Journal of
Linear and Topological Algebra
Vol. 04, *No.* 03, 2015, 165- 172

An algorithm for determining common weights by concept of membership function

S. Saati[a*], N. Nayebi[a]

[a] *Department of Mathematics, North Tehran Branch,*
Islamic Azad University, Tehran, Iran.

Received 29 April 2015; Revised 17 November 2015; Accepted 15 December 2015.

Abstract. Data envelopment analysis (DEA) is a method to evaluate the relative efficiency of decision making units (DMUs). In this method, the issue has always been to determine a set of weights for each DMU which often caused many problems. Since the DEA models also have the multi-objective linear programming (MOLP) problems nature, a rational relationship can be established between MOLP and DEA problems to overcome the problem of determining weights. In this study, a membership function was defined base on the results of *CCR* model and cross efficiency, and by using this membership function in a proposed model, we obtained a common set of weights for all DMUs. Finally, by solving a sample problem, the proposed algorithm was explained.

Keywords: Data envelopment analysis (DEA), Cross efficiency, Membership function, Common set of weights, Multi-objective programming problem.

1. Introduction

Charnes, Cooper, and Rhodes [2] presented CCR model to evaluate the efficiency of DMUs with several homogeneous inputs and outputs. In this model, a set of weights for each DMU is calculated and the CCR model determines the weights in such a way that the highest efficiency is obtained.

One of the weaknesses of CCR models is that it makes zero the weights of the DMUs weight are not in our interest. It means when the input increases and the related output decreases, the model considered the related clause to be equal zero in order to obtain the maximum efficiency. Because of these problems and also due to the fact that the basic

*Corresponding author.
E-mail address: s_saatim@iau-tnb.ac.ir (S. Saati).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

models are not able lo solve them, the calculation of common weights has attributed a lot of attention. The idea of common weights was presented by [5] for the first time and then it was developed by [6].

When it talks about a set of common weights for a set of DMUs, the issue is that how to determine these weights in order to achieve better evaluation for different DMUs. The evaluation of DMUs has investigated from different perspectives. Therefore, actually we faced a multi-objective programming problem.

Charnes and Cooper [1] had a major effect on the development of multi-objective linear programming. Some researchers also calculated a set of common weights by presenting a model and solving it ([3], [7])

One way for limiting weight changes among DMUs is using the cross efficiency method. By this method, the efficiency of each unit is calculated based on the weights of different units. The obtained efficiency by this method has more exact results in comparison with absolute efficiency method.

In this paper, the efficiency of DMUs was calculated using the DEA models; furthermore, the upper and lower bounds were determined by cross-efficiency method. Then, by using the results of the cross-efficiency in a linear programming model and applying fuzzy concept, a set of common weights was found.

In the first part of the present paper, DEA and common weights were elaborated briefly. Other required definitions were described in the next part. The proposed algorithm was presented in Part three. Moreover, the evaluation of the proposed algorithm through solving a numerical example is presented in Part five. Finally, the last part of the paper is dedicated to conclusions.

2. Initial definitions

Before presenting the algorithm, it is necessary to explain some definitions and basic concepts. It is also required to explain the relationships between these concepts and how they were used in present paper to achieve the goal (determining of common set of weights).

2.1 *Efficiency and DEA*

For the first time, DEA method was presented by Charnes, Cooper, and Rhodes [2] with the following model:

Consider n DMUs which are evaluated by m inputs and s outputs. Assume that x_{ij} and y_{rj} are their inputs and outputs values for i =1,...,m, r =1,...,s and j =1,...,n.

The seminal programming statement for the (input oriented) CCR model is:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rp} \\
\text{s.t}: \quad & \sum_{i=1}^{m} v_i x_{ip} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \qquad \forall\ j, \\
& u_r,\ v_i \geqslant 0 \qquad \forall\ r,\ i.
\end{aligned} \tag{1}
$$

In this model, v_i is the weight of input i and u_r is the weight of output r. v_i and u_r are determined in such a way that creates maximum efficiency for the unit. In this model, p is the index of DMU_p.

Despite the fact that CCR model provides applicable and usable information, the compensative nature of the model, necessitates to makes zero some weights in order to achieve the best efficiency and to avoid the equality of weights for the similar factor in DMUs ([2]).

