Journal of Linear and Topological Algebra
Vol. 11*, No.* 03*,* 2022*,* 215- 218
DOR: 20.1001.1.22520201.2022.11.03.6.1
DOI: 10.30495/JLTA.2022.695592

Topologically simple semihypergroup

H. M. Jafarabadi[a]

[a]*Department of Mathematics, Bardsir Branch, Islamic Azad University, Bardsir, Iran.*

Received 15 August 2022; Revised 28 September 2022, Accepted 30 September 2022.
Communicated by Hamidreza Rahimi

Abstract. In this note, we introduce the concept of topologically simple semihypergroup and determine when topological simplicity of a complete subsemihypergroup of a semihypergroup implies its simplicity from algebraic point of view.

Keywords: Hypergroupoid, hypergroup, semihypergroip, simple semihypergroup, topological semihypergroup.

2010 AMS Subject Classification: 20N20, 22A30.

1. Introduction and preliminaries

The theory of hyperstructures first emerged in the mid-1930s when the concept of algebraic hypergroups was introduced by Marty and Wall as a generalization of groups [10, 12]. However, the real origin of this concept actually dates back to the emergence of group theory in 1930 and in the Fribinius's studies. The theory of hypergroupoids (in the sense of Marty) is a well-developed branch of hyperstructure theory. By contrast, few studies have been conducted 0n this area concerning with topological structures. In fact, the theory of topological hypergroupoids was initiated in [2] via introducing the concept of topological transposition hypergroups with the help of employing pseudo and strong pseudo continuous hyperoperations. Later on, introducing topological hypergroupoid by Hoskava [7] led to improving this notion. Heidari et al. [5] stated the concept of topological hypergroups in the sense of Marty as a generalization of topological groups. Cristea and Hoskava [4] worked on the fuzzy pseudotopological hypergroupoids. They developed the concepts of topological hypergroupoid and pseudocontinuous or continuous hyperoperation into the fuzzy case. The topological isomorphism theorems of topological polygroups was proved by Heidari et al. [6]. The role of complete parts in topological polygroups has

---

E-mail address: hmjafarabadi@gmail.com (H. M. Jafarabadi).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

been investigated by Salehi Shadkami et al. [11]. Abbasizadeh and Davvaz [1] discussed the relation between two definitions of a fuzzy topological polygroup. The concept of topological complete hypergroup, which is a special class of topological hypergroups was introduced and studied by Manoranjan et al. [9].

In this paper, by considering the notion of a semihypergroup, we introduce the concept of a topologically simple semihypergroup. In particular, using complete semihypergroup, the relationship between topologically simple semihypergroup and simple semihypergroup from algebraic point of view is investigated, and finally some related results are indicated. Further, some theorems and definitions regarding hypergroupoids used through the paper are presented in this section. The preliminary concepts in this field can be found in [3].

A n-hyperoperation on nonempty set H is a map $\circ : H^n \to P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H. A hyperstructure (multivalued algebra) is a nonempty set endowed with a family of hyperoperations, and a hypergroupoid is a hyperstructure with a singleton family of hyperoperations.The hyperoperation of h_1 and h_2 as the elements of H is shown by $h_1 \circ h_2$ and called the hyperproduct of h_1 and h_2. If I and J are nonempty subsets of H, then $I \circ J = \bigcup\limits_{h_1 \in I h_2 \in J} h_1 \circ h_2$. When the hyperoperation of a hypergroupoid has associative property, the hypergroupoid is named a semihypergroup. Likewise, a semihypergroup $(H, \circ)$ is called hypergroup if

$$\forall h \in H, h \circ H = H \circ h = H.$$

A complete part of semihypergroup $(S, \circ)$ is the non-empty subset T of S that for any non-zero natural number n and for all $s_1, s_2, ..., s_n$ of S the following implication holds:

$$T \bigcap \prod_{i=1}^{n} s_i \neq \emptyset \Rightarrow \prod_{i=1}^{n} s_i \subseteq T$$

A left hyperideal of the semihypergroup S is a nonempty subset I of S with $SI \subseteq I$. Similarly, the right hyperideal can be defined. In addition, I is called (two-sided) hyperideal if it is both right and left hyperideal. If a semihypergroup S has no proper hyperideals, it is called simple. In fact, semihypergroup S is simple if and only if $SsS = S$ for all $s \neq 0$. It is clear that each hypergroup is a simple semihypergroup [8].

