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On m^{th}-autocommutator subgroup of finite abelian groups

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Abstract. Let G be a group and $Aut(G)$ be the group of automorphisms of G. For any natural number m, the m^{th}-autocommutator subgroup of G is defined as:

$$K_m(G) = \langle [g, \alpha_1, \dots, \alpha_m] | g \in G, \alpha_1, \dots, \alpha_m \in Aut(G) \rangle.$$

In this paper, we obtain the m^{th}-autocommutator subgroup of all finite abelian groups.

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1. Introduction

Let G be a group and $Aut(G)$ denote the group of automorphisms of G. As in [3], if $g \in G$ and $\alpha \in Aut(G)$, then the element $[g, \alpha] = g^{-1}\alpha(g)$ is an *autocommutator* of g and α. Hence, following [5] one may define the *autocommutator of weight m+1* ($m \geqslant 2$) inductively as:

$$[g, \alpha_1, \alpha_2, \dots, \alpha_m] = [[g, \alpha_1, \dots, \alpha_{m-1}], \alpha_m],$$

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for all $\alpha_1, \alpha_2, \ldots, \alpha_m \in Aut(G)$.
Now for any natural number m

$$K_m(G) = [G, \underbrace{Aut(G), \ldots, Aut(G)}_{m-times}] = \langle [g, \alpha_1, \alpha_2, \ldots, \alpha_m] | g \in G, \alpha_1, \ldots, \alpha_m \in Aut(G) \rangle,$$

which is called the m^{th}-*autocommutator subgroup* of G.
Throughout this paper we adopt additive notation for all abelian groups. To be brief, $([k]_n, [k']_m)$ of group $\mathbb{Z}_n \oplus \mathbb{Z}_m$ will be indicated as (k, k'), where $k \in \{0, 1, 2, \ldots, n-1\}$ and $k' \in \{0, 1, 2, \ldots, m-1\}$.

***Example* 1.1** i) Let n be a natural number. Then $K_m(\mathbb{Z}_{2^n}) = 2^m \mathbb{Z}_{2^n}$, for any natural number m.
ii) Let $G = D_{2n}$, dihedral group of order $2n$. Then one can check that $K_m(G) \cong 2^{m-1}\mathbb{Z}_n$, for any natural number m.

Proof. i) It is obvious by Lemma 2.2 of [5].
ii) We have $D_{2n} = \langle x, y \mid x^n = y^2 = (xy)^2 = 1 \rangle$ and $Aut(D_{2n}) = \{\alpha_{i,t} \mid 0 \leqslant i \leqslant n-1, 1 \leqslant t \leqslant n-1, (t, n) = 1\}$ such that $\alpha_{i,t}(x) = x^t$ and $\alpha_{i,t}(y) = x^i y$. So if n is even, then by induction on m, we have $K_m(D_{2n}) = \langle x^{2^{m-1}} \rangle \cong 2^{m-1}\mathbb{Z}_n$. If n is odd, then we have $K_m(D_{2n}) = \langle x \rangle \cong \mathbb{Z}_n \cong 2^{m-1}\mathbb{Z}_n$. ■

In [5] some properties of autocommutator subgroups of a finite abelian group are studied. The under example shows the m^{th}-autcommutator subgroup of a finite abelian group incorrectly concluded in the Theorem 2.5 of [5].

***Example* 1.2** Let $G = \mathbb{Z}_8 \oplus \mathbb{Z}_4$. Then by Theorem 2.5 of [5], $K_2(G) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. But if we define the automorphisms α and β of the G, given by $\alpha(a, b) = (a, a + b)$ and $\beta(a, b) = (a + 2b, b)$ for all $(a, b) \in G$, then we have $[(1, 0), \alpha, \beta] = (2, 0)$ and hence, $K_2(G)$ has an element of order 4. So $K_2(G) \neq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

In [6], we obtained the $K_m(\oplus_{i=1}^{k}\mathbb{Z}_{2^{n_i}})$ with $n_1 > n_2 > \ldots > n_k$ using a function which recursively defined in terms of the $n_1, \ldots, n_k$'s. In this paper we obtain the m^{th}-autocommutator subgroup of all finite abelian groups.

2. Preliminary Results

We begin with some useful results that will be used in the proof of our main results.

Lemma 2.1 ([6]) i) Let H and T be two arbitrary groups. Then for any natural number m,

$$K_m(H) \times K_m(T) \subseteq K_m(H \times T).$$

ii) Let H and T be finite groups such that $(|H|, |T|) = 1$. Then for any natural number m,

$$K_m(H) \times K_m(T) = K_m(H \times T).$$

Lemma 2.2 ([5], Lemma 2.2) If G is a finite cyclic group, then

$$K_m(G) = 2^m G, \text{ for any natural number m.}$$

Corollary 2.3 If G is a finite abelian group of odd order, then

$$K_m(G) = G, \text{ for any natural number m.}$$

Proof. It is obvious by Lemma 2.1 and Lemma 2.2. ■

Recall for any natural number n, if $G = \mathbb{Z}_{2^n}$, then $Aut(G)$ consists of all automorphisms $\alpha_i : g \mapsto ig$, where $1 \leqslant i < 2^n$ and i is an odd number. We know that the automorphism groups of finitely generated abelian groups are well-understood(see [4]). Now we have

