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On weakly eR-open functions

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Abstract. The main goal of this paper is to introduce and study a new class of function via the notions of e-θ-open sets and e-θ-closure operator which are defined by Özkoç and Aslım [10] called weakly eR-open functions and e-θ-open functions. Moreover, we investigate not only some of their basic properties but also their relationships with other types of already existing topological functions.

Keywords: e-Closed set, e-θ-open set, weakly eR-open function, e-θ-open function.

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1. Introduction and Preliminaries

Throughout the present paper, X and Y always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let X be a topological space and A a subset of X. The closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. The family of all closed sets of X is denoted $C(X)$. A subset A is said to be regular open [12] (resp. regular closed [12]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A point $x \in X$ is said to be δ-cluster point [13] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neigbourhood U of x. The set of all δ-cluster points of A is called the δ-closure [13] of A and is denoted by $cl_\delta(A)$. If $A = cl_\delta(A)$, then A is called δ-closed [13], and the complement of a δ-closed set is called δ-open [13]. A subset A is called semiopen [5] (resp. b-open [1], e-open [4], preopen [7], α-open [8]) if $A \subset cl(int(A))$ (resp. $A \subset cl(int(A)) \cup int(cl(A))$, $A \subset cl(int_\delta(A)) \cup int(cl_\delta(A))$, $A \subset int(cl(A))$,

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$A \subset int(cl(int(A)))$). The complement of a semiopen (resp. b-open, e-open, preopen, α-open) set is called semiclosed [5](resp. b-closed [1], e-closed [4], preclosed [7], α-closed [8]). The intersection of all e-closed sets of X containing A is called the e-closure [4] of A and is denoted by e-$cl(A)$. The union of all e-open sets of X contained in A is called the e-interior [4] of A and is denoted by e-$int(A)$. A subset A is said to be e-regular [10] if it is e-open and e-closed.

A point x of X is called a b-θ-cluster [11] (e-θ-cluster [10], θ-cluster [13]) point of A if $bcl(U) \cap A \neq \emptyset$ (e-$cl(U) \cap A \neq \emptyset$, $cl(U) \cap A \neq \emptyset$) for every b-open (e-open, open) set U of X containing x, respectively. The set of all b-θ-cluster (e-θ-cluster, θ-cluster) points of A is called the b-θ-closure [11] (e-θ-closure [10], θ-closure [13]) of A and is denoted by $bcl_\theta(A)$ (e-$cl_\theta(A)$, $cl_\theta(A)$), respectively. A subset A is said to be b-θ-closed [11] (e-θ-closed [10], θ-closed [13]) if $A = bcl_\theta(A)$ ($A = e$-$cl_\theta(A)$, $A = cl_\theta(A)$), respectively. The complement of a b-θ-closed (e-θ-closed, θ-closed) set is called a b-θ-open [11] (e-θ-open [10], θ-open [13]) set. A point x of X said to be a b-θ-interior [11] (e-θ-interior [10], θ-interior [13]) point of a subset A, denoted by $bint_\theta(A)$ (e-$int_\theta(A)$, $int_\theta(A)$), if there exists a b-regular (e-regular, regular) set U of X containing x such that $U \subset A$, respectively. The family of all e-open (resp. e-closed, e-regular, b-θ-open, e-θ-open, b-θ-closed, e-θ-closed) subsets of X is denoted by $eO(X)$ (resp. $eC(X)$, $eR(X)$, $B\theta O(X)$, $e\theta O(X)$, $B\theta C(X)$, $e\theta C(X)$). The family of all e-open (e-closed, e-regular, b-θ-open, e-θ-open, b-θ-closed, e-θ-closed) sets of X containing a point x of X is denoted by $eO(X,x)$ (resp. $eC(X,x)$, $eR(X,x)$, $B\theta O(X,x)$, $e\theta O(X,x)$, $B\theta C(X)$, $e\theta C(X,x)$). Also it is noted in [10] that

$$e\text{-regular} \Rightarrow e\text{-}\theta\text{-open} \Rightarrow e\text{-open}.$$

We shall use the well-known accepted language almost in the whole of the article.

Definition 1.1 A function $f : (X,\tau) \to (Y,\sigma)$ is called:
(a) contra e-θ-open if $f(U)$ is e-θ-closed in Y for each open set U of X.
(b) contra e-θ-closed if $f(U)$ is e-θ-open in Y for each closed set U of X.
(c) strongly continuous [6] if for every subset A of X, $f(cl(A)) \subset f(A)$.
(d) weakly BR-open [2] if $f(U) \subset bint_\theta(f(cl(U)))$ for each open set U of X.

2. Weakly *eR*-open Functions

In this section, we define the concept of weakly eR-open and investigate some basic properties of them.

Definition 2.1 A function $f : X \to Y$ is said to be weakly eR-open if $f(U) \subset e$-$int_\theta(f(cl(U)))$ for each open set U of X.