Lemma 1.1 [5] If (S, τ) be a topological space, then the family β including all $S_V = \{U \in P^*(S) : U \subseteq V, V \in \tau\}$ is a base for a topology on $P^*(S)$. This topology is shown by τ^*.

Definition 1.2 [5] Suppose $(S, \circ)$ is a semihypergroup and (S, τ) is a topological space. The triple $(S, \circ, \tau)$ is called topological semihypergroup if the mapping $(x, y) \mapsto x \circ y$ is continuous with respect to the product topology on $S \times S$ and the topology τ^* on $P^*(S)$. Moreover, topological semihypergroup $(H, \circ, \tau)$ is called topological hypergroup when the mapping $(x, y) \mapsto \frac{x}{y}$ is also continuous, where $\frac{x}{y} = \{z \in H : x \in z \circ y\}$

In topological groupoid theory the translation maps are homeomorphism. However, they are just continuous in topological hypergroupoid theory in general [11]. We say the topological hypergroupoid (H, τ) is a compact Hausdorff topological hypergroupoid if topological space (H, τ) is compact as well as Hausdorff [9].

Theorem 1.3 [9] Let H' be a subset of compact Hausdorff topological hypergroup H. Then $h\overline{H}' = \overline{hH'}$ for all $h \in H$.

Lemma 1.4 [9] Let $(H, \circ, \tau)$ be a topological complete hypergroup that every open subset of it is a complete part. Then, the translation maps are open maps.

2. Main results

In this section, we introduce the concept of topologically simple semihypergroup and determine when topological simplicity of a subsemihypergroup implies its simplicity from algebraic point of view.

Definition 2.1 Subsemihypergroup S' of topological semihypergroip S is called topologically simple if for every hyperideal of it such as I, we have $S' \subseteq \overline{I}$.

Definition 2.2 Let $(S, \circ)$ be a hypergroupoid and $x \in S$. We say x has infinite order $(O(x) = \infty)$ if for all $(m, n) \in Z^2$, where $m \neq n$, we have $x^n \bigcap x^m = \emptyset$, where $x^k = \underbrace{x \circ x \circ \cdots \circ x}$ (k times).

The following example indicates that a topologically simple semihypergroup is not simple algebraic, generally.

***Example* 2.3** Let $(H, \circ, \tau)$ be a compact Hausdorff topological complete hypergroup and $a \in H$ has infinite order. Then the subsemihypergroup $S = \bigcup_{i=1}^{\infty} a^i$ is a topologically simple subsemihypergroup of H that is not simple from algebraic point of view. In fact, according to the definition of simple semihypergroup, the subsemihypergroup S is not simple because $a \in S$ (as defined by S) and $a \notin SaS$ (a has the infinite order), therefore $SaS \neq S$. On the other hand, every hyperideal of S that is in the form of $I = \bigcup_{i=j}^{\infty} a^j (j \geq 2)$ has intersect with any infinite open subset of S which is seen in the form of $U = \bigcup_{i=j}^{\infty} a^j (j \geq 1)$. So I is dense in S, i.e. S is a topologically simple subsemihypergroup of H.

Based on the following lemma, the closure of any subsemihypergroup of a topological semihypergroup is again a subsemihypergroup.

Lemma 2.4 Let T be a subsemihypergroup of topological semihypergroup $(S, \circ, \tau)$. Then $\overline{T}$ is a subsemihypergroup of S, where the bar denotes closure.

Proof. Suppose $x, y \in \overline{T}$ and we have $x \circ y \not\subseteq \overline{T}$. Then since $\overline{T}$ is closed , there exist neighbourhoods V_x of x and V_y of y that $(V_x \circ V_y) = \emptyset$. On the other hand, since x and y are elements of $\overline{T}$, non-empty subsets $V_x \bigcap T$ and $V_y \bigcap T$ exist. If $a_x \in V_x \bigcap T$ and $a_y \in V_y \bigcap T$ then

$$\emptyset \neq a_x \circ a_y \subseteq (V_x \bigcap T) \circ (V_y \bigcap T) \subseteq T \subseteq \overline{T},$$

which is a contradiction. ■

The following corollary is immediate obtained by previous lemma and Example 2.3.

Corollary 2.5 Let $(S, \circ, \tau)$ be a topological complete semihypergroup and each topologically simple subsemihypergroup of S is simple. Then, no subsemihypergroup of S contains an element of infinite order.