Lemma 2.4 Let $G = \oplus_{i=1}^{k} \mathbb{Z}_{2^{n_i}}$ such that $n_1 > n_2 > \cdots > n_k$. Also let $\epsilon_i = (0, \dots 0, \underbrace{1}_{i}, 0, \dots, 0)$, for $i = 1, \dots, k$. Then for all $\alpha \in Aut(G)$ we have

$$\begin{aligned}
&\alpha(\epsilon_1) = (m_{11}, m_{12}, \dots, m_{1k})\\
&\alpha(\epsilon_2) = (2^{n_1-n_2} m_{21}, m_{22}, \dots, m_{2k})\\
&\alpha(\epsilon_3) = (2^{n_1-n_3} m_{31}, 2^{n_2-n_3} m_{32}, m_{33}, \dots, m_{3k})\\
&\quad\vdots\\
&\alpha(\epsilon_k) = (2^{n_1-n_k} m_{k1}, 2^{n_2-n_k} m_{k2}, \dots, 2^{n_{k-1}-n_k} m_{k(k-1)}, m_{kk})
\end{aligned}$$

where $m_{ij} \in \mathbb{Z}$ for all i, j and for all i, m_{ii} is odd.

Proof. We know that for $i = 1, \dots, k$, we have $|\epsilon_i| = 2^{n_i}$ Hence,
$\alpha(\epsilon_1) = (m_{11}, m_{12}, \dots, m_{1k})$
$\alpha(\epsilon_2) = (2^{n_1-n_2} m_{21}, m_{22}, m_{23}, \dots, m_{2k})$
$\alpha(\epsilon_3) = (2^{n_1-n_3} m_{31}, 2^{n_2-n_3} m_{32}, m_{33}, m_{34}, \dots, m_{3k})$
$\vdots$
$\alpha(\epsilon_k) = (2^{n_1-n_k} m_{k1}, 2^{n_2-n_k} m_{k2}, \dots, 2^{n_{k-1}-n_k} m_{k(k-1)}, m_{kk})$
for some $m_{ij} \in \{0, 1, 2, 3, \dots, 2^{n_j} - 1\}$. Now it is sufficient to prove m_{tt} is an odd number, for any natural number t. We use of induction on t. If $t = 1$, then clearly m_{11} is an odd number. Let $t > 1$ and assume $m_{t't'}$ is an odd number, for any natural number t' such that $t' < t$. Then we prove m_{tt} is an odd number. Assume that m_{tt} is an even number. Then for any natural number r such that $1 \leqslant r \leqslant k$ if $t < r < k$, then put $c_r = 0$ and if $r = t$, then put $c_r = 2^{n_t - 1}$. For $1 \leqslant r \leqslant t-1$ set $I_r = \{i \in \mathbb{Z} \mid 0 \leqslant i < r, c_{t-i} \neq 0\}$ and put

$$c_{t-r} = \begin{cases} 0 & \text{if } \sum_{s \in I_r} m_{(t-s)(t-r)} \text{ is an even number,} \\ 2^{n_{t-r}-1} & \text{if } \sum_{s \in I_r} m_{(t-s)(t-r)} \text{ is an odd number.} \end{cases}$$

Now $(c_1, \dots, c_k) \neq 0$, but $\alpha((c_1, \dots, c_k)) = 0$, which is a contradiction. Hence, m_{tt} is an odd number and this completes the proof. ■

Lemma 2.5 ([1]) For all natural numbers $k, n_1, n_2, \dots, n_k$ such that $n_1 > n_2 \geqslant \cdots \geqslant n_k$,

$$K_1(\mathbb{Z}_{2^{n_1}} \oplus \mathbb{Z}_{2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}) = 2\mathbb{Z}_{2^{n_1}} \oplus \mathbb{Z}_{2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}.$$

Lemma 2.6 ([7]) Suppose that $G = \oplus_{i=1}^{k} \mathbb{Z}_{2^{n_i}}$ with $k > 1$ and $n_1 = n_2 \geqslant n_3 \geqslant \cdots \geqslant n_k$. Then,

$$K_m(G) = G, \text{ for any natural number } m.$$

In [6], we obtained the $K_m(\oplus_{i=1}^k \mathbb{Z}_{2^{n_i}})$ with $n_1 > n_2 > ... > n_k$ using a function which recursively defined in terms of the $n_1, ..., n_k$'s.

Definition 2.7 We define for $i = 1, 2, ..., k$, $T_{0,i} = 0$ and for $m = 1, 2, ...$, that $T_{m,0} = \infty$. Now for $m \geqslant 0$ and $i = 1, 2, ..., k$, we have

$$T_{m+1,i} = \min\{T_{m,i-1}, T_{m,i} + 1, n_i - n_{i+1} + T_{m,i+1}, ..., n_i - n_k + T_{m,k}\}.$$

Theorem 2.8 ([6]) Suppose that $G = \oplus_{i=1}^k \mathbb{Z}_{2^{n_i}}$ with $n_1 > n_2 > ... > n_k$. Then,

$$K_m(G) = \oplus_{i=1}^k 2^{T_{m,i}} \mathbb{Z}_{2^{n_i}},$$

for any natural number m.

