-circle of a soft,

, we give the following example to show a soft mapping
 β .

be a soft metric space and $C(P_{e_1}^{x_0}, \tilde{r}), C(P_{e_2}^{y}, \tilde{\rho})$ be any *t* f sh $\frac{1}{r}$

mapping $T : X \to X$. At first, we give the following ϵ
which has two soft fixed-circles.
Example 3.20 Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space as
soft circles. Let us define a soft mapping $T : \tilde{X} \to \tilde{X}$ as (E,E) be a soft metric space and $C(P_e^x)$
soft mapping $T : \tilde{X} \to \tilde{X}$ as
 $E(E) = \begin{cases} P_e^x : P_e^x \in C(P_{e_1}^{x_0}, \tilde{r}) \cup C(P_{e_2}^y, \tilde{\rho}) \\ P_e^x : P_e^x \in C(P_{e_1}^{x_0}, \tilde{r}) \cup C(P_{e_2}^y, \tilde{\rho}) \end{cases}$ $\frac{\text{ac}}{\tilde{X}}$
, \tilde{r}

$$
T(P_e^x) = \begin{cases} P_e^x \, ; \, P_e^x \in C\left(P_{e_1}^{x_0}, \widetilde{r}\right) \widetilde{\cup} C\left(P_{e_2}^y, \widetilde{\rho}\right) \\ P_{e_3}^{\alpha} \, ; & \text{otherwise} \end{cases},
$$
 for all $P_e^x \in \widetilde{X}$, where $P_{e_3}^{\alpha}$ is a constant soft point such that \widetilde{d}

 $P_{e_3}^{\alpha}, P_{e_1}^{x_0}$ $\neq \tilde{r}$ and $\frac{f}{d\tilde{d}}$ $P_{e_3}^{\alpha}, P_{e_2}^y$ $\neq \tilde{\rho}$. Then *T* fixes soft circles both $C(P_{e_1}^{x_0}, \tilde{r})$ and $C(P_{e_2}^y)$ $T(P_e^{\alpha}) = \begin{cases} P_{e_3}^{\alpha} \\ P_{e_3}^{\alpha} \end{cases}$; otherwise
 $\widetilde{\epsilon}$ \widetilde{X} , where $P_{e_3}^{\alpha}$ is a constant soft point such that $\widetilde{d}(P_{e_3}^{\alpha}, \widetilde{\rho})$.
 $\neq \widetilde{\rho}$. Then *T* fixes soft circles both *C* $(P_{e_1}^{\alpha}, \$ $\frac{1}{\sqrt{r}}$
x

If we consider Example 3.20, then we say that the studying a uniqueness theorem of a soft fixed circle gains an importance. According to the above example, we prove the following theorem using the Banach type contraction [2]. If we consider Example 3.2
 Theorem 3.21 Let $(\tilde{X}, \tilde{d}, E)$ $(\tilde{X}, \tilde{d}, E)$ $(\tilde{X}, \tilde{d}, E)$
 Theorem 3.21 Let $(\tilde{X}, \tilde{d}, E)$ then we say that the studying a uniqueness theorem of ortance. According to the above example, we prove the nach type contraction [2].
be a soft metric space and $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping

which fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. If the condition rta
ach
e a
, \widetilde{r} $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ E be a soft metric space and T
 $(P_{e_1}^{x_0}, \tilde{r})$. If the condition
 $(T(P_e^x), T(P_{e_2}^y)) \leq \overline{h} \tilde{d} (P_e^x, P_{e_2}^y)$

$$
\widetilde{d}\left(T(P_e^x), T(P_{e_2}^y)\right) \widetilde{\leq} \overline{h}\widetilde{d}\left(P_e^x, P_{e_2}^y\right),\tag{9}
$$
\n
$$
(P_{e_1}^{x_0}, \widetilde{r}), P_{e_2}^y \widetilde{\in} \widetilde{X} \setminus C\left(P_{e_1}^{x_0}, \widetilde{r}\right) \text{ and some } \overline{0} \widetilde{\leqslant} \overline{h} \widetilde{\leqslant} \overline{1} \text{ by } T, \text{ then}
$$

