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Random fixed point theorems with an application to a random nonlinear integral equation

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Abstract. In this paper, stochastic generalizations of some fixed point for operators satisfying random contractively generalized hybrid and some other contractive condition have been proved. We discuss also the existence of a solution to a nonlinear random integral equation.

Keywords: Random fixed point, nonlinear random integral equation, contractively generalized hybrid.

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1. Introduction

Fixed point theory has the diverse applications in different branches of mathematics, statistics, engineering, and economics in dealing with the problems arising in approximation theory, potential theory, game theory, theory of differential equations, theory of integral equations, and others. Developments in the investigation on fixed points of non-expansive mappings, contractive mappings in different spaces like metric spaces, Banach spaces, Fuzzy metric spaces and cone metric spaces have almost been saturated. The study of random fixed point theorems was initiated by the Prague school of probabilistic in 1950's. The introduction of randomness leads to several new questions of measurability of solutions, probabilistic and statistical aspects of random solutions. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Hanš [9] and Špaček [26]. The survey article

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by Bharucha-Reid [7] in 1976 attracted the attention of several mathematicians and gave wings to this theory. The results of Špaček and Hanš in multi-valued contractive mappings was extended by Itoh [11]. By the same author random fixed point theorems with an application to random differential equations in Banach spaces are obtained. Mukherjee [16] gave a random version of Schauder's fixed point theorem on an atomic probability measure space. While Bharucha-Reid [6, 7] generalized Mukherjee's result on a general probability measure space. On the other hand, some authors [4, 17, 18, 23, 24] applied a random fixed point theorem to prove the existence of a solution in a Banach space of a random nonlinear integral equation. Sehgal and Waters [25] had obtained several random fixed point theorems including a random analogue of the classical results due to Rothe [20]. In some recent papers of Saha et al. [21, 22], some random fixed point theorems over separable Banach spaces and separable Hilbert spaces have been established.

On the other hand, the first fundamental fixed point theorem in deterministic form was due to S. Banach [3] in a metric space setting, this theorem runs as follows:

Theorem 1.1 (**Banach contraction principle**) Let (X, d) be a complete metric space, $c \in [0, 1)$ and let $T : X \to X$ be a mapping such that for each $x, y \in X$,

$$d(Tx, Ty) \leqslant cd(x, y).$$

Then T has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.

After this classical result, Kannan [13] gave a substantially new contractive mapping where the mapping T need not be continuous on X (but continuous at their fixed points, see [19]). He considered the contractive condition as follows: there exists a constant $b \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leqslant b[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$. A mapping $T : X \to X$ is said to be contractively nonspreading [8, 27] if there exists $\beta \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leqslant \beta[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$. A mapping $T : X \to X$ is called contractively hybrid [10] if there exists $\gamma \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leqslant \gamma[d(x, Ty) + d(y, Tx) + d(y, x)],$$

motivated by generalized hybrid mappings [14] in a Hilbert space, Takahashi et al. [10] introduced the concept of contractively generalized hybrid mappings on metric spaces and studied fixed point theorems for such mappings on complete metric spaces. Let (X, d) be a metric space, a mapping $T : X \to X$ is called contractively generalized hybrid [10] if there exist $\alpha, \beta \in \mathbb{R}$ and $\gamma \in [0, 1)$ such that

$$\alpha d(Tx, Ty) + (1 - \alpha)d(x, Ty) \leqslant \gamma\{\beta d(y, Tx) + (1 - \beta)d(y, x)\}, \tag{1.1}$$

for all $x, y \in X$, such a mapping T is also called contractively (α, β, γ)-generalized hybrid. For example, a contractively (α, β, γ)-generalized hybrid mapping is $r-$contractive for

$\alpha = 1$ and $\beta = 0$. It is contractively nonspreading for $\alpha = 1 + r$ and $\beta = 1$, see Takahashi et al. [10].

2. Preliminaries

In order to prove our main results, we need to recall the following concepts and results.

Let$(X, \sum)$ be a separable Banach space where $\sum$ is a σ-algebra of Borel subsets of X and let $(\Omega, \sum, \mu)$ denote a complete probability measure space with measure μ and $\sum$ be a σ-algebra of subsets of Ω. For more details one can see Joshi and Bose [12].

Definition 2.1 A mapping $x : \Omega \to X$ is said to be an $X-$valued random variable, if the inverse image under the mapping x of every Borel set B of X belongs to $\sum$, that is, $x^{-1}(B) \in \sum$ for all $B \in \sum$.

Definition 2.2 A mapping $x : \Omega \to X$ is said to be a finitely valued random variable, if it is constant on each of a finite number of disjoint sets $A_i \in \sum$ and is equal to 0 on $\Omega - \left(\bigcup\limits_{i=1}^{n} A_i\right)$. X is called a simple random variable if it is finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.

Definition 2.3 A mapping $x : \Omega \to X$ is said to be a strong random variable, if there exists a sequence $\{x_n(\omega)\}$ of simple random variables which converges to $x(\omega)$ almost surely, i.e., there exists a set $A_0 \in \sum$ with $(A_0) = 0$ such that $\lim\limits_{n\to\infty} x_n(\omega) = x(\omega)$, $\omega \in \Omega - A_0$.

Definition 2.4 A mapping $x : \Omega \to X$ is said to be weak random variable, if the function $x^*(x(\omega))$ is a real valued random variables for each $x^* \in X^*$, the space X^* denoting the first normed dual space of X.

Remark 1
(1) In a separable Banach space X, the notions of strong and weak random variables $x : \Omega \to X$ coincide and respect of such a space X, x is termed as a random variable (see Joshi and Bose [12, Corollary 1]).