The floor function of x , also called the greatest integer function, gives the largest integer less than or equal to x. In this paper we use of the symbol $\lfloor x \rfloor$.

Definition 2.9 Let $k, m, n_1, n_2, \ldots, n_k$, be natural numbers such that $n_1 > n_2 > \cdots > n_k$. Then put $T_{m1}^1 = m$ and for $k > 1$ put

$$T_{mj}^k = \begin{cases} \min\{m - j + 1, A_{mj}^k, B_{mj}^k\} & 1 \leqslant j \leqslant min\{m, k-1\} \\ \min\{m - j + 1, A_{mj}^k\} & j = k \leqslant m \\ 0 & m < j \leqslant k \end{cases}$$

where for $1 \leqslant j \leqslant \min\{m, k\}$

$$A_{mj}^k = \min\{(\lfloor \tfrac{m-j+1}{2} \rfloor)(n_i - n_{i+1}) \mid 1 \leqslant i \leqslant \min\{j, k-1\}\} + \tfrac{(-1)^{m+j}+1}{2},$$

and for $1 \leqslant j \leqslant \min\{m, k-1\}$

$$B_{mj}^k = \min\{n_j - n_{j+i} + (\lfloor \tfrac{m-j+1}{2} \rfloor - i)(n_{j+i} - n_{j+i+1}) \mid 0 \leqslant i \leqslant \min\{\lfloor \tfrac{m-j+1}{2} \rfloor, k-j-1\}\} + \tfrac{(-1)^{m+j}+1}{2}.$$

Also for $k > 1$ and $1 \leqslant j \leqslant \min\{m + 1, k - 1\}$, put

$$M_{mj} = (\lfloor \tfrac{m-j+2}{2} \rfloor)(n_j - n_{j+1}) + \tfrac{(-1)^{m+j+1}+1}{2}.$$

In order to prove main result, we need to prove some technical lemmas.

Lemma 2.10 i) If $k > 1$ and $1 \leqslant j \leqslant \min\{m + 1, k\}$, then

$$A_{(m+1)j}^k = \begin{cases} \min\{A_{m(j-1)}^k, M_{mj}\} & 2 \leqslant j \leqslant k-1, \\ A_{m(j-1)}^k & j = k, \\ M_{mj} & j = 1. \end{cases}$$

ii) If $k > 1$ and $1 \leqslant j \leqslant \min\{m + 1, k - 1\}$, then

$$B_{(m+1)j}^k = \begin{cases} \min\{n_j - n_{j+1} + B_{m(j+1)}^k, M_{mj}\} & 1 \leqslant j \leqslant \min\{m-1, k-2\}, \\ M_{mj} & Otherwise. \end{cases}$$

Proof. It is easy to check that the result is true. ■

Lemma 2.11 If $k > 1$ and $1 \leqslant j \leqslant k - 1$, then

$$T_{mj}^k \geqslant T_{m(j+1)}^k.$$

Proof. If $j \geqslant m$, then the result is obvious. If $j < m$ and $j = k-1$, then it is easy to check $T_{m(k-1)}^k \geqslant T_{mk}^k$. Now let $j < m$ and $j < k - 1$. Then $T_{mj}^k = \min\{m - j + 1, A_{mj}^k, B_{mj}^k\}$. If

$T^k_{mj} = m-j+1$, then result is obvious. If $T^k_{mj} = A^k_{mj} = (\lfloor \frac{m-j+1}{2} \rfloor)(n_{i_0} - n_{i_0+1}) + \frac{(-1)^{m+j}+1}{2}$, for some $1 \leqslant i_0 \leqslant j$ then $T^k_{mj} \geqslant (\lfloor \frac{m-j}{2} \rfloor)(n_{i_0} - n_{i_0+1}) + \frac{(-1)^{m+j+1}+1}{2} \geqslant A^k_{m(j+1)} \geqslant T^k_{m(j+1)}$. If $T^k_{mj} = B^k_{mj} = n_j - n_{j+i_0} + (\lfloor \frac{m-j+1}{2} \rfloor - i_0)(n_{j+i_0} - n_{j+i_0+1}) + \frac{(-1)^{m+j}+1}{2}$, for some $0 \leqslant i_0 \leqslant \min\{\lfloor \frac{m-j+1}{2} \rfloor, k-j-1\}$, then we have,
if $i_0 = 0$ then

$$\begin{aligned} T^k_{mj} &= (\lfloor \tfrac{m-j+1}{2} \rfloor)(n_j - n_{j+1}) + \tfrac{(-1)^{m+j}+1}{2} \\ &\geqslant (\lfloor \tfrac{m-j}{2} \rfloor)(n_j - n_{j+1}) + \tfrac{(-1)^{m+j+1}+1}{2} \\ &\geqslant A^k_{m(j+1)} \\ &\geqslant T^k_{m(j+1)}. \end{aligned}$$

Now if $i_0 \geqslant 1$, then we have one of the following cases.