is satisfied for all $P_e^x \xleftarrow{\sim} C$ $P_{e_1}^{x_0}, \widetilde{r}$ $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ is the unique soft fixed circle of T . fie
 $,\widetilde{r}$ $\frac{1}{n}$
 $\frac{1}{n}$ \overline{D} \setminus *C* $(P_{e_1}^{x_0}, \widetilde{r})$ and some $\overline{0} \leqslant \overline{h} \leqslant \overline{1}$ by *T*, then
T.
wo soft fixed circles *C* $(P_{e_1}^{x_0}, \widetilde{r})$ and *C* $(P_{e_2}^{y}, \widetilde{\rho})$ $(f \cdot f)$
ft, \widetilde{r}

Proof. Let us suppose that there exist two soft fixed circles $C(P_{e_1}^{x_0}, \tilde{r})$ and $C(P_{e_2}^{y})$ is satisfied for all $P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})$, $P_{e_2}^y \in Z$
 $C(P_{e_1}^{x_0}, \tilde{r})$ is the unique soft fixed circle of
 Proof. Let us suppose that there exist t

of the soft mapping *T*. If we take $P_{e_3}^u \in C$ *P P c*_{e₁}, \tilde{r}) and some $\overline{0} \leq \overline{h} \leq \overline{1}$ by *T*, then
 P r
 e soft fixed circles $C(P_{e_1}^{x_0}, \tilde{r})$ and $C(P_{e_2}^{y}, \tilde{\rho})$
 $P_{e_1}^{x_0}, \tilde{r}$) and $P_{e_4}^{y} \in C(P_{e_2}^{y}, \tilde{\rho})$ with P_{e then using the inequality (9), we get

$$
\tilde{d}(T(P_{e_3}^u), T(P_{e_4}^v)) = \tilde{d}(P_{e_3}^u, P_{e_4}^v) \leq \overline{h}\tilde{d}(P_{e_3}^u, P_{e_4}^v),
$$
\nwhere $\tilde{d}(T(P_{e_3}^u), T(P_{e_4}^v)) = \tilde{d}(P_{e_3}^u, P_{e_4}^v) \leq \overline{h}\tilde{d}(P_{e_3}^u, P_{e_4}^v)$.

 $\widetilde{d}(T(P_{e_3}^u), T(P_{e_4}^v)) = \widetilde{d}(P_{e_3}^u, P_{e_4}^v) \leqslant \overline{h}\widetilde{d}(P_{e_3}^u, P_{e_4}^v),$
a contradiction because of $\overline{0} \leqslant \overline{h} \leqslant \overline{1}$. Thereby, we have $P_{e_3}^u = P_{e_4}^v$ for all $P_{e_3}^u \in C$ $P_{e_1}^{x_0}, \widetilde{r}),$ $, \widetilde{r}$ $\tilde{d}(T(P_{e_3}^u), T(P_{e_4}^v)) = \tilde{d}(P_{e_3}^u, P_{e_4}^v) \leqslant \overline{h}$
 e contradiction because of $\overline{0} \leqslant \overline{h} \leqslant \overline{1}$. Thereby, we have P_e^v
 $P_{e_4}^v \in C(P_{e_2}^y, \tilde{\rho})$ and so *T* has a unique soft fixed circ $P_{e_4, \widetilde{r}}^{v}$

 $P_{e_4}^v \n\t\tilde{\in} C(P_{e_2}^y, \tilde{\rho})$ and so *T* has a unique soft fixed circle $C(P_{e_1}^{x_0}, \tilde{r})$.

To obtain the uniqueness theorem, the choice of the contractive condition is not unique.

