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On the girth of the annihilating-ideal graph of a commutative ring

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Abstract. The annihilating-ideal graph of a commutative ring R is denoted by $\mathbb{AG}(R)$, whose vertices are all nonzero ideals of R with nonzero annihilators and two distinct vertices I and J are adjacent if and only if $IJ = 0$. In this article, we completely characterize rings R when $\mathrm{gr}(\mathbb{AG}(R)) \neq 3$.

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1. Introduction

Throughout this paper all rings are assumed to be commutative with identity $1 \neq 0$. The notion of a zero divisor graph was first introduced by I. Beck in [7], who let all the elements of R be vertices and two distinct vertices x and y are adjacent if and only if $xy = 0$. He mainly discussed the coloring of the zero divisor graph. After that many authors studied the zero divisor graph with some slight different in their definitions. For a fairly complete survey on the topic see [3]. Some years later, experts generalized results of the classic zero divisor graph theory to noncommutative rings ([10]) and recently to module theory ([6]). Some authors assigned other graphs to rings such as co-maximal ideal graph, total graph, unit graph, etc. (see, for example, [2, 4, 5]).

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For a commutative ring R, let $\mathbb{A}(R)$ be the set of all ideals with nonzero annihilators and $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$. In [8], the concept of the annihilating-ideal graph for a commutative ring R was introduced. It is a simple undirected graph, denoted by $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}(R)^*$ and two distinct vertices I and J are adjacent in case $IJ = 0$.

A graph G is said to be connected if there exists a path between any two distinct vertices of G. For distinct vertices x and y of G, let $d(x, y)$ be the length of a shortest path from x to y, if there is no such path, we put $d(x, y) = \infty$ and let $d(x, x) = 0$. The diameter of G is

$$\text{diam}(G) = \sup\{d(x, y) \mid \quad x \text{ and } y \text{ are distinct vertices of } G\}.$$

The girth of G, denoted by $\text{gr}(G)$, is the length of the shortest cycle in G and if G contains no cycles then $\text{gr}(G) = \infty$. In [8, Theorem 2.1], it was shown that for a commutative ring R, the annihilating-ideal graph $\mathbb{AG}(R)$ is always connected with $\text{diam}(\mathbb{AG}(R)) \leq 3$ and $\text{gr}(\mathbb{AG}(R)) = 3, 4$ or ∞.

A graph G is bipartite if the vertex set of G can be partitioned into two subsets A and B such that no edge has both ends in any one subset. A bipartite graph G is said to be complete in case every vertex is adjacent to every other vertices that are not in the same subset. A complete bipartite graph with parts A and B such that $|A| = m$ and $|B| = n$ is denoted by $K_{m,n}$. A star graph is a complete bipartite graph $K_{1,n}$. Let $\bar{K}_{n,2}$ be the graph formed by joining the complete bipartite graph $G_1 = K_{n,2}$ (with vertex set $A \cup B$, $|A| = n$ and $|B| = 2$) to the star graph $G_2 = K_{1,n}$ by identifying the center of G_2 and a point of B.

In Section 2, we investigate when $\text{gr}(\mathbb{AG}(R)) = 4$. We prove that for a commutative reduced ring R, $\text{gr}(\mathbb{AG}(R)) = 4$ if and only if $\mathbb{AG}(R) = K_{m,n}$ for some infinite cardinals m and n. Next we show that for a commutative ring R with $Nil(R) \neq 0$, $\text{gr}(\mathbb{AG}(R)) = 4$ if and only if $\mathbb{AG}(R) = \bar{K}_{n,2}$ for some infinite cardinal n. Moreover, R has nontrivial idempotents.

Section 3 concerns with the case when $\text{gr}(\mathbb{AG}(R)) = \infty$. For a commutative ring R, it turns out that $\text{gr}(\mathbb{AG}(R)) = \infty$ if and only if $\mathbb{AG}(R)$ is a star graph or $\mathbb{AG}(R) = \bar{K}_{1,2}$ depending on whether or not R is a reduced ring. Finally we determine the girth of the annihilating-ideal graph of polynomial ring $R[x]$ and power series ring $R[[x]]$ in term of $\text{gr}(\mathbb{AG}(R))$.