(case1) $n_{j+i_0} - n_{j+i_0+1} = 1$. Hence,

$$\begin{aligned} T^k_{mj} &\geqslant 1 + n_{j+1} - n_{j+i_0} + \lfloor \tfrac{m-j+1}{2} \rfloor - i_0 + \tfrac{(-1)^{m+j}+1}{2} \\ &\geqslant 1 + n_{j+1} - n_{j+i_0} + \lfloor \tfrac{m-j}{2} \rfloor - i_0 + \tfrac{(-1)^{m+j+1}+1}{2} \\ &= n_{j+1} - n_{j+i_0} + (\lfloor \tfrac{m-j}{2} \rfloor - (i_0 - 1))(n_{j+i_0} - n_{j+i_0+1}) + \tfrac{(-1)^{m+j+1}+1}{2} \\ &\geqslant B^k_{m(j+1)} \\ &\geqslant T^k_{m(j+1)}. \end{aligned}$$

(case 2) $n_j - n_{j+i_0} < 2i_0$. Hence, $n_{j+s} - n_{j+s+1} = 1$, for some $0 \leqslant s \leqslant i_0 - 1$. Now we have

(case 2.1) If $s = 0$, then $n_j - n_{j+1} = 1$. Hence,

$$\begin{aligned} T^k_{mj} &\geqslant i_0 + (\lfloor \tfrac{m-j+1}{2} \rfloor - i_0) + \tfrac{(-1)^{m+j}+1}{2} \\ &\geqslant \lfloor \tfrac{m-j}{2} \rfloor + \tfrac{(-1)^{m+j+1}+1}{2} \\ &= \lfloor \tfrac{m-j}{2} \rfloor (n_j - n_{j+1}) + \tfrac{(-1)^{m+j+1}+1}{2} \\ &\geqslant A^k_{m(j+1)} \\ &\geqslant T^k_{m(j+1)}. \end{aligned}$$

(case 2.2) If $s \geqslant 1$, then

$$\begin{aligned} T^k_{mj} &= n_j - n_{j+1} + n_{j+1} - n_{j+s} + n_{j+s} - n_{j+i_0} + (\lfloor \tfrac{m-j+1}{2} \rfloor - i_0)(n_{j+i_0} - n_{j+i_0+1}) \\ &\quad + \tfrac{(-1)^{m+j}+1}{2} \geqslant 1 + n_{j+1} - n_{j+s} + i_0 - s + \lfloor \tfrac{m-j+1}{2} \rfloor - i_0 + \tfrac{(-1)^{m+j}+1}{2} \\ &\geqslant n_{j+1} - n_{j+s} + \lfloor \tfrac{m-j}{2} \rfloor - s + 1 + \tfrac{(-1)^{m+j+1}+1}{2} \\ &= n_{j+1} - n_{j+s} + (\lfloor \tfrac{m-j}{2} \rfloor - (s-1))(n_{j+s} - n_{j+s+1}) + \tfrac{(-1)^{m+j+1}+1}{2} \\ &\geqslant B^k_{m(j+1)} \\ &\geqslant T^k_{m(j+1)}. \end{aligned}$$

(case 3) $n_j - n_{j+i_0} \geqslant 2i_0$ and $n_{j+i_0} - n_{j+i_0+1} \geqslant 2$. Hence,

$$T^k_{mj} \geqslant 2i_0 + (\lfloor \tfrac{m-j+1}{2} \rfloor - i_0) \times 2 + \tfrac{(-1)^{m+j}+1}{2} = m - j + 1 > T^k_{m(j+1)}.$$

These imply the result. ■

The following lemmas are proved in a similar way with the previous lemma and are left to reader.

Lemma 2.12 i) If $k > 2$ and $1 \leqslant j \leqslant k-2$, then

$$T^k_{m(j+2)} + n_{j+1} - n_{j+2} \geqslant T^k_{m(j+1)}.$$

ii) If $k > 1$ and $1 \leqslant j \leqslant \min\{m, k-1\}$, then

$$n_j - n_{j+1} + m - j \geqslant \min\{m - j + 2, M_{mj}\}.$$

iii) If $k > 1$ and $1 \leqslant j \leqslant \min\{m, k-1\}$, then

$$M_{mj} \geqslant n_j - n_{j+1} + T^k_{m(j+1)}.$$

Lemma 2.13 For $k > 1$ and $2 \leqslant j \leqslant \min\{m+1, k\}$,
i) if $2 \leqslant j \leqslant \min\{m-1, k-2\}$, then

$$B^k_{m(j-1)} \geqslant \min\{m-j+2, A^k_{m(j-1)}, M_{mj}, n_j - n_{j+1} + B^k_{m(j+1)}\}.$$