For example, using the Kannan To obtain the uniqueness theorem, the choice of the contractive condition is not unique. For example, using the Kannan type contraction [17], we prove the following theorem. \hat{X} and \hat{X} and \tilde{X} , \tilde{d} , E

be a soft metric space and $T : \tilde{X} \to \tilde{X}$ be a soft mapping

Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space and $T : \tilde{X} \to \tilde{X}$ be a soft mapping

fit circle $C(P_{e_1}^{x_0}, \tilde{r})$. If the condition
 $(T(P_e^x), T(P_{e_2}^y)) \le$ which fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. If the condit[ion](#page-14-14) $\begin{aligned} \sup_{\mathbf{y} \in \mathbb{R}} \mathbf{y} \neq \mathbf{z} \ \mathbf{z}, \widetilde{r} \end{aligned}$ ¹
sc
 \tilde{d} metric space and $T : \widetilde{X} \to \widetilde{X}$ be
the condition
 $\widetilde{d}(T(P_e^x), P_e^x) + \widetilde{d}(T(P_{e_2}^y), P_{e_2}^y))$ $\sqrt{c_1}$

$$
\widetilde{d}\left(T(P_e^x), T(P_{e_2}^y)\right) \widetilde{\leq} \overline{h}\left[\widetilde{d}\left(T(P_e^x), P_e^x\right) + \widetilde{d}\left(T(P_{e_2}^y), P_{e_2}^y\right)\right]
$$
\n(10)
\n
$$
\text{all } P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right), P_{e_2}^y \widetilde{\in} \widetilde{X} \setminus C\left(P_{e_1}^{x_0}, \widetilde{r}\right) \text{ and some } \overline{0} \widetilde{\leq} \overline{h} \widetilde{<} \frac{1}{2} \text{ by } T, \text{ then}
$$

 $\widetilde{d}\left(T(P_e^x)\right)$ is satisfied for all
 $P_e^x\ \widetilde{\in}\ C$ $P_{e_1}^{x_0}, \widetilde{r}$ $\frac{1}{2}$ by *T*, then $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ is the unique soft fixed circle of *T*. fie
 $,\widetilde{r}$ \bar{h}
r $\overline{D} \setminus C(P_{e_1}^{x_0}, \widetilde{r})$ and some $\overline{0} \leqslant \overline{h} \leqslant \frac{1}{2}$ by *T*, then
T.
wo soft fixed circles $C(P_{e_1}^{x_0}, \widetilde{r})$ and $C(P_{e_2}^{y}, \widetilde{\rho})$

Proof. Let us suppose that there exist two soft fixed circles $C(P_{e_1}^{x_0}, \tilde{r})$ and $C(P_{e_2}^{y})$ e_1 is satisfied for all $P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})$, $P_{e_2}^y \in C(P_{e_1}^{x_0}, \tilde{r})$ is the unique soft fixed circle of **Proof.** Let us suppose that there exist to the soft mapping *T*. If we take $P_{e_3}^u \in C$ *P C (P*^{*x*₀}, \tilde{r}) and some $0 \le h < \frac{1}{2}$ by *T*, then
 P.
 p soft fixed circles $C(P_{e_1}^{x_0}, \tilde{r})$ and $C(P_{e_2}^{y}, \tilde{\rho})$
 $P_{e_1}^{x_0}, \tilde{r}$) and $P_{e_4}^{v} \in C(P_{e_2}^{y}, \tilde{\rho})$ with $P_{e_3}^{u} \neq P_{e_$ $(F$
ft
, \widetilde{r} then using the inequality (10), we get $\frac{d}{d}$ f we take $P_{e_3}^u \in C(P_{e_1}^{x_0}, \tilde{r})$ and $P_{e_4}^v \in C$
 \widetilde{q} (10), we get
 $= \widetilde{d}(P_{e_3}^u, P_{e_4}^v) \leq \overline{h} \left[\widetilde{d}(T(P_{e_3}^u), P_{e_3}^u) + \widetilde{d} \right]$ P_e^2
 \tilde{d}

$$
\widetilde{d}\left(T(P_{e_3}^u), T(P_{e_4}^v)\right) = \widetilde{d}\left(P_{e_3}^u, P_{e_4}^v\right) \widetilde{\leq} \overline{h}\left[\widetilde{d}\left(T(P_{e_3}^u), P_{e_3}^u\right) + \widetilde{d}\left(T(P_{e_4}^v), P_{e_4}^v\right)\right] = \overline{0},
$$
\na contradiction. Thereby, we have