(2) If X is a separable Banach space then the $\sigma-$algebra generated by the class of all spherical neighourhoods of X is equal to the $\sigma-$algebra of Borel subsets of X. Hence every strong and also weak random variable is measurable in the sense of Definition 2.1.

Let Y be another Banach space. We also need the following definitions as cited in Joshi and Bose [12].

Definition 2.5 A mapping $F : \Omega \times X \to Y$ is said to be a random mapping if $F(\omega, x) = Y(\omega)$ is a $Y-$valued random variable for every $x \in X$.

Definition 2.6 A mapping $F : \Omega \times X \to Y$ is said to be a continuous random mapping if the set of all $\omega \in \Omega$ for which $F(\omega, x)$ is a continuous function of x has measure one.

Definition 2.7 An equation of the type $F(\omega, x(\omega)) = x(\omega)$, where $F : \Omega \times X \to Y$ is a random mapping, is called a random fixed point equation.

Definition 2.8 Any mapping $x : \Omega \to X$ which satisfies the random fixed point equation $F(\omega, x(\omega)) = x(\omega)$ almost surely is said to be a wide sense solution of the fixed point equation.

Definition 2.9 Any $X-$valued random variable $x(\omega)$ which satisfies $\mu\{\omega : F(\omega, x(\omega)) = x(\omega)\} = 1$ is said to be a random solution of the fixed point equation or a random fixed point of F.

Remark 2 *A random solution is a wide sense solution of the fixed point equation, but the converse is not necessarily true. This is evident from the following example as found under Joshi and Bose [12, Remark 1].*

Our main aim of this paper is to define the random analogue of a (α, β, γ)-generalized hybrid and thereby prove the stochastic version of the deterministic fixed point theorem in a separable Banach space. Also some more random fixed point theorems have been established in separable Banach space to investigate this relatively new field of research extensively with application.

Now, we define the random version of a $(\alpha, \beta, \gamma)-$generalized hybrid and then establish a random fixed point theorem for $(\alpha, \beta, \gamma)-$generalized hybrid.

3. Random analogue of (α, β, γ)-generalized hybrid

Definition 3.1 Let X be a separable Banach space and $(\Omega, \sum, \mu)$ a complete probability measure space. Then $T : \Omega \times X \to X$ is called a random contractively generalized hybrid if there exist a finitely real valued random variables $\alpha(\omega), \beta(\omega)$ and $\gamma(\omega)$ such that

$$\alpha(\omega) \|T(\omega, x_1) - T(\omega, x_2)\| \leqslant \gamma(\omega)\Big(\beta(\omega) \|x_2 - T(\omega, x_1)\| + (1 - \beta(\omega)) \|x_1 - x_2\| \Big) \\ - (1 - \alpha(\omega)) \|x_1 - T(\omega, x_2)\|, \tag{3.1}$$

for all $x_1, x_2 \in X$.

Theorem 3.2 Let X be a separable Banach space and $(\Omega, \sum, \mu)$ a complete probability measure space. Let $T : \Omega \times X \longrightarrow X$ be a continuous operator satisfying (3.1) almost surely, where $0 \leqslant \gamma(\omega) < 1$ is a real valued random variable and $\gamma(\omega).\beta(\omega) < \alpha(\omega)$ almost surely. Then there exists a random fixed point of T.

Proof. Let $A = \{\omega \in \Omega : T(\omega, x)$ is a continuous function of $x\}$,

$$B = \Big\{\omega \in \Omega : 0 \leqslant \gamma(\omega) < 1\Big\} \cap \Big\{\omega \in \Omega : \gamma(\omega).\beta(\omega) < \alpha(\omega)\Big\},$$

$$C_{x_1,x_2} = \Big\{\omega \in \Omega : \alpha(\omega) \|T(\omega, x_1) - T(\omega, x_2)\| \leqslant \gamma(\omega)\big(\beta(\omega) \|x_2 - T(\omega, x_1)\| \\ +(1 - \beta(\omega)) \|x_1 - x_2\| \big) - (1 - \alpha(\omega)) \|x_1 - T(\omega, x_2)\| \Big\}.$$

Let S be a countable dense subset of X, we now prove that

$$\bigcap_{x_1,x_2 \in X} (C_{x_1,x_2} \cap A \cap B) = \bigcap_{s_1,s_2 \in X} (C_{s_1,s_2} \cap A \cap B).$$

Let $\omega \in \bigcap_{s_1,s_2\in X}(C_{s_1,s_2}\cap A\cap B)$. Then for all $s_1, s_2 \in X$,

$$\begin{aligned}\alpha(\omega)\left\|T(\omega,s_1)-T(\omega,s_2)\right\| \leqslant \gamma(\omega)\{&\beta(\omega)\left\|s_2-T(\omega,s_1)\right\| \\ &+(1-\beta(\omega))\left\|s_1-s_2\right\|\}-(1-\alpha(\omega))\left\|s_1-T(\omega,s_2)\right\|,\end{aligned}$$

note that for all $x_1, x_2 \in X$,

$$\left\|s_1-s_2\right\| \leqslant \left\|s_1-x_1\right\|+\left\|x_1-x_2\right\|+\left\|x_2-s_2\right\|, \tag{3.2}$$

$$\left\|s_1-T(\omega,s_1)\right\| \leqslant \left\|s_1-x_1\right\|+\left\|x_1-T(\omega,x_1)\right\|+\left\|T(\omega,x_1)-T(\omega,s_1)\right\|, \tag{3.3}$$

$$\left\|s_2-T(\omega,s_2)\right\| \leqslant \left\|s_2-x_2\right\|+\left\|x_2-T(\omega,x_2)\right\|+\left\|T(\omega,x_2)-T(\omega,s_2)\right\|, \tag{3.4}$$

$$\left\|s_1-T(\omega,s_2)\right\| \leqslant \left\|s_1-x_1\right\|+\left\|x_1-T(\omega,x_2)\right\|+\left\|T(\omega,x_2)-T(\omega,s_2)\right\|, \tag{3.5}$$

$$\left\|s_2-T(\omega,s_1)\right\| \leqslant \left\|s_2-x_2\right\|+\left\|x_2-T(\omega,x_1)\right\|+\left\|T(\omega,x_1)-T(\omega,s_1)\right\|. \tag{3.6}$$