ii) If $j = k-1$ or $m \leqslant j \leqslant k-2$, then

$$B^k_{m(j-1)} \geqslant \min\{m-j+2, A^k_{m(j-1)}, M_{mj}\}.$$

iii) If $j = k$, then

$$B^k_{m(j-1)} \geqslant A^k_{m(j-1)}.$$

Lemma 2.14 For $k > 1$, $m > 1$ and $1 \leqslant j \leqslant \min\{m-1, k\}$,
i) if $2 \leqslant j \leqslant k-2$, then

$$T^k_{mj} + 1 \geqslant \min\{m-j+2, A^k_{m(j-1)}, M_{mj}, n_j - n_{j+1} + B^k_{m(j+1)}\}.$$

ii) If $j = k-1$ such that $k > 2$, then

$$T^k_{mj} + 1 \geqslant \min\{m-j+2, A^k_{m(j-1)}, M_{mj}\}.$$

iii) If $j = k$, then

$$T^k_{mj} + 1 \geqslant \min\{m-j+2, A^k_{m(j-1)}\}.$$

iv) If $j = 1$ such that $k = 2$, then

$$T^k_{mj} + 1 \geqslant \min\{m-j+2, M_{mj}\}.$$

v) If $j = 1$ such that $k > 2$, then

$$T^k_{mj} + 1 \geqslant \min\{m-j+2, M_{mj}, n_j - n_{j+1} + B^k_{m(j+1)}\}.$$

Lemma 2.15 For $k > 1$, $m > 1$ and $1 \leqslant j \leqslant \min\{m-1, k-1\}$,
i) if $2 \leqslant j \leqslant k-2$, then

$$n_j - n_{j+1} + A^k_{m(j+1)} \geqslant \min\{m-j+2, A^k_{m(j-1)}, M_{mj}, n_j - n_{j+1} + B^k_{m(j+1)}\}.$$

ii) If $j = k-1$ such that $k > 2$ then

$$n_j - n_{j+1} + A^k_{m(j+1)} \geqslant \min\{m-j+2, A^k_{m(j-1)}, M_{mj}\}.$$

iii) If $j = 1$ and $k = 2$, then

$$n_j - n_{j+1} + A^k_{m(j+1)} \geqslant \min\{m-j+2, M_{mj}\}.$$

iv) If $j = 1$ and $k > 2$, then

$$n_j - n_{j+1} + A^k_{m(j+1)} \geqslant \min\{m-j+2, M_{mj}, n_j - n_{j+1} + B^k_{m(j+1)}\}.$$

Finally we will prove the following corollary, that is useful in the proof of the main result.

Corollary 2.16 If $k > 1$ and $1 \leqslant j \leqslant k$, then

$$T^k_{(m+1)j} = \begin{cases} \min\{T^k_{m(j-1)}, T^k_{mj}+1, n_j - n_{j+1} + T^k_{m(j+1)}\} & 2 \leqslant j \leqslant k-1, \\ \min\{T^k_{mj}+1, n_j - n_{j+1} + T^k_{m(j+1)}\} & j = 1, \\ \min\{T^k_{m(j-1)}, T^k_{mj}+1\} & j = k. \end{cases}$$

Proof. We prove the result for case $2 \leqslant j \leqslant \min\{m-1, k-2\}$.

$$
\begin{aligned}
T^k_{(m+1)j} &= \min\{m-j+2, A^k_{(m+1)j}, B^k_{(m+1)j}\} \text{ (by Definition 2.9)}\\
&= \min\{m-j+2, A^k_{m(j-1)}, n_j-n_{j+1}+B^k_{m(j+1)}, M_{mj}\} \text{ (by Lemma 2.10)}\\
&= \min\{m-j+2, A^k_{m(j-1)}, n_j-n_{j+1}+B^k_{m(j+1)}, n_j-n_{j+1}+m-j, M_{mj}\}\\
&\quad \text{(by Lemma 2.12(ii))}\\
&= \min\{m-j+2, A^k_{m(j-1)}, B^k_{m(j-1)}, n_j-n_{j+1}+B^k_{m(j+1)}, n_j-n_{j+1}+m-j\\
&\quad , M_{mj}\} \text{ (by Lemma 2.13(i))}\\
&= \min\{m-j+2, A^k_{m(j-1)}, B^k_{m(j-1)}, T^k_{mj}+1, n_j-n_{j+1}+B^k_{m(j+1)}\\
&\quad , n_j-n_{j+1}+m-j, M_{mj}\} \text{ (by Lemma 2.14(i))}\\
&= \min\{m-j+2, A^k_{m(j-1)}, B^k_{m(j-1)}, T^k_{mj}+1, n_j-n_{j+1}+A^k_{m(j+1)}\\
&\quad , n_j-n_{j+1}+B^k_{m(j+1)}, n_j-n_{j+1}+m-j, M_{mj}\}\text{(by Lemma 2.15(i))}\\
&= \min\{T^k_{m(j-1)}, T^k_{mj}+1, n_j-n_{j+1}+T^k_{m(j+1)}, M_{mj}\}\text{(by Definition 2.9)}\\
&= \min\{T^k_{m(j-1)}, T^k_{mj}+1, n_j-n_{j+1}+T^k_{m(j+1)}\}\text{(by Lemma 2.12(iii))}.
\end{aligned}
$$

The other cases are proved in a similar way. ■

3. Main Results

Let G be a finite abelian group and m be a natural number. Then $G = H \oplus T$, where T is a finite abelian group of odd order and H is trivial group or a finite abelian 2-group. Hence, $K_m(G) = K_m(H) \oplus T$, by Lemma 2.1(ii) and Corollary 2.3. Therefore without loss of generality, we restrict attention to the case where G is a finite abelian 2-group.