For a commutative ring R, let $Nil(R)$ be the set of all nilpotent elements of R. If I is an ideal of R, we denote the annihilator of I in R by $\text{ann}_R(I)$.

2. Rings with $\text{gr}(\mathbb{AG}(R)) = 4$

In this section we characterize rings R for which $\text{gr}(\mathbb{AG}(R)) = 4$. First we prove the following useful lemma about cycles of odd length in the annihilating-ideal graphs.

Lemma 2.1 For a commutative ring R, if $\text{gr}(\mathbb{AG}(R)) = 4$, then $\mathbb{AG}(R)$ contains no cycle of odd length.

Proof. We prove by induction on the length of a cycle. Obviously $\mathbb{AG}(R)$ contains no cycle of length 3. Now suppose that there is no cycle of length 3, 5, 7,..., $2k-1$ in $\mathbb{AG}(R)$. We show that there dose not exist a cycle of length $n := 2k+1$ in $\mathbb{AG}(R)$. By contrary, suppose that $I_1 - I_2 - ... - I_n - I_1$ is a cycle of length n in $\mathbb{AG}(R)$. Note that $I_1I_3 \neq 0$ and consider the closed path $I_1I_3 - I_4 - I_5 - ... - I_n - I_1I_3$ of length $n-2$. If $I_1I_3 \neq I_j$ for $4 \leqslant j \leqslant n$, then we would have a cycle of length $n-2$ which contradicts our induction hypothesis. Now suppose that $I_1I_3 = I_j$ for some $4 \leqslant j \leqslant n$. We have two cases:

Case *i*. If j is even, then $I_2 - I_3 - I_4 - ... - I_j - I_2$ is a cycle of odd length less than $2k+1$, which is impossible.

Case *ii*. If j is odd, then $I_j - I_{j+1} - ... - I_n - I_1 - I_2 - I_j$ is a cycle of odd length less than $2k+1$ and this is impossible.

Therefore, in $\mathbb{AG}(R)$ there exist no cycle of odd length. ■

It is known that a connected graph is bipartite if and only if it contains no cycle of odd length.

Proposition 2.2 Let R be a commutative ring such that $\mathbb{AG}(R)$ has at least two vertices. If $\text{gr}(\mathbb{AG}(R)) \neq 3$, then $\mathbb{AG}(R)$ is a bipartite graph.

Proof. We must have $\text{gr}(\mathbb{AG}(R)) = 4$ or ∞. Therefore, by Lemma 2.1, there exists no cycle of odd length in $\mathbb{AG}(R)$. Since $\mathbb{AG}(R)$ is always a connected graph, it should be a bipartite graph. ■

In the following, we have another result considering when the annihilating-ideal graph of a reduced ring R is a (complete) bipartite graph. Note that this result is similar to [9, Theorem 2.3], but its proof is not the same as that one.

Proposition 2.3 Let R be a commutative reduced ring. Then $\mathbb{AG}(R)$ is a bipartite graph if and only if there exist nonzero prime ideals P and Q of R with $P \cap Q = 0$.

Proof. Suppose that there exist nonzero prime ideals P and Q of R with $P \cap Q = 0$. Let

$$X := \{I \mid I \quad \text{is a nonzero ideal of } R \text{ contained in } P\}$$

and

$$Y := \{J \mid J \quad \text{is a nonzero ideal of } R \text{ contained in } Q\}.$$