Let $x_1, x_2 \in X$, we have

$$\begin{aligned}\alpha(\omega)\left\|T(\omega,x_1)-T(\omega,x_2)\right\| &\leqslant \alpha(\omega)\left\|T(\omega,x_1)-T(\omega,s_1)\right\|+\alpha(\omega)\left\|T(\omega,s_1)-T(\omega,s_2)\right\| \\ &\quad+\alpha(\omega)\left\|T(\omega,s_2)-T(\omega,x_2)\right\| \\ &\leqslant \alpha(\omega)\left\|T(\omega,x_1)-T(\omega,s_1)\right\|+\alpha(\omega)\left\|T(\omega,s_2)-T(\omega,x_2)\right\| \\ &\quad+\gamma(\omega)\{\beta(\omega)\left\|s_2-T(\omega,s_1)\right\|+(1-\beta(\omega))\left\|s_1-s_2\right\|\} \\ &\quad-(1-\alpha(\omega))\left\|s_1-T(\omega,s_2)\right\|,\end{aligned}$$

using (3.2), (3.5) and (3.6) we have

$$\begin{aligned}\alpha(\omega)\left\|T(\omega,x_1)-T(\omega,x_2)\right\| &\leqslant \alpha(\omega)\left\|T(\omega,x_1)-T(\omega,s_1)\right\|+\alpha(\omega)\left\|T(\omega,s_2)-T(\omega,x_2)\right\| \\ &\quad+\gamma(\omega).\beta(\omega)[\left\|s_2-x_2\right\|+\left\|x_2-T(\omega,x_1)\right\|+\left\|T(\omega,x_1)-T(\omega,s_1)\right\|] \\ &\quad+\gamma(\omega)(1-\beta(\omega))[\left\|s_1-x_1\right\|+\left\|x_1-x_2\right\|+\left\|x_2-s_2\right\|] \\ &\quad-(1-\alpha(\omega))[\left\|s_1-x_1\right\|+\left\|x_1-T(\omega,x_2)\right\|+\left\|T(\omega,x_2)-T(\omega,s_2)\right\|]\end{aligned}$$

$$\begin{aligned}&\leqslant (\alpha(\omega)+\gamma(\omega).\beta(\omega))\left\|T(\omega,x_1)-T(\omega,s_1)\right\|+(2\alpha(\omega)-1)\left\|T(\omega,s_2)-T(\omega,x_2)\right\| \\ &\quad+\gamma(\omega)\{\beta(\omega)\left\|x_2-T(\omega,x_1)\right\|+(1-\beta(\omega))\left\|x_1-x_2\right\|\}-(1-\alpha(\omega))\left\|x_1-T(\omega,x_2)\right\| \\ &\quad+[\gamma(\omega).\beta(\omega)+\gamma(\omega)(1-\beta(\omega))]\left\|x_2-s_2\right\|+[\gamma(\omega)(1-\beta(\omega))-(1-\alpha(\omega))]\left\|s_1-x_1\right\| \\ &< 2\alpha(\omega)\left\|T(\omega,x_1)-T(\omega,s_1)\right\|+(2\alpha(\omega)-1)\left\|T(\omega,s_2)-T(\omega,x_2)\right\| \\ &\quad+\gamma(\omega)\{\beta(\omega)\left\|x_2-T(\omega,x_1)\right\|+(1-\beta(\omega))\left\|x_1-x_2\right\|\}-(1-\alpha(\omega))\left\|x_1-T(\omega,x_2)\right\| \\ &\quad+\gamma(\omega)\left\|x_2-s_2\right\|+(\gamma(\omega)-1)\left\|s_1-x_1\right\|.\end{aligned}$$

Since for a particular $\omega \in \Omega$, $T(\omega, x)$ is a continuous function of x, so for any $\epsilon > 0$, there exists $\delta_i(x_i) > 0$ $(i = 1, 2)$ such that

$$\|T(\omega, x_1) - T(\omega, s_1)\| < \frac{\epsilon}{8\alpha(\omega))}, \text{ whenever } \|x_1 - s_1\| < \delta_1(x_1) = \frac{\epsilon}{4(\gamma(\omega) - 1)},$$

and

$$\|T(\omega, x_2) - T(\omega, s_2)\| < \frac{\epsilon}{4(2\alpha(\omega) - 1)}, \text{ whenever } \|x_2 - s_2\| < \delta_2(x_2) = \frac{\epsilon}{4\gamma(\omega)},$$

if we take $\rho_1 = \min(\delta_1, \frac{\varepsilon}{4})$ and $\rho_2 = \min(\delta_2, \frac{\varepsilon}{4})$, for such a choic of ρ_1 and ρ_2, we get

$$\alpha(\omega) \|T(\omega, x_1) - T(\omega, x_2)\| \leqslant \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \gamma(\omega)\{\beta(\omega) \|x_2 - T(\omega, x_1)\| \\ +(1 - \beta(\omega)) \|x_1 - x_2\|\} - (1 - \alpha(\omega)) \|x_1 - T(\omega, x_2)\|.$$