Theorem 3.1 Let $k, n_1, \ldots, n_k$ are natural numbers and $G = \oplus_{j=1}^k \mathbb{Z}_{2^{n_j}}$. Then using the notations in the Definition 2.9,
i) if $n_1 > n_2 > \cdots > n_k$, then

$$K_m(G) = \oplus_{i=1}^k 2^{T^k_{mj}} \mathbb{Z}_{2^{n_j}}.$$

ii) If $k > 1$ and $n_1 > n_2 > \cdots > n_{t-1} > n_t = n_{t+1} \geqslant \cdots \geqslant n_k$, for some natural number $1 \leqslant t < k$, then

$$K_m(G) = \begin{cases} G & t = 1,\\ (\oplus_{j=1}^{t-1} 2^{T'_{mj}} \mathbb{Z}_{2^{n_j}}) \oplus \mathbb{Z}_{2^t} \oplus \cdots \oplus \mathbb{Z}_{2^k} & t > 1,\end{cases}$$

where $T'_{mj} = \min\{T^{t-1}_{mj}, n_j - n_t\}$.

Proof. i) If $k = 1$, then the result is true by Lemma 2.2. Now for $k > 1$ we use of induction on m. If $m = 1$, then the result is true by Lemma 2.5. Let $m > 1$ and assume the result to be true for m, the induction hypothesis. Let $(a_1, \ldots, a_k) \in G$ and $\alpha_1, \ldots, \alpha_{m+1} \in Aut(G)$. Then due to the induction hypothesis $[(a_1, \ldots, a_k), \alpha_1, \ldots, \alpha_{m+1}] = [(b_1, \ldots, b_k), \alpha_{m+1}]$ such that $b_j = 2^{T^k_{mj}} b'_j$ for some $b'_j \in \mathbb{Z}_{2^{n_j}}$. Hence, by Lemma 2.4 , $[(b_1, \ldots, b_k), \alpha_{m+1}] = (c_1, \ldots, c_k)$, where $c_j = m_{1j}b_1 + m_{2j}b_2 + \cdots + m_{(j-1)j}b_{j-1} + (-1 + m_{jj})b_j + 2^{n_j - n_{j+1}} m_{(j+1)j} b_{j+1} + 2^{n_j - n_{j+2}} m_{(j+2)j} b_{j+2} + \cdots + 2^{n_j - n_k} m_{kj} b_k$. Now if $2 \leqslant j \leqslant k-1$, then by Lemma 2.11 and Lemma 2.12(i), we have $c_j = 2^{min\{T^k_{m(j-1)}, T^k_{mj}+1, n_j - n_{j+1} + T^k_{m(j+1)}\}} c'_j$ for some $c'_j \in \mathbb{Z}_{2^{n_j}}$ and hence, $c_j = 2^{T^k_{(m+1)j}} c'_j$ by Corollary 2.16. Similarly, if $j = 1$ or $j = k$, then $c_j = 2^{T^k_{(m+1)j}} c'_j$, for some $c'_j \in \mathbb{Z}_{2^{n_j}}$. Hence,

$$K_{m+1}(G) \subseteq \oplus_{j=1}^k 2^{T^k_{(m+1)j}} \mathbb{Z}_{2^{n_j}}.$$

Now, for the reverse conclusion if $T^k_{(m+1)j} \geqslant n_j$ then $2^{T^k_{(m+1)j}}\mathbb{Z}_{2^{n_j}} = \langle 0\rangle$. Hence, for $1 \leqslant j \leqslant k$ let $T^k_{(m+1)j} < n_j$. Now if $1 < j < k$ then by Corollary 2.16 we have one of the following cases.

(Case 1) If $T^k_{(m+1)j} = T^k_{m(j-1)}$, then we define the automorphism α of G, given by $\alpha(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_k) = (a_1, \ldots, a_{j-1}, a_{j-1} + a_j, a_{j+1} \ldots, a_k)$. Hence, $(0, \ldots, 0, \underbrace{2^{T^k_{(m+1)j}}}_{j}, 0, \ldots, 0) = [(0, \ldots, 0, \underbrace{2^{T^k_{m(j-1)}}}_{j-1}, 0, \ldots, 0), \alpha]$. Now by induction hypothesis $[(0, \ldots, 0, \underbrace{2^{T^k_{m(j-1)}}}_{j-1}, 0, \ldots, 0), \alpha] \in [K_m(G), A]$. So

$(0, \ldots, 0, \underbrace{2^{T^k_{(m+1)j}}}_{j}, 0, \ldots, 0) \in K_{m+1}(G)$.