\n
$$
P_{e_3}^u = P_{e_4}^v \text{ for all } P_{e_3}^u \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right), P_{e_4}^v \widetilde{\in} C\left(P_{e_2}^y, \widetilde{\rho}\right) \text{ and}
$$

so *T* has a unique soft fixed circle $C(P_{e_1}^{x_0}, \tilde{r})$. $\begin{array}{c} \hline h \ v^v \ e_{4} \ , \widetilde r \end{array}$ a contradiction. Thereby, we lso *T* has a unique soft fixed c
Using Chatterjea type cont
Theorem 3.23 Let $(\tilde{X}, \tilde{d}, E)$ $\overline{\mathbf{a}}$ be $C^{s}(P_{e_1}^{x_0}, \tilde{r})$.

be a soft metric space and $T : \tilde{X} \to \tilde{X}$ be a soft mapping circle O

Using Chatterjea type contraction [4], we obtain another uniqueness theorem.

Let $(\widetilde{X}, \widetilde{d}, E)$ be a so
oft circle $C(P_{e_1}^{x_0}, \widetilde{r})$. I
 $(T(P_e^x), T(P_{e_2}^y)) \widetilde{\leqslant} \overline{h}$ which fixes the soft circle $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$. [If](#page-14-15) the condition $\frac{1}{2}$ e z sc
 \tilde{d} ft metric space and $T : \widetilde{X} \to \widetilde{X}$ be

f the condition
 $\left[\widetilde{d} \left(T(P_e^x), P_{e_2}^y \right) + \widetilde{d} \left(T(P_{e_2}^y), P_e^x \right) \right]$ *d*ecall $\sqrt{c_1}$

$$
\widetilde{d}\left(T(P_e^x), T(P_{e_2}^y)\right) \widetilde{\leq} \overline{h}\left[\widetilde{d}\left(T(P_e^x), P_{e_2}^y\right) + \widetilde{d}\left(T(P_{e_2}^y), P_e^x\right)\right]
$$
\n
$$
\text{and } P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right), P_{e_2}^y \widetilde{\in} \widetilde{X} \setminus C\left(P_{e_1}^{x_0}, \widetilde{r}\right) \text{ and some } \overline{0} \widetilde{\leq} \overline{h} \widetilde{<} \frac{1}{2} \text{ by } T, \text{ then}
$$
\n
$$
\text{and } P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right), P_{e_2}^y \widetilde{\in} \widetilde{X} \setminus C\left(P_{e_1}^{x_0}, \widetilde{r}\right) \text{ and some } \overline{0} \widetilde{\leq} \overline{h} \widetilde{<} \frac{1}{2} \text{ by } T, \text{ then}
$$

 $\label{eq:2.1} \widetilde{d}\left(T(P_e^x),\right.$ is satisfied for all
 $P_e^x\ \widetilde{\in}\ C$ $P_{e_1}^{x_0}, \widetilde{r}$ $\frac{1}{2}$ by *T*, then $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ is the unique soft fixed circle of T . fie
 $,\widetilde{r}$ \bar{h}
r $\overline{D} \setminus C(P_{e_1}^{x_0}, \widetilde{r})$ and some $\overline{0} \leq \overline{h} \leq \frac{1}{2}$ by *T*, then
T.
wo soft fixed circles $C(P_{e_1}^{x_0}, \widetilde{r})$ and $C(P_{e_2}^{y}, \widetilde{\rho})$