As $\epsilon > 0$ is arbitrary, it follow that

$$\alpha(\omega) \|T(\omega, x_1) - T(\omega, x_2)\| \leqslant \gamma(\omega)\{\beta(\omega) \|x_2 - T(\omega, x_1)\| \\ +(1 - \beta(\omega)) \|x_1 - x_2\|\} - (1 - \alpha(\omega)) \|x_1 - T(\omega, x_2)\|.$$

Thus $\omega \in \bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B)$, which implies that

$$\bigcap_{s_1, s_2 \in X} (C_{s_1, s_2} \cap A \cap B) \subset \bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B),$$

also, similar to above proof, we have

$$\bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B) \subset \bigcap_{s_1, s_2 \in X} (C_{s_1, s_2} \cap A \cap B),$$

and so

$$\bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B) = \bigcap_{s_1, s_2 \in X} (C_{s_1, s_2} \cap A \cap B).$$

Let $N = \bigcap_{s_1, s_2 \in X} (C_{s_1, s_2} \cap A \cap B)$, then $\mu(N) = 1$ and for each $\omega \in N$, $T(\omega, x)$ is a deterministic continuous operator satisfying the mapping referred to in [10] and hence, this has a wide sense solution $x(\omega)$. The randomness and measurability of $x(\omega)$ can be proved by generating an approximating sequence of random variable $x_n(\omega)$ as follows: Let $x_\circ(\omega)$ be a random variable, let $x_1(\omega) = T(\omega, x_\circ(\omega))$, then it follows that $x_1(\omega)$ is a random variable, then we consider $x_{n+1}(\omega) = T(\omega, x_n(\omega))$, by repeated iteration, it gives that $\{x_n(\omega)\}$ is a sequence of random variable convergence to $x(\omega)$. This implies that $x(\omega)$ is measurable and unique random fixed point of T. ■

In the following theorem, we prove the stochastic version of deterministic fixed point theorem for a general contractive mapping and some other related results.

Theorem 3.3 Let X be a separable Hilbert space and $(\Omega, \sum, \mu)$ be a complete propability measurable space. Let $T : \Omega \times X \longrightarrow X$ be a continuous operator such that for $\omega \in \Omega$, T satisfy the following condition:

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leqslant \alpha(\omega) \ \max \left\{ \begin{array}{c} \|x_1 - x_2\|, \frac{\beta(\omega)}{2}[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|], \\ \frac{\gamma(\omega)}{2}[\|x_1 - T(\omega, x_2)\| + \|x_2 - T(\omega, x_1)\|] \end{array} \right\}, \tag{3.7}$$

for all $x_1, x_2 \in X$ where $\alpha(\omega), \beta(\omega)$ and $\gamma(\omega)$ are nonnegative real valued random varibles such that $\beta(\omega), \gamma(\omega) \in (0, 1)$, $\alpha(\omega) > 0$ and $\alpha(\omega).\beta(\omega)$, $\alpha(\omega).\gamma(\omega) < \alpha(\omega)$ almost surely. Then T has a unique random fixed point in X.

Proof. Let $A = \{\omega \in \Omega : T(\omega, x) \text{ is a continuous function of } x\}$,

$$B = \{\omega \in \Omega : \alpha(\omega) > 0\} \cap \{\omega \in \Omega : 0 < \beta(\omega), \gamma(\omega) < 1\}$$
$$\cap \{\omega \in \Omega : \alpha(\omega).\beta(\omega), \alpha(\omega).\gamma(\omega) < \alpha(\omega)\},$$

$$C_{x_1,x_2} = \left\{ \begin{array}{c} \omega \in \Omega : \|T(\omega, x_1) - T(\omega, x_2)\| \\ \leqslant \alpha(\omega) \max \left\{ \begin{array}{c} \|x_1 - x_2\|, \frac{\beta(\omega)}{2}[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|] \\ , \frac{\gamma(\omega)}{2}[\|x_1 - T(\omega, x_2)\| + \|x_2 - T(\omega, x_1)\|] \end{array} \right\} \end{array} \right\}.$$

Let S be a countable dense subset of X, we now prove that

$$\bigcap_{x_1,x_2 \in X} (C_{x_1,x_2} \cap A \cap B) = \bigcap_{s_1,s_2 \in X} (C_{s_1,s_2} \cap A \cap B).$$

Then for all $s_1, s_2 \in X$,

$$\|T(\omega, s_1) - T(\omega, s_2)\| \leqslant \alpha(\omega) \max \left\{ \begin{array}{c} \|s_1 - s_2\|, \frac{\beta(\omega)}{2}[\|s_1 - T(\omega, s_1)\| + \|s_2 - T(\omega, s_2)\|] \\ , \frac{\gamma(\omega)}{2}[\|s_1 - T(\omega, s_2)\| + \|s_2 - T(\omega, s_1)\|] \end{array} \right\}. \tag{3.8}$$

Now, we examine the following cases:

Case(i). Suppose

$$\|T(\omega, s_1) - T(\omega, s_2)\| = \alpha(\omega) \|s_1 - s_2\|,$$

now,

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leqslant \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_1) - T(\omega, s_2)\| + \|T(\omega, s_2) - T(\omega, x_2)\|$$

$$\leqslant \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\| + \alpha(\omega) \|s_1 - s_2\|. \tag{3.9}$$