We show that $\mathbb{AG}(R)$ is a complete bipartite graph with vertex set $X \cup Y$. Let I be an arbitrary vertex in $\mathbb{AG}(R)$. Since $\mathbb{AG}(R)$ is a connected graph, there is a vertex J adjacent to I and so $IJ = 0$. As P is a prime ideal of R, $I \subseteq P$ or $J \subseteq P$. If $I \subseteq P$ then $I \in X$. If $I \nsubseteq P$, then $J \subseteq P$. Since $J \nsubseteq Q$, we must have $I \subseteq Q$ and hence $I \in Y$. Therefore, $\mathbb{A}(R)^* \subseteq X \cup Y$. Note that $P \cap Q = 0$ implies that $X \cap Y = \emptyset$ and any vertex in X is adjacent to any vertex in Y. Also vertices in X or vertices in Y are not adjacent to each other. For if I_1 and I_2 are vertices in X with $I_1I_2 = 0$, then $I_1I_2 \subseteq Q$ implies that $I_1 \subseteq Q$ or $I_2 \subseteq Q$. Thus $I_1 \subseteq P \cap Q$ or $I_2 \subseteq P \cap Q$, which is impossible. Therefore, $\mathbb{AG}(R)$ is a complete bipartite graph with the vertex set $X \cup Y$.

Conversely, suppose that $\mathbb{AG}(R)$ is a bipartite graph. Therefore, $\mathbb{A}(R)^* = X \cup Y$ and $X \cap Y = \emptyset$. Let $P := \sum_{I \in X} I$ and $Q := \sum_{J \in Y} J$. First we show that P is a prime ideal of R. Suppose that a, b are nonzero elements of R with $ab \in P$. Then there exist a vertex I in X such that $ab \in I$. Since $\mathbb{AG}(R)$ is a bipartite graph, there is a vertex $J \in Y$ with $IJ = 0$, thus $abJ = 0$. Now we have the following two cases:

Case *i*. $bJ = 0$. Then $RbJ = 0$ and $Rb \neq J$, observe that R is a reduced ring. As $\mathbb{AG}(R)$ is a bipartite graph and $J \in Y$, we have $Rb \in X$ and so $b \in P$.

Case *ii*. $bJ \neq 0$. We claim that $bJ \in Y$ and $a \in P$. We have $IJ = 0$ and so $I(bJ) = 0$. Since R is reduced, $I \neq bJ$ and hence $bJ \in Y$. Also $(Ra)(bJ) = 0$ implies that

$Ra \neq bJ$ and $Ra \in X$, thus $a \in P$.

Therefore, P is a nonzero prime ideal of R. Similarly, we can prove that Q is a prime ideal of R. Now we show that $P \cap Q = 0$. Let $0 \neq x \in P \cap Q$. Then there is a vertex $I \in X$ with $x \in I$. As $\mathbb{AG}(R)$ is a bipartite graph, there is a vertex $J \in Y$ with $IJ = 0$. Thus $(Rx)J = 0$ and so $Rx \in X$. Similarly, we can show that $Rx \in Y$. Therefore, $Rx \in X \cap Y = \emptyset$ which is impossible. Hence $P \cap Q = 0$. ■

Now we characterize rings R for which $\mathrm{gr}(\mathbb{AG}(R)) = 4$. Two cases can be happened depending on whether or not R is a reduced ring.

Theorem 2.4 Let R be a commutative reduced ring. The following statements are equivalent.

(a) $\mathrm{gr}(\mathbb{AG}(R)) = 4$

(b) There exist nonzero prime ideals P and Q of R which are not minimal ideals such that $P \cap Q = 0$.

(c) $\mathbb{AG}(R) = K_{m,n}$, for some infinite cardinals m and n.

Proof. (a)$\Longrightarrow$(b) By Proposition 2.2, $\mathbb{AG}(R)$ is a bipartite graph. Since R is a reduced ring, Proposition 2.3 implies that there exist nonzero prime ideals P and Q of R with $P \cap Q = 0$. Note that as in the proof of Proposition 2.3, if one of the prime ideals P or Q is a minimal ideal of R, then $\mathbb{AG}(R)$ would be a star graph which does not contain a cycle of length 4.

(b)$\Longrightarrow$(c) Let

$$X := \{I \mid I \text{ is a nonzero ideal of } R \text{ contained in P}\}$$

and

$$Y := \{J \mid J \text{ is a nonzero ideal of } R \text{ contained in Q}\}.$$

If $|X| = m$ and $|Y| = n$, then as in the proof of Proposition 2.3, $\mathbb{AG}(R) \cong K_{m,n}$. We claim that m and n are infinite cardinals. By way of contrary, suppose that $m < \infty$. Thus P contains finitely many nonzero ideals of R and so there exists a minimal ideal I of R contained in P. Clearly, $M := ann_R(I)$ is a maximal ideal of R contained in Q, and hence $M = Q$. Therefore, $R = P \oplus Q$ and hence $P \cong R/Q$. Thus P is a minimal ideal of R and this is a contradiction.