Definition 2.2 A function $f : X \to Y$ is said to be e-θ-open if $f(U)$ is e-θ-open in Y for each open set U of X.

It is clear to see that every e-θ-open function is a weakly eR-open. However, a weakly eR-open function need not be e-θ-open as shown by the following example.

***Example* 2.3** Let $X = \{a,b,c,d\}$ and

$$\tau = \{\emptyset, X, \{a,d\}\} \quad \text{and} \quad \sigma = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}\}.$$

The identity function $f : (X,\tau) \to (X,\sigma)$ is weakly eR-open, but it is not e-θ-open.

The notions of weakly eR-open function and weakly BR-open function are independent as shown by the following examples.

***Example* 2.4** Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. The identity function $f : (X, \tau) \to (X, \tau)$ is weakly eR-open, but it is not weakly BR-open.

***Example* 2.5** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$. $f = \{(a, d), (b, c), (c, b), (d, d)\}$ is weakly BR-open, but it is not weakly eR-open.

Lemma 2.6 [10] Let A be a subset of a space X. Then:
(1) $e\text{-}cl_\theta(A) = \cap\{V|(A \subset V)(V \in eR(X))\}$.
(2) $x \in e\text{-}cl_\theta(A)$ iff $A \cap U \neq \emptyset$ for each e-regular set U of X containing x.
(3) $e\text{-}cl_\theta(A)$ is e-θ-closed.
(4) Any intersections of e-θ-closed sets is e-θ-closed and any union of e-θ-open sets is e-θ-open.
(5) A is e-θ-open in X if and only if for each $x \in A$ there exists an e-regular set U containing x such that $x \in U \subset A$.

Theorem 2.7 Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent:
(a) f is weakly eR-open,
(b) $f(int_\theta(A)) \subset e\text{-}int_\theta(f(A))$ for every subset A of X,
(c) $int_\theta(f^{-1}(B)) \subset f^{-1}(e\text{-}int_\theta(B))$ for every subset B of Y,
(d) $f^{-1}(e\text{-}cl_\theta(B)) \subset cl_\theta(f^{-1}(B))$ for every subset B of Y,
(e) $f(int(F)) \subset e\text{-}int_\theta(f(F))$ for each closed subset F of X,
(f) $f(int(cl(U))) \subset e\text{-}int_\theta(f(cl(U)))$ for each open subset U of X,
(g) $f(U) \subset e\text{-}int_\theta(f(cl(U)))$ for every regular open subset U of X,
(h) $f(U) \subset e\text{-}int_\theta(f(cl(U)))$ for every α-open subset U of X,
(i) For each $x \in X$ and each open set U of X containing x, there exists an e-θ-open set V of Y containing $f(x)$ such that $V \subset f(cl(U))$.

Proof. $(a) \Rightarrow (b)$: Let A be any subset of X and $x \in int_\theta(A)$.

$$\begin{aligned} x \in int_\theta(A) &\Rightarrow (\exists U \in \mathcal{U}(x))(x \in U \subset cl(U) \subset A) \\ &\left.\begin{array}{r}\Rightarrow (\exists U \in \mathcal{U}(x))(f(x) \in f(U) \subset f(cl(U)) \subset f(A)) \\ f \text{ is weakly } eR\text{-open}\end{array}\right\} \Rightarrow \\ &\Rightarrow f(U) \subset e\text{-}int_\theta(f(cl(U))) \subset e\text{-}int_\theta(f(A)) \\ &\Rightarrow f(x) \in e\text{-}int_\theta(f(A)) \\ &\Rightarrow x \in f^{-1}(e\text{-}int_\theta(f(A))). \end{aligned}$$

$(b) \Rightarrow (c)$: Let B be any subset of Y.

$$\left.\begin{array}{r} B \subset Y \Rightarrow f^{-1}(B) \subset X \\ (b) \end{array}\right\} \begin{array}{l} \Rightarrow f\left(int_\theta(f^{-1}(B))\right) \subset e\text{-}int_\theta(f(f^{-1}(B))) \subset e\text{-}int_\theta(B) \\ \Rightarrow int_\theta(f^{-1}(B)) \subset f^{-1}(e\text{-}int_\theta(B)). \end{array}$$

$(c) \Rightarrow (d)$: Let B be any subset of Y.

$$\left.\begin{array}{r} B \subset Y \Rightarrow Y \setminus B \subset Y \\ (c) \end{array}\right\} \begin{array}{l} \Rightarrow int_\theta(f^{-1}(Y \setminus B)) \subset f^{-1}(e\text{-}int_\theta(Y \setminus B)) \\ \Rightarrow int_\theta(X \setminus f^{-1}(B)) \subset f^{-1}(Y \setminus e\text{-}cl_\theta(B)) \\ \Rightarrow X \setminus cl_\theta(f^{-1}(B)) \subset X \setminus f^{-1}(e\text{-}cl_\theta(B)) \\ \Rightarrow f^{-1}(e\text{-}cl_\theta(B)) \subset cl_\theta(f^{-1}(B)). \end{array}$$