Using (3.2) and (3.9), we have

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leqslant \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\|$$

$$+\alpha(\omega)[\|s_1 - x_1\| + \|x_1 - x_2\| + \|x_2 - s_2\|], \tag{3.10}$$

since for a particular $\omega \in \Omega$, $T(\omega, x)$ is a continuous function of x, so for any $\epsilon > 0$, there exists $\delta_i(x_i) > 0$ $(i = 1, 2)$ such that

$$\|T(\omega, x_1) - T(\omega, s_1)\| < \frac{\epsilon}{4}, \text{ whenever } \|x_1 - s_1\| < \delta_1(x_1), \tag{3.11}$$

and

$$\|T(\omega, x_2) - T(\omega, s_2)\| < \frac{\epsilon}{4}, \text{ whenever } \|x_2 - s_2\| < \delta_2(x_2), \tag{3.12}$$

where

$$\delta = \delta_1(x_1) = \delta_2(x_2) = \frac{\epsilon}{4\alpha(\omega)}, \tag{3.13}$$

by choosing $\rho = \min(\delta, \frac{\epsilon}{4})$ then from (3.10), we have

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leqslant \frac{\epsilon}{4} + \frac{\epsilon}{4} + \alpha(\omega)[\frac{\epsilon}{4\alpha(\omega)} + \|x_1 - x_2\| + \frac{\epsilon}{4\alpha(\omega)}]$$

$$= \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \alpha(\omega) \|x_1 - x_2\| .$$

As $\epsilon > 0$ is arbitrary, it follow that

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leqslant \alpha(\omega) \|x_1 - x_2\| . \tag{3.14}$$

Case(ii). Suppose

$$\|T(\omega, s_1) - T(\omega, s_2)\| = \frac{\alpha(\omega).\beta(\omega)}{2}[\|s_1 - T(\omega, s_1)\| + \|s_2 - T(\omega, s_2)\|],$$

now,

$$\begin{aligned}\|T(\omega, x_1) - T(\omega, x_2)\| &\leqslant \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_1) - T(\omega, s_2)\| + \|T(\omega, s_2) - T(\omega, x_2)\| \\ &\leqslant \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\|\end{aligned}$$

$$+\frac{\alpha(\omega).\beta(\omega)}{2}[\|s_1 - T(\omega, s_1)\| + \|s_2 - T(\omega, s_2)\|]. \tag{3.15}$$

By using (3.3), (3.4) and (3.15), by routine calculation, we get

$$\begin{aligned}\|T(\omega,x_1)-T(\omega,x_2)\| &\leqslant \|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,x_2)-T(\omega,s_2)\|\\ &\quad+\frac{\alpha(\omega).\beta(\omega)}{2}[\|s_1-x_1\|+\|x_1-T(\omega,x_1)\|+\|T(\omega,x_1)-T(\omega,s_1)\|\\ &\quad+\|s_2-x_2\|+\|x_2-T(\omega,x_2)\|+\|T(\omega,x_2)-T(\omega,s_2)\|]\\ &=(1+\frac{\alpha(\omega).\beta(\omega)}{2})[\|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,x_2)-T(\omega,s_2)\|]\\ &\quad+\frac{\alpha(\omega).\beta(\omega)}{2}[\|s_1-x_1\|+\|s_2-x_2\|]\\ &\quad+\frac{\alpha(\omega).\beta(\omega)}{2}[\|x_1-T(\omega,x_1)\|+\|x_2-T(\omega,x_2)\|]\\ &<(\frac{2+\alpha(\omega)}{2})[\|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,x_2)-T(\omega,s_2)\|]\\ &\quad+\frac{\alpha(\omega)}{2}[\|s_1-x_1\|+\|s_2-x_2\|]\end{aligned}$$

$$+\frac{\alpha(\omega).\beta(\omega)}{2}[\|x_1-T(\omega,x_1)\|+\|x_2-T(\omega,x_2)\|], \tag{3.16}$$

since for a particular $\omega\in\Omega$, $T(\omega,x)$ is a continuous function of x, so for any $\epsilon>0$, there exists $\delta_i(x_i)>0$ $(i=1,2)$ such that

$$\|T(\omega,x_1)-T(\omega,s_1)\|<\frac{\epsilon}{2(2+\alpha(w))},\ \text{whenever}\ \|x_1-s_1\|<\delta_1(x_1),$$

and

$$\|T(\omega,x_2)-T(\omega,s_2)\|<\frac{\epsilon}{2(2+\alpha(w))},\ \text{whenever}\ \|x_2-s_2\|<\delta_2(x_2),$$

where

$$\delta=\delta_1(x_1)=\delta_2(x_2)=\frac{\epsilon}{2\alpha(\omega)},$$

by choosing $\rho=\min(\delta,\frac{\epsilon}{4})$ and from (3.16), we get

$$\begin{aligned}\|T(\omega,x_1)-T(\omega,x_2)\| &\leqslant \frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\alpha(\omega).\beta(\omega)}{2}[\|x_1-T(\omega,x_1)\|+\|x_2-T(\omega,x_2)\|]\\ &=\epsilon+\frac{\alpha(\omega).\beta(\omega)}{2}[\|x_1-T(\omega,x_1)\|+\|x_2-T(\omega,x_2)\|].\end{aligned}$$

As $\epsilon>0$ is arbitrary, it follow that

$$\|T(\omega,x_1)-T(\omega,x_2)\|\leqslant\frac{\alpha(\omega).\beta(\omega)}{2}[\|x_1-T(\omega,x_1)\|+\|x_2-T(\omega,x_2)\|] \tag{3.17}$$

Case(iii). Suppose

$$\|T(\omega,s_1)-T(\omega,s_2)\|=\frac{\alpha(\omega).\gamma(\omega)}{2}[\|s_1-T(\omega,s_2)\|+\|s_2-T(\omega,s_1)\|],$$

now,

$$\begin{aligned}\|T(\omega,x_1)-T(\omega,x_2)\| &\leqslant \|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_1)-T(\omega,s_2)\|+\|T(\omega,s_2)-T(\omega,x_2)\|\\ &\leqslant \|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_2)-T(\omega,x_2)\|\\ &+\frac{\alpha(\omega).\gamma(\omega)}{2}[\|s_1-T(\omega,s_2)\|+\|s_2-T(\omega,s_1)\|]. \qquad (3.18)\end{aligned}$$