Proof. Let us suppose that there exist two soft fixed circles $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ and $C\left(P_{e_2}^{y}, \tilde{\rho}\right)$ is satisfied for all $P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})$, $P_{e_2}^y \in C(P_{e_1}^{x_0}, \tilde{r})$ is the unique soft fixed circle of
Proof. Let us suppose that there exist to the soft mapping *T*. If we take $P_{e_3}^u \in C$ *P C (P*^{*x*₀}, \tilde{r}) and some $0 \le h < \frac{1}{2}$ by *T*, then
 P.
 p soft fixed circles $C(P_{e_1}^{x_0}, \tilde{r})$ and $C(P_{e_2}^{y}, \tilde{\rho})$
 $P_{e_1}^{x_0}, \tilde{r}$) and $P_{e_4}^{v} \in C(P_{e_2}^{y}, \tilde{\rho})$ with $P_{e_3}^{u} \neq P_{e_$ $(F$
ft
, \widetilde{r} then using the inequality (11), we get $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ of \widetilde{d} g *T*. If we take $P_{e_3}^u \in C(P_{e_1}^{x_0}, \tilde{r})$ and $P_{e_4}^v \in C(P_{e_2}^y, \tilde{\rho})$ with *F*

quality (11), we get
 $= \tilde{d}(P_{e_3}^u, P_{e_4}^v) \leq \overline{h} \left[\tilde{d}(T(P_{e_3}^u), P_{e_4}^v) + \tilde{d}(T(P_{e_4}^v), P_{e_3}^u) \right] = 2\overline{h}\tilde{d}$ $\widetilde{\tilde{e}}$
 \widetilde{d}

$$
\widetilde{d}\left(T(P_{e_3}^u), T(P_{e_4}^v)\right) = \widetilde{d}\left(P_{e_3}^u, P_{e_4}^v\right) \widetilde{\leq} \overline{h}\left[\widetilde{d}\left(T(P_{e_3}^u), P_{e_4}^v\right) + \widetilde{d}\left(T(P_{e_4}^v), P_{e_3}^u\right)\right] = \overline{2h}\widetilde{d}\left(P_{e_3}^u, P_{e_4}^v\right),
$$
\na contradiction because of $\overline{0} \widetilde{\leq} \overline{h} \widetilde{\leq} \frac{1}{2}$. Thereby, we have $P_{e_3}^u = P_{e_4}^v$ for all $P_{e_3}^u \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$,

 $P_{e_1}^{x_0}, \widetilde{r}),$ $d(T(P_{e_3}^u), T(P_{e_4}^v)) = d(P_{e_3}^u, P_{e_4}^v) \le \overline{h} \left[d(T(P_{e_3}^u), P_{e_4}^v) + d \right]$

a contradiction because of $\overline{0} \le \overline{h} \le \frac{\overline{1}}{2}$. Thereby, we have $P_{e_4}^v \in C(P_{e_2}^y, \overline{\rho})$ and so *T* has a unique soft fixed c $P_{e_1}^{x_0}, \widetilde{r}$. $\begin{array}{c} \mathcal{V} \ e_4 \ \hline \ e_4 \ \hline \ \eta \end{array}$

3.3 *Some consequences*

Now, we give the following remarks.