2.2 *Cross efficiency*

Cross efficiency was proposed by [8] for the first time. From the time being introduced, many articles have been presented to develop and use this concept ([9]).

A set of weights is obtained from measuring the efficiency of unit p. If we calculate the efficiencies of other units by using these weights, we will have the table of cross efficiency.

Table 1. The table of cross efficiency

DMU	DMU_1	DMU_2	$\cdots$	DMU_n
DMU_1	θ_{11}	θ_{12}	$\cdots$	θ_{1n}
DMU_2	θ_{21}	θ_{22}	$\cdots$	θ_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
DMU_n	θ_{n1}	θ_{n2}	$\cdots$	θ_{nn}

In Table 1, the CCR model is run for each DMU. Considering the obtained weights, the efficiency of other units are calculated. θ_{ij} is the efficiency of DMU_j by using optimum weights of DMU_i. Since we used the obtained weights from Model 1 for solving MOLP problems, it seems necessary first to explain MOLP.

2.3 *Multi-objective linear programming (MOLP)*

The general form of a multi-objective linear programming problem is as follow:

$$
\begin{aligned}
& \max \{f_1(x), ..., f_k(x)\} \\
& \text{s.t}: \quad x \in X
\end{aligned}
$$

In this problem, the functions $f_1(x), ..., f_k(x)$ are k linear functions and X is the

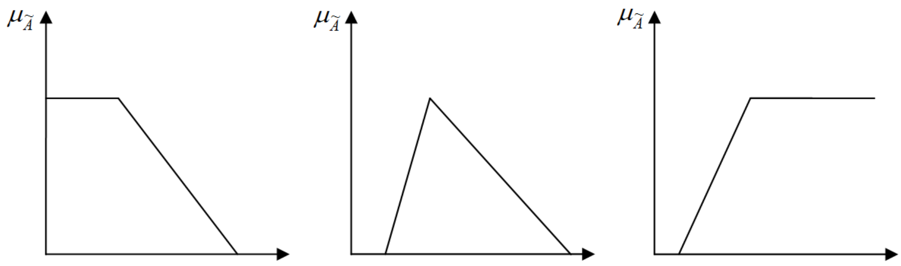

Figure 1. Types of linear membership function

feasible region. There are various methods for solving multi-objective linear programming problems [4]. Due to the nature of multi-objective linear problems, it may be impossible to obtain optimum answer. Hence, all methods are resolved to find a satisfactory answer. Since satisfaction is a relative adjective, so it seems that fuzzy logic is more suitable for solving these problems. Because of the conceptual compliance of this method with the nature of multi-objective linear programming problems, the concept of fuzzy logic is explained in following section.

2.4 *Fuzzy set*

For each fuzzy set $\widetilde{A}$, we define a crisp set X and a membership function. In this case, the fuzzy set $\widetilde{A}$ is represented as follows:

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x))|x \in X, \mu_{\widetilde{A}}(x) : X \to [0, 1]\}$$

It is clear that the basis of fuzzy logic is a membership function which is defined for the set.

This function shows how a member belongs to the set. The simplest form of this function is a linear form. Figure 1 shows some examples of membership functions in a linear mode.

3. Proposed algorithm

In the last two decades, researchers paid much attention to decision making models. In these decisions, some objectives which might sometimes be opposite are considered together. Since the related algorithms are based on mathematical logic, and highly compatible with the way of thinking and mental processes in human being, they are very efficient and their usage caused solving many decision making problems.

Many solutions have been proposed for solving multi-objective problems, but fuzzy solution is more desirable due to its certain fuzzy answer. In this paper, we used this method for finding a satisfactory answer.

In the proposed algorithm, we calculated the efficiency using data envelopment analysis; then for solving the multi-objective linear problems, we found the minimum and maximum efficiencies among the calculated efficiencies and by using the fuzzy method we wrote a membership function. To implement this method, we applied the following algorithm.

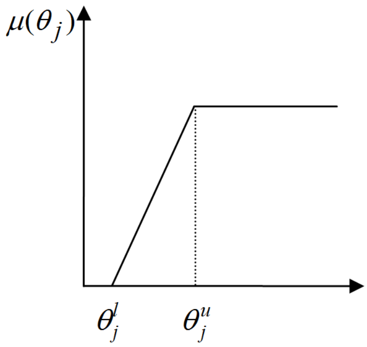

Figure 2. Membership function for DMUs efficiencies

Step1) by using CCR model (Model 1) we obtain the efficiencies of DMUs. This model was written for DMU_p.