By using (3.5), (3.6) and (3.18), by routine check-up, we get

$$\begin{aligned}\|T(\omega,x_1)-T(\omega,x_2)\| &\leqslant \|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_2)-T(\omega,x_2)\|\\ &\quad+\frac{\alpha(\omega).\gamma(\omega)}{2}[\|s_1-x_1\|+\|x_1-T(\omega,x_2)\|+\|T(\omega,x_2)-T(\omega,s_2)\|\\ &\quad+\|s_2-x_2\|+\|x_2-T(\omega,x_1)\|+\|T(\omega,x_1)-T(\omega,s_1)\|]\\ &=(1+\frac{\alpha(\omega).\gamma(\omega)}{2})[\|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,x_2)-T(\omega,s_2)\|]\\ &\quad+\frac{\alpha(\omega).\gamma(\omega)}{2}[\|s_1-x_1\|+\|s_2-x_2\|]\\ &\quad+\frac{\alpha(\omega).\gamma(\omega)}{2}[\|x_1-T(\omega,x_2)\|+\|x_2-T(\omega,x_1)\|]\\ &<(\frac{2+\alpha(\omega))}{2})[\|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,x_2)-T(\omega,s_2)\|]\\ &\quad+\frac{\alpha(\omega)}{2}[\|s_1-x_1\|+\|s_2-x_2\|]\\ &\quad+\frac{\alpha(\omega).\gamma(\omega)}{2}[\|x_1-T(\omega,x_2)\|+\|x_2-T(\omega,x_1)\|],\end{aligned}$$

again choose $\rho=\min(\delta,\frac{\epsilon}{4})$ and by the same method of Case ii, we have

$$\begin{aligned}\|T(\omega,x_1)-T(\omega,x_2)\| &\leqslant \frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\alpha(\omega).\gamma(\omega)}{2}[\|x_1-T(\omega,x_2)\|+\|x_2-T(\omega,x_1)\|]\\ &=\epsilon+\frac{\alpha(\omega).\gamma(\omega)}{2}[\|x_1-T(\omega,x_2)\|+\|x_2-T(\omega,x_1)\|].\end{aligned}$$

As $\epsilon>0$ is arbitrary, it follow that

$$\|T(\omega,x_1)-T(\omega,x_2)\|\leqslant\frac{\alpha(\omega).\gamma(\omega)}{2}[\|x_1-T(\omega,x_2)\|+\|x_2-T(\omega,x_1)\|]. \qquad (3.19)$$

Combining (3.14), (3.17) and (3.19), we get

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leqslant \alpha(\omega) \ \max \left\{ \begin{array}{c} \|x_1 - x_2\|, \frac{k(\omega)}{2}[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_2)\|], \\ \frac{\gamma(\omega)}{2}[\|x_1 - T(\omega, x_2)\| + \|x_2 - T(\omega, x_1)\|] \end{array} \right\}.$$

Thus $\omega \in \bigcap_{x_1,x_2 \in X} (C_{x_1,x_2} \cap A \cap B)$, which implies that

$$\bigcap_{s_1,s_2 \in X} (C_{s_1,s_2} \cap A \cap B) \subset \bigcap_{x_1,x_2 \in X} (C_{x_1,x_2} \cap A \cap B),$$

also, similar to above proof, we have

$$\bigcap_{x_1,x_2 \in X} (C_{x_1,x_2} \cap A \cap B) \subset \bigcap_{s_1,s_2 \in X} (C_{s_1,s_2} \cap A \cap B),$$

and so

$$\bigcap_{x_1,x_2 \in X} (C_{x_1,x_2} \cap A \cap B) = \bigcap_{s_1,s_2 \in X} (C_{s_1,s_2} \cap A \cap B).$$

Let $N = \bigcap_{s_1,s_2 \in X} (C_{s_1,s_2} \cap A \cap B)$, then $\mu(N) = 1$ and for each $\omega \in N$, $T(\omega, x)$ is a deterministic continuous operator satisfying the general contractive condition and hence, this has a wide sense solution $x(\omega)$. The randomness and measurability of $x(\omega)$ can be proved by generating an approximating sequence of random variable $x_n(\omega)$ as follows: Let $x_\circ(\omega)$ be a random variable, let $x_1(\omega) = T(\omega, x_\circ(\omega))$. Then it follows that $x_1(\omega)$ is a random variable, then we consider $x_{n+1}(\omega) = T(\omega, x_n(\omega))$. By repeated iteration, it gives that $\{x_n(\omega)\}$ is a sequence of random variable convergence to $x(\omega)$. This implies that $x(\omega)$ is measurable and unique random fixed point of T. ■

4. Application to a random nonlinear integral equation

In this section, we apply Theorem 3.3 to prove the existence of a solution in a Banach space of a random nonlinear integral equation of the form:

$$x(t;\omega) = h(t;\omega) + \lambda(\omega) \int_S k(t, s;\omega) f(s, x(s;\omega)) d\mu_\circ(s), \tag{4.1}$$

where,
(i) S is a locally compact metric space with metric d on $S \times S$, $\mu_\circ$ is a complete $\sigma-$finite measure defined on the collection of Borel subsets of S,
(ii) $\omega \in \Omega$, where ω is a supporting set of the probability measure space $(\Omega, \sum, \mu)$,
(iii) $x(t;\omega)$ is an unknown vector-valued random variable for each $t \in S$,
(iv) $h(t;\omega)$ is the stochastic free term defined for $t \in S$,
(v) $k(t, s;\omega)$ is the stochastic kernel defined for t and s in S,
(vi) $f(t, x)$ is vector-valued function of $t \in S$ and x.
The integral equation (4.1) in stochastic version is a similar to Fredholm integral equation of the second kind in deterministic.