(c)$\Longrightarrow$(a) It is clear. ■

Note that for the nonzero rings R_1 and R_2, the prime ideals of the ring $R_1 \times R_2$ are of the form $R_1 \times P_2$ and $P_1 \times R_2$ where P_i's are prime ideals of R_i's.

Corollary 2.5 Let R be a commutative reduced ring with nontrivial idempotents, then $\mathrm{gr}(\mathbb{AG}(R)) = 4$ if and only if $R = R_1 \times R_2$ where R_1 and R_2 are integral domains which are not fields.

Proof. Let $R = R_1 \times R_2$ where R_1 and R_2 are nonzero rings. Suppose that $\mathrm{gr}(\mathbb{AG}(R)) = 4$, then by Theorem 2.4, there exist nonzero prime ideals P and Q of R which are not minimal ideals such that $P \cap Q = 0$. According to the above fact about prime ideals of the ring $R = R_1 \times R_2$, we should have $P = R_1 \times 0$ and $Q = 0 \times R_2$, therefore, R_1 and R_2 are integral domains which are not fields. ■

In the following, we consider non-reduced rings R with $\mathrm{gr}(\mathbb{AG}(R)) = 4$. First we prove the next lemma.

Lemma 2.6 Let R be a commutative ring with $Nil(R) \neq 0$. If $\mathrm{gr}(\mathbb{AG}(R)) = 4$, then R has a nontrivial idempotent, i.e., we have $R = R_1 \times R_2$ for nonzero rings R_1 and R_2.

Proof. Let I ba a nonzero ideal of R with $I^2 = 0$. Since any two distinct nonzero ideals contained in I are adjacent and $\mathrm{gr}(\mathbb{AG}(R)) \neq 3$, there exists at most one nonzero ideal properly contained in I. so without of generality, we can assume that I is a minimal ideal of R. Let $M := ann_R(I)$, then M is a maximal ideal of R and $M \neq I$, otherwise, (R, M) would be a local ring and $\mathbb{AG}(R) \cong K_1$. Suppose that J and K are two nonzero distinct ideals of R such that $JK = 0$, $J \neq I$ and $K \neq I$. Since $JK \subseteq M$, one of the ideals J or K, say J must be contained in M and hence $IJ = 0$. Note that $IK \neq 0$, because there does not exist any triangle in $\mathbb{AG}(R)$. Thus $d(I, K) = 2$. Therefore, for any vertex L in $\mathbb{AG}(R)$, we have $d(I, L) = 0$, 1 or 2. Now put

$$X := \{J \in \mathbb{A}(R)^* \mid d(I, J){=}0 \text{ or } 2\}$$

and

$$Y := \{J \in \mathbb{A}(R)^* \mid d(I, J){=}1\}.$$

Then $\mathbb{A}(R)^* = X \cup Y$ and $X \cap Y = \emptyset$. It is easy to observe that $\mathbb{AG}(R)$ is a bipartite graph with two parts X and Y. Now let $P := \sum_{J \in X} J$ and $Q := \sum_{J \in Y} J$. Clearly, $Q = M$ and $I \subseteq P \cap Q$. We claim that $P \cap Q$ is a nil ideal of R. By the contrary, let $x \in P \cap Q$ and that x is not nilpotent. Since $x \in P$ and x is not nilpotent, there exists $J \in X$ such that $x \in J$ and $d(I, J) = 2$. Now the connectivity of $\mathbb{AG}(R)$ implies that for some ideal $K \in Y$, we have $JK = 0$ and so $(Rx)K = 0$. As x is not nilpotent and $K \in Y$, we have $Rx \neq K$ and $Rx \in X$. Since $x \in Q = M$ is not nilpotent, $(Rx)I = 0$ and thus $Rx \in Y$, we conclude that $Rx \in X \cap Y = \emptyset$, which is a contradiction. Therefore, $P \cap Q$ is a nil ideal of R and $P + Q = R$. Consequently $R/P \times R/Q$ and hence the ring $R/P \cap Q$ has a nontrivial idempotent. It is known that idempotents can be lifted modulo nil ideals, therefore, R has a nontrivial idempotent too. ■