Step2) when we solve CCR model, a set of weights is obtained for DMU_p. If we use these weights to calculate the efficiencies of other DMUs, maximum and minimum efficiency for each unit are determined by using cross efficiency and we define:

$$\theta_j^l = \min_{1\leqslant j\leqslant n} \theta_{ij} \qquad \theta_j^u = \max_{1\leqslant j\leqslant n} \theta_{ij}$$

Step3) Regarding to θ_j^l and θ_j^u, we define membership function $\mu(\theta_j)$ as follow:

$$\mu(\theta_j) = \begin{cases} 1 & \theta_j \geqslant \theta_j^u \\ \frac{\theta_j-\theta_j^l}{\theta_j^u-\theta_j^l} & \theta_j^l < \theta_j < \theta_j^u \\ 0 & \theta_j \leqslant \theta_j^l \end{cases}$$

where $\theta_j = \sum_{r=1}^{s} u_r y_{rj} \ \ (j = 1, ..., n)$. This membership function was illustrated in Figure 2.

Step4) In this step, we try to maximize the values of $\mu(\theta_j)$. It means that we let each objective functions reaches its optimum value as much as possible. However, all CCR (Model 1) restrictions are valid for all DMUs, so we proposed following model:

$$\begin{aligned} \max \quad & \lambda & & (2)\\ \text{s.t}: \quad & \lambda \leqslant \mu(\theta_j) & \forall\ j \\ & \sum_{i=1}^{m} v_i x_{ij} \leqslant 1 & \forall\ j \\ \sum_{r=1}^{s} u_r y_{rj} - & \sum_{i=1}^{m} v_i x_{ij} \leq 0 & \forall\ j, \\ & u_r,\ v_i \geqslant 0 & \forall\ r,\ i. \end{aligned}$$

In the proposed algorithm, a max-min problem is solved. In fact, we were after an answer that with regard to it, all membership functions reach their greatest value. To solve the problem, Model 2 was used. In this model, $\lambda \leqslant \mu(\theta_j) \ \ (j = 1, ..., n)$ maximize the minimum value of membership functions.

In Model 2, λ shows the maximum satisfaction of DMUs derived from the obtained weights of inputs and outputs.

By solving this model using linear programming software, a set of common weights is obtained.

4. Numerical example

In Table 2, we have 5 DMUs with 2 inputs and 1 output. The data of this table was extracted from [?].

In Table 2, the first column indicates the DMUs, columns 2 and 3 indicate input 1 and 2 respectively and column 4 shows the output. The last column (column 5) includes the units' efficiencies which obtained from Model 1. With regard to Model 1, D_5 is not efficient.

Table 2. Data and efficiencies of DMUs

DMU	I_1	I_2	O_1	θ
D_1	2	12	1	1.0000
D_2	2	8	1	1.0000
D_3	5	5	1	1.0000
D_4	10	4	1	1.0000
D_5	10	6	1	0.7500

In Table 3, the second and third columns include the weights of the first and the second inputs respectively and the next column shows the weights of generated outputs. As it can be observed, when CCR model (Model 1) is used to calculate the efficiency of the units, some weights become zero; moreover, different weights are obtained for each unit.

Table 3. Factor weights of DMUs

DMU	v_1	v_2	u_1
D_1	0.5000	0.0000	1.0000
D_2	0.5000	0.0000	1.0000
D_3	0.1000	0.1000	1.0000
D_4	0.0000	0.2500	1.0000
D_5	0.0250	0.1250	0.7500

In Table 4, the efficiencies of the under evaluation units according to the cross-efficiency is shown.

Table 4. The Cross Efficiencies Matrix

DMU	D_1	D_2	D_3	D_4	D_5
D_1	1.0000	1.0000	0.4000	0.2000	0.2000
D_2	1.0000	1.0000	0.4000	0.2000	0.2000
D_3	0.7143	1.1000	1.0000	0.7143	0.6250
D_4	0.3333	0.5000	0.8000	1.0000	0.6667
D_5	0.4839	0.7143	1.0000	1.0000	0.7500
θ_j^l	0.3333	0.5000	0.4000	0.2000	0.2000
θ_j^u	1.0000	1.0000	1.0000	1.0000	0.7500

Using data presented in Table 4 and solving Model 2, fuzzy answer $\lambda = 0.2960$ with common weights was obtained. Table 5 shows the results.