We shall further assume that S is the union of a decreasing sequence of countable family of compact sets $\{C_n\}$ having the properties that $C_1 \subset C_2 \subset ..$ and that for any other compact set S there is a C_i which contains it (see [2]).
we will the steps of Lee and Padjett [15] with necessary modification as required for the more general settings.

Definition 4.1 We define the space $C(S, L_2(\Omega, \sum, \mu))$ to be the space of all continuous functions from S into $L_2(\Omega, \sum, \mu)$ with the topology of uniform convergence on compacta i.e. for each fixed $t \in S$, $x(t;\omega)$ is a vector valued random variable such that

$$\|x(t;\omega)\|^2_{L_2(\Omega,\sum,\mu)} = \int\limits_{\Omega} |x(t;\omega)|^2 \, d\mu(\omega) < \infty.$$

It may be noted that $C(S, L_2(\Omega, \sum, \mu))$ is locally convex space (see [5]) whose topologies defined by a countable family of seminorms given by

$$\|x(t;\omega)\|_n = \sup_{t \in C_n} \|x(t;\omega)\|_{L_2(\Omega,\beta,\mu)}, n = 1, 2, ..$$

Moreover $C(S, L_2(\Omega, \sum, \mu))$ is complete relative to this topology since $L_2(\Omega, \sum, \mu)$ is complete. We will consider the function $h(t;\omega)$ and $f(t, x(t;\omega))$ to be in the space $C(S, L_2(\Omega, \sum, \mu))$ with respect to the stochastic kernel. We assume that for each pair (t, s), $k(t, s; \omega) \in L_\infty(\Omega, \beta, \mu)$ and denote the norm by

$$|||k(t, s; \omega)||| = \|k(t, s; \omega)\|_{L_\infty(\Omega,\sum,\mu)} = \mu - ess \sup_{\omega \in \Omega} |k(t, s; \omega)|.$$

Also we will suppose that $k(t, s; \omega)$ is such that $|||k(t, s; \omega)||| . \|x(t;\omega)\|_{L_2(\Omega,\sum,\mu)}$ is $\mu_\circ$-integrable with respect to s for each $t \in S$ and $x(s;\omega)$ in $C(S, L_2(\Omega, \sum, \mu))$, hence there exists a real valued function G defined $\mu_\circ-$a.e. on S, so that $G(S) \|x(s;\omega)\|_{L_2(\Omega,\sum,\mu)}$ is $\mu_\circ-$integrable so that for each pair $(t, s) \in S \times S$,

$$|||k(t, u; \omega) - k(s, u; \omega)||| . \|x(u;\omega)\|_{L_2(\Omega,\beta,\mu)} \leqslant G(u) \|x(u;\omega)\|_{L_2(\Omega,\beta,\mu)}$$

$\mu_\circ-$a.e. Further, for almost all $s \in S$, then $k(t, s; \omega) : S \to L_\infty(\Omega, \sum, \mu)$ will be continuous in t.

We now define the random integral operator T on $C(S, L_2(\Omega, \sum, \mu))$ by

$$(Tx)(t;\omega) = \lambda(\omega) \int_S k(t, s; \omega) x(s;\omega)) d\mu_\circ(s), \quad |\lambda(\omega)| < 1 \tag{4.2}$$

where the integral is a Fredholm integral. Moreover, we have that for each $t \in S$, $(Tx)(t;\omega) \in L_2(\Omega, \sum, \mu)$ and that $(Tx)(t;\omega)$ is continuous in mean square by Lebesgue's dominated convergence theorem. So $(Tx)(t;\omega) \in C(S, L_2(\Omega, \sum, \mu))$.

Definition 4.2 [5] Let B and D be Banach spaces. The pair (B, D) is said to be admissible with respect to a random operator $T(\omega)$ if $T(\omega)(B) \subset D$.

Lemma 4.3 [15] The linear operator T defined by (4.2) is continuous from $C(S, L_2(\Omega, \sum, \mu))$ into itself.

Lemma 4.4 [15] If T is a continuous linear operator from $C(S, L_2(\Omega, \sum, \mu))$ into itself and $B, D \subset C(S, L_2(\Omega, \sum, \mu))$ are Banach spaces stronger than $C(S, L_2(\Omega, \sum, \mu))$ such that (B, D) is admissible with respect to T, then T is continuous from B into D.

Remark 3 *[15] From Lemmas 4.3 and 4.4, it follows that:*
(1) The operator T defined by (4.2) is a bounded linear operator from B into D.
(2) By a random solution of the equation (4.1) we will mean a function $x(x;\omega)$ in $C(S, L_2(\Omega, \sum, \mu))$ which satisfies the equation (4.1) $\mu-$a.e.