- (1) Theorem 3.6 (resp. Theorem 3.8, Theorem 3.10, Theorem 3.12, Theorem 3.14 and Theorem 3.16) guarantees the existence of a soft fixed circle. Also, they can be considered as soft fixed-point theorems to show the existence of a soft fixed
point in case the soft fixed circle has only one soft point. point in case the soft fixed circle has only one soft point. and Theorem 3.16) guarantees the existence of a soft fix-
be considered as soft fixed-point theorems to show the
point in case th[e so](#page-10-1)ft fixed circle has only one soft point
The condition *(i)* given in Theorem 3.6 guarant $\begin{array}{c} \text{e:} \ \text{e:} \ \text{the} \ \text{as} \ \text{r:} \ \pi \end{array}$ rcie
enc
, \widetilde{r}
- (2) The cond[itio](#page-6-1)n (*i*) given in T[heo](#page-7-0)rem 3.6 gu[aran](#page-7-1)tees that $T(P_e^x)$ $T(P_e^x)$ $T(P_e^x)$ is not in [the](#page-9-1) $P_{e_1}^{x_0}$, \tilde{r}). Similarly, the condition *(ii)* given in Theorem 3.6 guarantees that $T(P_e^x)$ is not in the interior of the soft circle $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ for each $P_e^x \in C\left(P_{e_1}^{x_0}, \tilde{r}\right)$. Consequently, $T\left(C\left(P_{e_1}^{x_0}, \tilde{r}\right) \right)$ *, r*e $\begin{array}{c} \text{only} \ \text{in } 3.6 \ \text{for} \ \text{guara} \ \text{in } \widetilde{e} \ \widetilde{e} \ \text{in} \end{array}$ $\begin{array}{c} \text{for } \ \text{and } P_{\text{c}} \ \text{th} \ \overline{\widetilde{r}} \end{array}$ t
 d
 ri
 ri
 \widetilde{r} T ne
exte
cond
of th
 $\widetilde{\subseteq} C$ $P_{e_1}^{x_0}, \widetilde{r}$. $\inf_{\substack{\text{of}\ \sigma:\ \text{in}\ \mathbb{R}^n}}$ for the soft circle $C(P_{e_1}^{x_0}, \tilde{r})$ f[or e](#page-6-1)ach $P_e^x \tilde{\in} C(P_{e_1}^{x_0})$
 $\tilde{\subseteq} C(P_{e_1}^{x_0}, \tilde{r})$.

The condition (*i*) given in Theorem 3.8 guarant

rior of the soft circle $C(P_{e_1}^{x_0}, \tilde{r})$ for each $P_e^x \tilde{\in} C$ $\frac{\partial \mathbf{r}}{\partial \mathbf{r}}$
 $\frac{\partial \mathbf{r}}{\partial \mathbf{r}}$ *r*
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 , r̃
- (3) The condition (*i*) given in Theorem 3.8 guarantees that $T(P_e^x)$ is not in the inte- $P_{e_1}^{x_0}$, \tilde{r}). Similarly, the condition (*ii*) given in Theorem 3.8 guarantees that $T(P_e^x)$ is not in the exterior of the The condition (*i*) given in Theorem 3.8

rior of the soft circle $C(P_{e_1}^{x_0}, \tilde{r})$ for each

(*ii*) given in Theorem 3.8 guarantees

soft circle $C(P_{e_1}^{x_0}, \tilde{r})$ for each $P_e^x \in C$ gi⁻
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 $,\widetilde{r}$ guarantees that $T(P_e^x)$ is not in the exterior of t

cch $P_e^x \in C(P_{e_1}^{x_0}, \tilde{r})$. Consequently, $T(C(P_{e_1}^{x_0}, \tilde{r}))$

Theorem 3.10 guarantees that $T(P_e^x)$ is not in t
 $(P_{e_1}^{x_0}, \tilde{r})$ for each $P_e^x \in C(P_{e_1}^{x_0$ ran $\widetilde{\in}$ *(*
 \widetilde{T} _{, \widetilde{r})} $\frac{1}{\alpha}$
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⊆ $C(P_{e_1}^{x_0}, \widetilde{r}).$ $\begin{array}{c} \text{and} \ \text{then} \ \text{def} \ \text{$ $\begin{array}{l} \hbox{and} \ \mathbf{p}x \ e \ \hbox{or} \ \mathbf{p}x \ \mathbf{p}y \ \mathbf{p$ $\begin{array}{c} \text{en} \ \text{at} \ \text{at} \ \text{at} \ \text{on} \ \end{array}$ $\frac{1}{2}$
- (4) The condition (*i*) given in Theorem 3.10 guarantees that $T(P_e^x)$ is not in the exterior of the soft circle $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ $C\left(P_{e_1}^{x_0}, \tilde{r}\right)$ for each $P_e^x \tilde{\in} C\left(P_{e_1}^{x_0}, \tilde{r}\right)$. Also, the condition (*ii*) given in Theorem 3.10 implies that $T(P_e^x)$ can be lies on or exterior or interior of the soft circle $C(P_{e_1}^{x_0}, \tilde{r})$. Hence, $T(P_e^x)$ should be lies on or interior of the soft *n* i
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 , r̃ circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. *n*
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Th
; \widetilde{r} $\widetilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x)$ $\widetilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x)$ $\widetilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x)$
 $\widetilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x)$
- (5) If we consider $L = \overline{-1}$ in the condition (*i*) of Theorem 3.12, then we obtain