Table 5. The Input and Output Weights

Input weights	Output weights
v_1=0.0556	u_1 =0.6481
v_2=0.0741	

Considering the above weights and using Model 1, the efficiencies of the units were calculated and presented in Table 6.

Table 6. The Efficiencies by Model 2

DMU	Efficiency
D_1	0.6480
D_2	0.9210
D_3	1.0000
D_4	0.7610
D_5	0.6480

Comparing the results shown in Table 6 with the results obtained from Saati's [7] study, the following items can be seen.

- In Table 6, the D_3 has an efficiency equal one, while in [7] D_2 is efficient.

- In Table 6, efficiencies of 3 units which were obtained using Model 2 weights, are higher. This can be interpreted as increasing the units' satisfactions.

- In [7], the upper and lower bounds have been determined and the maximum efficiency within the interval of these bounds has been calculated, where as in this paper a set of common weights was obtained and the efficiencies of the units were calculated considering these weights. This finding indicates that the method used in the present paper is more precise.

5. Conclusions

In the proposed method, not only the relationship between DEA and MOLP problems was described but also a method for solving multi-objective problems was presented in which the efficient unit can be determined by calculating a set of common weights. It should be noted that Model 1 also has alternative optimal solutions, which led to achieving different results in Table 6. The existence of other optimal solutions for linear programming problems, especially in data envelopment analysis models are amongst the issues which can cause problems. To solve these problems in data envelopment analysis, a common set of weights is used. In this paper, we used fuzzy concept in order to reduce the impact of this type of solutions. As the provided membership function and Model 2 indicate, the weights of inputs and outputs are calculated with a same degree of membership. In fact, this degree of membership reflects the impact of other optimal solutions in DEA models. Furthermore, using fuzzy concept, the obtained answer was more desirable as compared with the situation where only CCR model is used to solve the problem. This is due to the fact that for solving problems using fuzzy method, all objective functions have the same level of satisfaction from the obtained result. This is the advantage of using this method compared to other methods. Additionally, using this method, we can achieve a set of common weights that all units have the same satisfaction level in reaching the optimum value of the objective function, which is an important issue in multi-objective problems.

References

[1] A. Charnes, W. W. Cooper, Management models and industrial applications of linear programming, John Wiley, New York, 1961.
[2] A. Charnes, W. W. Cooper and E. Rhodes, Measuring the efficiency of decision making units, European journal of operational research. (1978), 2, 429-444.
[3] G. R. Jahanshahloo, A. Memariani, F. Hosseinzadeh Lotfi and H. Z. Rezai, A note on some of DEA models and finding efficiency and complete ranking using common set of weights, Applied mathematics and computation. (2005), 166, 265-281.
[4] J. Lu, G. Zhang, D. Ruan and F. Wu, Multi-objective group decision making methods software and applications with fuzzy set techniques. Imperial collage press. 2007.
[5] Y. Roll, W. Cook and B. Golany, Controlling factor weights in data envelopment analysis. IIE Trans, (1991), 23, 2-9.
[6] T. Roll, B. Golany, Alternate methods of treating factor weights in data envelopment analysis, Omega, (1993), 21, 99-109.
[7] M. S. Saati, Determining a common set of weights in DEA by solving a linear programming, Journal of industrial engineering international, (2008), 6, 51-56.
[8] T. R. Sexton, R. H. Silkman and A. J. Hogan, Data envelopment analysis: critique and extensions. In: R.H. Silkman (Ed.), Measuring efficiency: An assessment of data envelopment analysis. San Francisco, CA, Jossey-Bass, 1986.
[9] Y. M. Wang, K. P. Chin, Some alternative models for DEA cross efficiency evaluation, Production economics, (2010), 128, 332-33.
[10] Y. M. Wand, K. S. Chin and Y. Luo, Cross efficiency evaluation based on ideal and anti-ideal decision making units, Expert systems with applications, (2011), 38, 10312-10319.