Now we are in a position to prove theorem concerning the existence of a random solution of the equation (4.1) as the following:

Theorem 4.5 We consider the stochastic integral equation (4.1) subject to the following conditions:
(a) B and D are Banach spaces stronger than $C(S, L_2(\Omega, \sum, \mu))$ such that (B, D) is admissible with respect to the integral operator defined by (4.2),
(b) $h(t;\omega) \in D$,
(c) $x(t;\omega) \to f(t, x(t;\omega))$ is an operator from the set

$$Q(\rho) = \{x(t;\omega) : x(t;\omega) \in D, \|x(t;\omega)\|_D \leqslant \rho\}$$

into the space B satisfying

$$\|f(t, x_1(t;\omega)) - f(t, x_2(t;\omega))\|_B \leqslant \alpha(\omega)\ \max \left\{ \begin{array}{c} \|x_1(t;\omega) - x_2(t;\omega)\|_D\,, \\ \frac{\beta(\omega)}{2}[\|x_1(t;\omega) - f(t, x_1(t;\omega))\|_D \\ + \|x_2(t;\omega) - f(t, x_2(t;\omega))\|_D], \\ \frac{\gamma(\omega)}{2}[\|x_1(t;\omega) - f(t, x_2(t;\omega))\|_D \\ + \|x_2(t;\omega) - f(t, x_1(t;\omega))\|_D] \end{array} \right\}, \tag{4.3}$$

for $x_1(t;\omega), x_2(t;\omega) \in Q(\rho)$, where $\alpha(\omega), \beta(\omega)$ and $\gamma(\omega)$ are nonnegative real valued random variable such that $\beta(\omega), \gamma(\omega) \in (0, 1)$, $\alpha(\omega) > 0$ almost surely.
Then there exists a unique random solution of (4.1) in $Q(\rho)$, provided $\alpha(\omega).\beta(\omega) < \alpha(\omega)$, $\alpha(\omega).\gamma(\omega) < \alpha(\omega)$ and

$$\|h(t;\omega)\|_D + (\frac{2+\alpha(\omega)}{2-\alpha(\omega)})c(\omega)\, \|f(t;0)\|_B \leqslant \rho(1 - \frac{c(\omega)\alpha(\omega)}{2-\alpha(\omega)}),$$

where $c(\omega)$ is the norm of the operator $T(\omega)$.

Proof. Define the operator $U(\omega)$ from $Q(\rho)$ into D by

$$(Ux)(t;\omega) = h(t;\omega) + \lambda(\omega) \int_S k(t, s;\omega) f(s, x(s;\omega)) d\mu_\circ(s).$$

So,

$$\begin{aligned} \|(Ux)(t;\omega)\|_D &\leqslant \|h(t;\omega)\|_D + c(\omega)\, \|f(t, x(t;\omega))\|_B \\ &\leqslant \|h(t;\omega)\|_D + c(\omega)\, \|f(t;0)\|_B + c(\omega)\, \|f(t, x(t;\omega)) - f(t, 0)\|_B\,. \end{aligned}$$

Applying condition (4.3) of this theorem, we get

$$\|f(t,x(t;\omega))-f(t,0)\|_B \leqslant \alpha(\omega)\max\{\|x(t;\omega)\|_D\,, \frac{\beta(\omega)}{2}[\|x(t;\omega)-f(t,x(t;\omega))\|_D+\|f(t,0)\|_D], \frac{\gamma(\omega)}{2}[\|x(t;\omega)-f(t,0))\|_D+\|f(t,x(t;\omega))\|_D]\}.$$

By the same manner of three cases of Theorem 3.2 and by simple proof, we have

$$\|(Ux)(t;\omega)\|_D \leqslant \|h(t;\omega)\|_D + c(\omega)\,\|f(t;0)\|_B + c(\omega)\alpha(\omega)\rho < \rho \text{ from case (i)}, \tag{4.4}$$

also,

$$\|(Ux)(t;\omega)\|_D \leqslant \|h(t;\omega)\|_D + \frac{c(\omega)\alpha(\omega)}{2-\alpha(\omega)}\rho + (\frac{2+\alpha(\omega)}{2-\alpha(\omega)})c(\omega)\,\|f(t,0)\|_B < \rho \text{ from cases (ii), (iii)}. \tag{4.5}$$

Then by (4.4) and (4.5), we have $(Ux)(t;\omega) \in Q(\rho)$, then for $x_1(t;\omega)$, $x_2(t;\omega) \in Q(\rho)$, we have by condition (c)

$$\begin{aligned}
\|(Ux_1)(t;\omega)-(Ux_2)(t;\omega)\|_D &= |\lambda(\omega)|\left\|\int_S k(t,s;\omega)[f(s,x_1(s;\omega)-f(s,x_2(s;\omega))]d\mu_\circ(s)\right\|_D \\
&\leqslant \left\|\int_S k(t,s;\omega)[f(s,x_1(s;\omega)-f(s,x_2(s;\omega))]d\mu_\circ(s)\right\|_D \quad \text{since } |\lambda(\omega)|<1 \\
&< c(\omega)\,\|f(t,x_1(t;\omega)-f(t,x_2(t;\omega))\|_B \\
&\leqslant \alpha(\omega)\ \max\{\|x_1(t;\omega)-x_2(t;\omega)\|_D\,, \\
&\quad \frac{\beta(\omega)}{2}[\|x_1(t;\omega)-(Ux_1)(t;\omega))\|_D+\|x_2(t;\omega)-(Ux_2)(t;\omega))\|\,, \\
&\quad \frac{\gamma(\omega)}{2}[\|x_1(t;\omega)-(Ux_2)(t;\omega))\|_D+\|x_2(t;\omega)-(Ux_1)(t;\omega))\|_D]\}.
\end{aligned}$$

Therefore $U(\omega)$ is a random contractive nonlinear operator on $Q(\rho)$ hence, by Theorem 3.3 there exists a random fixed point of $U(\omega)$, which is the random solution of equation (4.1). ■

Remark 4 *If we take $h(t;\omega)=0$ and $\lambda(\omega)=1$ in Equation (4.1), we have Fredholm equation of the first kind. By a similar method of Theorem 4.5 there exists a random fixed point of $U(\omega)$, which is the random solution of it.*

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