Theorem 2.7 Let R be a commutative ring with $Nil(R) \neq 0$. Then the following statements are equivalent:

(a) $\mathrm{gr}(\mathbb{AG}(R)) = 4$
(b) $R \cong D \times S$, where D is an integral domain which is not a field and (S, M) is a local ring such that $M^2 = 0$ and M is a minimal ideal of S.
(c) $\mathbb{AG}(R) = \bar{K}_{n,2}$ for some infinite cardinal n.

Proof. (a)$\Longrightarrow$(b) Since $\mathrm{gr}(\mathbb{AG}(R)) = 4$ and $Nil(R) \neq 0$, Lemma 2.6 implies that R has a nontrivial idempotent, so $R = R_1 \times R_2$, for some nonzero rings R_1 and R_2. As $Nil(R) \neq 0$, there is a nonzero ideal I of R with $I^2 = 0$. Let $I := I_1 \times I_2$ where I_i is an ideal of R_i for $i = 1, 2$. If I_1 and I_2 are nonzero ideals, then $I_1 \times 0 - I_1 \times I_2 - 0 \times I_2 - I_1 \times 0$ would be a triangle in $\mathbb{AG}(R)$. Therefore, we can assume that $I_1 = 0$ and $I_2 \neq 0$. If there exists a nonzero ideal I' of R_2 distinct from I_2 such that $I' I_2 = 0$, then $0 \times I_2 - 0 \times I' - R_1 \times 0 - 0 \times I_2$ is a triangle in $\mathbb{AG}(R)$, but $\mathrm{gr}(\mathbb{AG}(R)) = 4$. We conclude that I_2 is a minimal ideal of R_2 and $I_2 = ann_{R_2}(I_2)$ is a maximal ideal of R_2, thus R_2 is a local ring and I_2 is the only nontrivial ideal of R_2. Now we show that R_1 is an integral domain, otherwise if K and J are two nonzero ideals in R_1 with $KJ = 0$, then $K \times 0 - J \times I_2 - 0 \times I_2 - K \times 0$ would be a triangle in $\mathbb{AG}(R)$. Note that R_1 is not a field, since there is a cycle of length 4 in $\mathbb{AG}(R)$.

(b)$\Longrightarrow$(c) Suppose that $R = D \times S$, where D is an integral domain which is not a field and (S, M) is a local ring such that $M^2 = 0$ and M is a minimal ideal of S. Consider the following three subsets of $\mathbb{A}(R)^*$.

$$X := \{I \times 0 \mid I \text{ is a nonzero ideal of } D\}$$

$$Y := \{0 \times S, 0 \times M\}$$

$$Z := \{I \times M \mid I \text{ is a nonzero ideal of } D\}.$$

Thus $\mathbb{A}(R)^* = X \cup Y \cup Z$ and every vertex in X is adjacent to every vertex in Y and also every vertex in Z is adjacent to only $o \times M$. If $|X| = n$, then $\mathbb{AG}(R) \cong \bar{K}_{n,2}$.

(c)$\Longrightarrow$(a) It is obvious. ■

As a consequence of Theorem 2.4 and Theorem 2.7, the complete bipartite graph $K_{m,n}$ with $m, n > 1$ can be realized as an annihilating-ideal graph $\mathbb{AG}(R)$ if and only if m and n are infinite cardinals. The following example exhibits such rings.

***Example* 2.8** Let F be a field and $R = \frac{F[x,y]}{\langle xy \rangle}$, then R is a reduced ring with two nonzero prime ideals $P = \frac{\langle x \rangle}{\langle xy \rangle}$ and $Q = \frac{\langle y \rangle}{\langle xy \rangle}$ such that $P \cap Q = 0$. We have $\mathrm{gr}(\mathbb{AG}(R)) = 4$ and $\mathbb{AG}(R) \cong K_{m,n}$ for infinite cardinals m and n.