(Case 2) If $T^k_{(m+1)j} = T^k_{mj} + 1$, then we define the automorphism α of G, given by $\alpha(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_k) = (a_1, \ldots, a_{j-1}, 3a_j, a_{j+1}, \ldots, a_k)$. Hence, $(0, \ldots, 0, \underbrace{2^{T^k_{(m+1)j}}}_{j}, 0, \ldots, 0) = [(0, \ldots, 0, \underbrace{2^{T^k_{mj}}}_{j}, 0, \ldots, 0), \alpha]$. Now by induction hypothesis $[(0, \ldots, 0, \underbrace{2^{T^k_{mj}}}_{j}, 0, \ldots, 0), \alpha] \in [K_m(G), A]$. So

$(0, \ldots, 0, \underbrace{2^{T^k_{(m+1)j}}}_{j}, 0, \ldots, 0) \in K_{m+1}(G)$.

(Case 3) If $T^k_{(m+1)j} = n_j - n_{j+1} + T^k_{m(j+1)}$, then we define the automorphism α of G,given by $\alpha(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_k) = (a_1, \ldots, a_{j-1}, a_j + 2^{n_j - n_{j+1}}a_{j+1}, a_{j+1}, \ldots, a_k)$. Hence, $(0, \ldots, 0, \underbrace{2^{T^k_{(m+1)j}}}_{j}, 0, \ldots, 0) = [(0, \ldots, 0, \underbrace{2^{T^k_{m(j+1)}}}_{j+1}, 0, \ldots, 0), \alpha]$.

Now by induction hypothesis $[(0, \ldots, 0, \underbrace{2^{T^k_{m(j+1)}}}_{j+1}, 0, \ldots, 0), \alpha] \in [K_m(G), A]$.

So $(0, \ldots, 0, \underbrace{2^{T^k_{(m+1)j}}}_{j}, 0, \ldots, 0) \in K_{m+1}(G)$.

Similarly if $j = 1$ or $j = k$, then $(2^{T^k_{(m+1)1}}, 0, \ldots, 0) \in K_{m+1}(G)$ and $(0, \ldots, 0, 2^{T^k_{(m+1)k}}) \in K_{m+1}(G)$. Hence,

$$\oplus_{j=1}^{k} 2^{T^k_{(m+1)j}}\mathbb{Z}_{2^{n_j}} \subseteq K_{m+1}(G).$$

So the result is true by induction.

ii) If $t = 1$, then the result is true, by Lemma 2.6. If $t > 1$, then we prove the result by induction on m. If $m = 1$, then by Lemma 2.5 the result is true. Let $m > 1$ and assume the result to be true for m, the induction hypothesis. Also let $(a_1, \ldots, a_k) \in G$ and $\alpha_1, \ldots, \alpha_{m+1} \in Aut(G)$. Then by induction hypothesis $[(a_1, \ldots, a_k), \alpha_1, \ldots, \alpha_{m+1}] = [(2^{T'_{m1}}b_1, \ldots, 2^{T'_{m(t-1)}}b_{t-1}, b_t, \ldots, b_k), \alpha_{m+1}]$ for some $b_j \in \mathbb{Z}_{2^{n_j}}$. Now by Lemma 2.4 and a similar way to the part (i) we conclude $[(2^{T'_{m1}}b_1, \ldots, 2^{T'_{m(t-1)}}b_{t-1}, b_t, \ldots, b_k), \alpha_{m+1}] = (2^{T'_{(m+1)1}}c_1, \ldots, 2^{T'_{(m+1)(t-1)}}c_{t-1}, c_t, \ldots, c_k)$ for some $c_j \in \mathbb{Z}_{2^{n_j}}$. Hence,

$$K_{m+1}(G) \subseteq (\oplus_{j=1}^{t-1} 2^{T'_{(m+1)j}}\mathbb{Z}_{2^{n_j}}) \oplus \mathbb{Z}_{2^{n_t}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}.$$

Now, for the reverse conclusion by Lemma 2.1(i) and Lemma 2.6 we have, $K_{m+1}(\mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_{t-1}}}) \oplus \mathbb{Z}_{2^{n_t}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}} \subseteq K_{m+1}(G)$. Hence, by the part (i) we have $2^{T^{t-1}_{(m+1)1}}\mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus 2^{T^{t-1}_{(m+1)(t-1)}}\mathbb{Z}_{2^{n_{t-1}}} \oplus \mathbb{Z}_{2^{n_t}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}} \subseteq K_{m+1}(G)$. Therefore if $1 \leqslant j < t$

such that $T'_{(m+1)j} = T^{t-1}_{(m+1)j}$, then
$(0,\ldots,0,\underbrace{2^{T'_{(m+1)j}}}_{j},0,\ldots,0) \in K_{m+1}(G)$. Now if $1 \leqslant j < t$ such that $T'_{(m+1)j} = n_j - n_t$, then we define the automorphism α of the group G, given by
$\alpha(a_1,\ldots,a_k) = (a_1,\ldots,a_{j-1},a_j + 2^{n_j-n_t}a_t, a_{j+1},\ldots,a_k)$. Hence,
$(0,\ldots,0,\underbrace{2^{n_j-n_t}}_{j},0,\ldots,0) = [(0,\ldots,0,\underbrace{1}_{t},0,\ldots,0),\alpha]$. Now by induction hypothesis
$[(0,\ldots,0,\underbrace{1}_{t},0,\ldots,0),\alpha] \in [K_m(G),A]$. So
$(0,\ldots,0,\underbrace{2^{n_j-n_t}}_{j},0,\ldots,0) \in K_{m+1}(G)$. These imply that