$$
\widetilde{d}\left(P_e^x, T(P_e^x)\right) \,\,\widetilde{\leq}\,\,\varphi(P_e^x)-\varphi(T(P_e^x)).
$$

Therefore, the condition (*i*) of Theorem 3.6 (resp. The[orem](#page-8-1) 3.10) is satisfied. On the other hand, the condition (*ii*) of Theorem 3.6 is the same as the condition (*ii*) of Theorem 3.12. Also, if the condition (*ii*) of Theorem 3.12 is satisfied, then the condition (*ii*) of Theorem 3.10 is satisfied. *[d](#page-8-1)*(*P*_e. *d*) *d f* the condition (*ii*) of Theorem 3.
 d f Theorem 3.10 is satisfied.
 d \overline{P} in the condition (*i*) of Theorem 3.14, $\tilde{d}(P_e^x, T(P_e^x)) \leq \varphi(P_e^x) + \varphi(T(P_e^x)) - \overline{2}\tilde{r}$.

(6) If we consider $\overline{L} = -\overline{1}$ in the condition (*i*[\) o](#page-6-1)f Theorem 3.14, [then](#page-7-1) we obtain

$$
\widetilde{d}\left(P_e^x,T(P_e^x)\right) \,\,\widetilde{\leq}\,\,\varphi(P_e^x)+\varphi(T(P_e^x))-\overline{2}\widetilde{r}.
$$

Hence, the condition (*i*) of Theorem 3.8 is satisfied. [Also,](#page-9-1) the condition (*ii*) of Theorem 3.8 is the same as the condition (*ii*) of Theorem 3.14. *, r*eso,
 *r*eso,
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4.
⊆*C* $\frac{1}{r}, \widetilde{r}$

- (7) The condition (*i*) of Theorem 3.16 shows that $T(C(P_{e_1}^{x_0}, \tilde{r}))$ $P_{e_1}^{x_0}, \widetilde{r}$.
- (8) Theorem 3.21 guarantees the uniqueness of a soft fixed circle. Also, it can be considered as a soft fixed-point result [to s](#page-7-0)how the uniqueness of a soft fixed point in case th[e so](#page-7-0)ft fixed circle has only one soft point. On the [othe](#page-9-1)r hand, to obtain a new uniqueness theorem of [a sof](#page-10-1)t fixed circle, different contractions should be used.

4. Conclusion and future work

In this paper, we introduced the notion of a soft fixed circle and proved some soft fixedcircle theorems on soft metric spaces with some illustrative examples. The importance of working with a fixed-circle problem with the help of soft set is to increase the number of fixed points of a mapping by means of parametrization. In this way, these theorems can be used in various fields of application such as neural networks, activation functions etc. Finally, we can leave the following open problem ideas:

Problem 1: Can the obtained theoretical results in this paper be applied to other areas such as engineering, activation functions, neural networks, discontinuity?

Problem 2: Can new solutions to the fixed-circle problem be found using the fuzzy soft set theory?

Problem 3: Is it possible to study other fixed-circle theorems using different contractive conditions on soft metric spaces?

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