3. Rings with $\mathrm{gr}(\mathbb{AG}(R)) = \infty$

The aim of this section is to study rings whose annihilating-ideal graph have no cycles. First we consider reduced rings.

Theorem 3.1 For a commutative reduced ring R, the following statements are equivalent:

(a) $\mathrm{gr}(\mathbb{AG}(R)) = \infty$
(b) $R \cong F \times D$ where F is a field and D is an integral domain.
(c) $\mathbb{AG}(R)$ is a star graph.

Proof. (a)$\Longrightarrow$(b) Suppose that $\mathrm{gr}(\mathbb{AG}(R)) = \infty$, then by Proposition 2.2, $\mathbb{AG}(R)$ is a bipartite graph. Since R is a reduced ring and there is no cycle in $\mathbb{AG}(R)$, Proposition 2.3 implies that there are nonzero prime ideals P and Q of R with $P \cap Q = 0$ and one of the ideals P or Q is a minimal ideal of R, say P. Note that $M := ann_R(P)$ is a maximal ideal of R. We have $PM = 0 \subseteq Q$ and $P \nsubseteq Q$, hence $M = Q$. Therefore, $R \cong (R/P) \times (R/Q)$, where R/P is an integral domain and R/Q is a field.

(b)$\Longrightarrow$(c)and (c)$\Longrightarrow$(a) are clear. ■

Now we characterize non-reduced rings R for which $\mathrm{gr}(\mathbb{AG}(R)) = \infty$. Two cases can be occurred depending on whether or not R has nontrivial idempotents.

Proposition 3.2 Let R be a commutative non-reduced ring with a nontrivial idempotent. The following statements are equivalent:

(a) $\mathrm{gr}(\mathbb{AG}(R)) = \infty$
(b) $R \cong F \times S$ where F is a field and (S, M) is a local ring such that $M^2 = 0$ and M is a minimal ideal of S.

(c) $\mathbb{AG}(R) \cong \bar{K}_{1,2}$

Proof. It is similar to the proof of Theorem 2.7. ■

Proposition 3.3 Let R be a commutative non-reduced ring with no nontrivial idempotent, then $\mathrm{gr}(\mathbb{AG}(R)) = \infty$ if and only if $\mathbb{AG}(R)$ is a star graph or a singleton.

Proof. If $\mathbb{AG}(R)$ is a star graph or a singleton, clearly, $\mathrm{gr}(\mathbb{AG}(R)) = \infty$. Now suppose that $\mathrm{gr}(\mathbb{AG}(R)) = \infty$. There is a nonzero ideal I of R with $I^2 = 0$. As in the proof of Lemma 2.6 , we can assume that I is a minimal ideal of R. Thus $M := ann_R(I)$ is a maximal ideal of R. If $I = M$, then $\mathbb{AG}(R)$ is a singleton. So suppose that $I \neq M$, we claim that $\mathbb{AG}(R)$ is a star graph with center I. By the contrary, suppose that K and L are two distinct vertices in $\mathbb{AG}(R)$ such that $KL = 0$ and $K \neq I \neq L$. Since $KL = 0 \subseteq M$, we have $K \subseteq M$ or $L \subseteq M$. Assume that $K \subseteq M$, thus $IK = 0$. Now if $0 \neq J \subsetneq K$, then $I - K - L - J - I$ would be a cycle of length 3 or 4 but $\mathrm{gr}(\mathbb{AG}(R)) = \infty$. So K must be a minimal ideal of R. Since R has no nontrivial idempotents, $K^2 = 0$ and hence $I - K - (K + I) - I$ is a cycle in $\mathbb{AG}(R)$. Therefore, we get a contradiction. Consequently, $\mathbb{AG}(R)$ is a star graph with center I. ■

In the sequel, we present some examples of rings R for which $\mathbb{AG}(R)$ is a star graph. But first we need the following lemma.