Theorem 2.6 Let $(S, \circ, \tau)$ be a compact Haussdorf topological complete semihypergroup that every open subset of S is a complete part. Then S is a topologically simple semihypergroup if and only if for all x in S, we have $\overline{S \circ x \circ S} = S$.

Proof. Suppose $(S, \circ, \tau)$ be a compact Haussdorf topological complete semihypergroup that every open subset of S is a complete part. Additionally, suppose S is topologically simple and x is an arbitrary element of S. Then $S \circ x \circ S$ is a hyperideal of S and this hyperideal has to be dense in S via definition of topologically simple semihypergroup. So, $\overline{S \circ x \circ S} = S$.

Conversely, suppose for every element such as x in S, $\overline{S \circ x \circ S} = S$ and I be a non-zero (two-sided) hyperideal of S. If i is a non-zero element of I, then by Theorem 1.3, we have

$$S = \overline{S \circ i \circ S} \subseteq \overline{S \circ I \circ S} \subseteq \overline{I}.$$

Hence, $\overline{I} = S$, and so S is topologically simple semihypergroup. ■

Theorem 2.7 Let $(H, \circ, \tau)$ be a compact Haussdorf topological complete semihypergroup that every open subset of S is a complete part. Then subsemihypergroup S of H is topologically simple if and only if $\overline{S}$ is simple (algebraic) subsemihypergroup of H.

Proof. We first suppose S is a topologically simple subsemihypergroup of H, I is a non-zero proper hyperideal of $\overline{S}$ and i is an element of I. By Lemma 1.4, $\overline{S} \circ i \circ \overline{S}$ is a closed proper hyperideal of $\overline{S}$. Therefore, $(\overline{S} \circ i \circ \overline{S}) \bigcap S$ is also a closed proper hyperideal of S, a fact contradicting the topological simplicity of S.
By contrast, suppose $\overline{S}$ is simple subsemihypergroup of H. By Theorem 1.3 the closure of every hyperideal of S is a hyperideal of $\overline{S}$. Thus, if I is a non-empty hyperideal of S then $\overline{I}$ is a hyperideal of $\overline{S}$. Since $\overline{S}$ is simple, we have $\overline{S} = \overline{I}$. As a result, $S \subseteq \overline{S} = \overline{I}$, means S is topologically simple semihypergroup. ■

Acknowledgments

The author is grateful to the referees for their meticulous reading and valuable suggestions for improving the paper.

References

[1] N. Abbasizadeh, B. Davvaz, Intuitionistic fuzzy topological polygroups, Int. J. Anal. Appl. 12 (2) (2016), 163-179.
[2] R. Ameri, Topological transposition hypergroups, Ital. J. Pure Appl. Math. 13 (2003), 171-176.
[3] P. Corsini, V. Leoreanu, Applications of Hyperstructures Theory, Dordrecht, Kluwer Academic Publisher, 2003.
[4] I. Cristea, S. Hoskova, Fuzzy pseudotopological hypergroupoids, Iran. J. Fuzzy Syst. 6 (4) (2009), 11-19.
[5] D. Heidari, B. Davvaz, S. Modarres, Topological hypergroups in the sense of Marty, Commun. Algebra. 42 (2014), 4712-4721.
[6] D. Heidari, B. Davvaz, S. Modarres, Topological polygroups, Bull. Malays. Math. Sci. Soc. 39 (2) (2016), 707-721.
[7] S. Hoskava-Mayerova, Topological hypergroupoid, Comput. Math. Appl. 64 (2012), 2845-2849.
[8] H. M. Jafarabadi, N. H. Sarmin, M. R. Molaei, Completely simple and regular semihypergroups, Bull. Malays. Math. Sci. Soc. 35 (2) (2012), 335-343.
[9] S. Manoranjan, D. Kousik, B. Davvaz, On topological complete hypergroups, Filomat. 31 (16) (2017), 5045-5056.
[10] F. Marty, Sur Uni Generalization de la Notion de Group, 8th Congress Math Scandenaves, Stockholm, 1934.
[11] M. S. Shadkami, M. R. Ahmadi, B. Davvaz, The role of complete part in topological polygroups, Int. J. Anal. Appl. 11 (2016), 54-60.
[12] H. S. Wall, Hypergroups, Amer. J. Math. 59 (1937), 77-98.