$$(\oplus_{j=1}^{t-1} 2^{T'_{(m+1)j}}\mathbb{Z}_{2^{n_j}}) \oplus \mathbb{Z}_{2^{n_t}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}} \subseteq K_{m+1}(G)$$

and lead to the result. ■

Corollary 3.2 Let $k, m, n_1, n_2, \ldots, n_k$, be natural numbers such that $k > 1$, $n_1 > n_2 > \cdots > n_k$ and $n_1 - n_2 = 1$. Then

$$K_m(\mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}) = \begin{cases} \oplus_{j=1}^{k} 2^{\lfloor \frac{m-j+2}{2} \rfloor}\mathbb{Z}_{2^{n_j}} & m \geqslant k, \\ (\oplus_{j=1}^{m} 2^{\lfloor \frac{m-j+2}{2} \rfloor}\mathbb{Z}_{2^{n_j}}) \oplus \mathbb{Z}_{2^{m+1}} \oplus \cdots \oplus \mathbb{Z}_{2^k} & m < k. \end{cases}$$

Proof. It is evident $T^k_{mj} = \lfloor \frac{m-j+1}{2} \rfloor + \frac{(-1)^{m+j}+1}{2} = \lfloor \frac{m-j+2}{2} \rfloor$, for $1 \leqslant j \leqslant \min\{m,k\}$. ■

Corollary 3.3 Let $k, m, n_1, n_2, \ldots, n_k$, be natural numbers such that $k > 1$ and $n_s - n_{s+1} \geqslant 2$, for any natural number $1 \leqslant s < k$. Then

$$K_m(\mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}) = \begin{cases} \oplus_{j=1}^{k} 2^{m-j+1}\mathbb{Z}_{2^{n_j}} & m \geqslant k, \\ (\oplus_{j=1}^{m} 2^{m-j+1}\mathbb{Z}_{2^{n_j}}) \oplus \mathbb{Z}_{2^{m+1}} \oplus \cdots \oplus \mathbb{Z}_{2^k} & m < k. \end{cases}$$

Proof. It is easy to check $T^k_{mj} = m - j + 1$, for $1 \leqslant j \leqslant \min\{m,k\}$. ■

Corollary 3.4 Let $G = \oplus_{i=1}^{k}\mathbb{Z}_{2^{n_i}}$ such that
i) $n_1 > n_2 > \cdots > n_k$. Then

$$K_{2n_1-1}(G) = \langle 0 \rangle.$$

ii) $n_1 > n_2 > \cdots > n_{t-1} > n_t = n_{t+1} \geqslant \cdots \geqslant n_k$, for some natural number $1 \leqslant t < k$. Then

$$K_m(G) \neq \langle 0 \rangle, \text{ for any natural number } m.$$

Proof. *i)* Let $m = 2n_1 - 1$. Then $m \geqslant k$. Hence $m - j + 1 \geqslant n_j$ and $A^k_{mj} \geqslant \lfloor \frac{m-j+1}{2} \rfloor + \frac{(-1)^{m+j}+1}{2} = \lfloor \frac{m-j+2}{2} \rfloor \geqslant n_j$, for $1 \leqslant j \leqslant k$. Also $B^k_{mj} \geqslant i + \lfloor \frac{m-j+1}{2} \rfloor - i + \frac{(-1)^{m+j}+1}{2} = \lfloor \frac{m-j+2}{2} \rfloor \geqslant n_j$, for $1 \leqslant j \leqslant k-1$. Therefore $T^k_{mj} \geqslant n_j$, for $1 \leqslant j \leqslant k$ and hence, $2^{T^k_{mj}}\mathbb{Z}_{2^{n_j}} = \langle 0 \rangle$.
ii) It is obvious by Theorem 3.1(ii). ■

At last we conclude the autocommutator subgroups of all finite abelian groups of orders 32 and 24.

***Example* 3.5** Let G be a finite abelian group of order 32. Then

G	$K_1(G)$	$K_2(G)$	$K_3(G)$	$K_4(G)$	$K_n(G)(n \geqslant 5)$
$\mathbb{Z}_{32}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_8$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\langle 0 \rangle$
$\mathbb{Z}_{16} \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\langle 0 \rangle$	$\langle 0 \rangle$
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\langle 0 \rangle$
$\mathbb{Z}_8 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$
$\mathbb{Z}_4 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$
$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$

***Example* 3.6** Let G be a finite abelian group of order 24. then

G	$K_1(G)$	$K_2(G)$	$K_n(G)(n \geqslant 3)$
$\mathbb{Z}_8 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_3$
$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_3$
$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3$

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