Lemma 3.4 Let R be a commutative ring with $|\mathbb{A}(R)^*| > 1$. If R has a minimal ideal P such that P is a prime ideal of R, then $\mathbb{AG}(R)$ is a star graph.

Proof. If I and J are nonzero ideals of R with $IJ = 0$, then $I = P$ or $J = P$. Thus P is the only vertex in $\mathbb{AG}(R)$ which is adjacent to every other vertices. ■

Note that the converse of Lemma 3.4 is not true in general. For the counterexample, see the following.

Example **3.5** (a) The annihilating-ideal graph of the rings $\mathbb{Z}_{p^4}$ where p is a prime number and $\frac{K[x]}{\langle x^4 \rangle}$ where K is a field, are isomorphic to $K_{1,2}$.

(b) Let $R = \frac{F[x,y]}{\langle x^2,xy \rangle}$ where F is a field. Then $\frac{\langle x \rangle}{\langle x^2,xy \rangle}$ is a minimal ideal of R which is a prime ideal. Then by Lemma 3.4, $\mathbb{AG}(R)$ is an infinite star graph.

(c) Let $R = \frac{F[x,y]}{\langle x^2,y^2 \rangle}$ where F is a field of characteristic 2. The annihilating-ideal graph of R is a star graph. If $\bar{x} = x + \langle x^2, y^2 \rangle$ and $\bar{y} = y + \langle x^2, y^2 \rangle$, then $\langle \bar{x}\bar{y} \rangle$ is the vertex which is adjacent to every other vertices. Note that $\langle \bar{x}\bar{y} \rangle$ is a minimal ideal of R which is not a prime ideal and $\mathbb{AG}(R)$ is a finite star graph if and only if F is a finite field.

As an application of our results about $\mathrm{gr}(\mathbb{AG}(R))$, we can obtain $\mathrm{gr}(\mathbb{AG}(R[x]))$ in terms of $\mathrm{gr}(\mathbb{AG}(R))$.

Theorem 3.6 Let R be a commutative ring which is not an integral domain.

(a) If R is non-reduced, then $\mathrm{gr}(\mathbb{AG}(R[x])) = 3$
(b) If R is reduced and $\mathrm{gr}(\mathbb{AG}(R)) = 3$, then $\mathrm{gr}(\mathbb{AG}(R[x])) = 3$.
(c) If R is reduced and $\mathrm{gr}(\mathbb{AG}(R)) \neq 3$, then $\mathrm{gr}(\mathbb{AG}(R[x])) = 4$.

Similar results hold for $\mathbb{AG}(R[[x]])$.

Proof. (a) Suppose that R is a non-reduced ring, then there is a nonzero ideal I of R with $I^2 = 0$. Now consider the cycle $I[x] - xI[x] - x^2I[x] - I[x]$ in $\mathbb{AG}(R[x])$ which implies that $\mathrm{gr}(\mathbb{AG}(R[x])) = 3$.

(b)Assume that R is a reduced ring with $\mathrm{gr}(\mathbb{AG}(R)) = 3$. So there are three distinct nonzero ideals I, J and K in R such that $I - J - K - I$ is a cycle in $\mathbb{AG}(R)$. Obviously, $I[x] - J[x] - K[x] - I[x]$ is a triangle in $\mathbb{AG}(R[x])$ and hence $\mathrm{gr}(\mathbb{AG}(R[x])) = 3$.

(c) Suppose that R is a reduced ring with $\mathrm{gr}(\mathbb{AG}(R)) \neq 3$, so by Proposition 2.2, $\mathbb{AG}(R)$ is a bipartite graph. Now Proposition 2.3 implies that $P \cap Q = 0$, for some nonzero prime ideals P and Q of R. Observe that $P[x]$ and $Q[x]$ are two nonzero prime ideals of $R[x]$ with $P[x] \cap Q[x] = 0$. Again by Proposition 2.3 , $\mathbb{AG}(R[x])$ is a bipartite graph, and $P[x]-Q[x]-xP[x]-xQ[x]-P[x]$ is a cycle of length 4 in $\mathbb{AG}(R[x])$. Therefore, $\mathrm{gr}(\mathbb{AG}(R[x])) = 4$. ■

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