$(d) \Rightarrow (e)$: Let F be any closed set of X.

$$\left.\begin{array}{r} F \in C(X) \Rightarrow Y \setminus f(F) \subset Y \\ (d) \end{array}\right\} \Rightarrow$$
$$\Rightarrow f^{-1}(e\text{-}cl_\theta(Y \setminus f(F))) \subset cl_\theta(f^{-1}(Y \setminus f(F))) = cl_\theta(X \setminus f^{-1}(f(F))) \subset cl_\theta(X \setminus F)$$
$$\Rightarrow f^{-1}(Y \setminus e\text{-}int_\theta(f(F))) \subset cl_\theta(X \setminus F) = X \setminus int_\theta(F)$$
$$\left.\begin{array}{r} \Rightarrow X \setminus f^{-1}(e\text{-}int_\theta(f(F))) \subset X \setminus int_\theta(F) \\ F \in C(X) \Rightarrow int_\theta(F) = int(F) \end{array}\right\} \Rightarrow f\,(int(F)) \subset e\text{-}int_\theta(f(F)).$$

$(e) \Rightarrow (f)$, $(f) \Rightarrow (g)$: Obvious.
$(g) \Rightarrow (h)$: Let U be any α-open set of X.

$$\left.\begin{array}{r} U \in \alpha O(X) \Rightarrow (U \subset int(cl(int(U))))(int(cl(int(U))) \in RO(X)) \\ (g) \end{array}\right\} \Rightarrow$$
$$\begin{array}{rl} \Rightarrow f(U) \subset f(int(cl(int(U)))) & \subset e\text{-}int_\theta(f(cl(int(cl(int(U)))))) \\ & = e\text{-}int_\theta(f(cl(int(U)))) \subset e\text{-}int_\theta(f(cl(U))). \end{array}$$

$(h) \Rightarrow (i)$: Straightforward.
$(i) \Rightarrow (a)$: Let U be an open set in X and $y \in f(U)$.

$$\left.\begin{array}{r} (U \in \tau)(y \in f(U)) \\ (i) \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists V \in e\theta O(Y, y))(V \subset f(cl(U))) \\ y \in V \subset e\text{-}int_\theta(f(cl(U))) \end{array}\right\} \Rightarrow$$
$$\Rightarrow f(U) \subset e\text{-}int_\theta(f(cl(U))).$$ ■

Theorem 2.8 Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective function. Then the following statements are equivalent:
(a) f is weakly eR-open,
(b) For each $x \in X$ and each open set U of X containing x, there exists an e-regular set V containing $f(x)$ such that $V \subset f(cl(U))$,
(c) $e\text{-}cl_\theta(f\,(int(cl(U)))) \subset f\,(cl(U))$ for each subset U of X,
(d) $e\text{-}cl_\theta(f\,(int(F))) \subset f\,(F)$ for each regular closed subset F of X,
(e) $e\text{-}cl_\theta(f\,(U)) \subset f\,(cl(U))$ for each open subset U of X,
(f) $e\text{-}cl_\theta(f\,(U)) \subset f\,(cl(U))$ for each preopen subset U of X,
(g) $f\,(U) \subset e\text{-}int_\theta(f\,(cl(U)))$ for each preopen subset U of X,
(h) $f^{-1}\,(e\text{-}cl_\theta(B)) \subset cl_\theta(f^{-1}\,(B))$ for each subset B of Y,
(i) $e\text{-}cl_\theta(f\,(U)) \subset f\,(cl_\theta(U))$ for each subset U of X,
(j) $e\text{-}cl_\theta(f\,(int(cl_\theta(U)))) \subset f\,(cl_\theta(U))$ for each subset U of X.

Proof. $(a) \Rightarrow (b)$: Let $x \in X$ and U be any open subset of X containing x.

$$\left.\begin{array}{r} x \in U \in \tau \\ (a) \end{array}\right\} \Rightarrow \left.\begin{array}{r} f(x) \in f\,(U) \subset e\text{-}int_\theta(f\,(cl(U))) \in e\theta O(Y, f(x)) \\ \text{Lemma 2.6(5)} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists V \in eR(Y, f(x)))(V \subset e\text{-}int_\theta(f\,(cl(U))) \subset f\,(cl(U))).$$

$(b) \Rightarrow (c)$: Let $x \in X$ and $U \subset X$.

$$\begin{aligned} f(x) \in Y \setminus f(cl(U)) = f(X \setminus cl(U)) &\Rightarrow x \in X \setminus cl(U) \\ &\Rightarrow (\exists G \in \mathcal{U}(x))(G \cap U = \emptyset) \\ &\left.\begin{array}{r}\Rightarrow (\exists G \in \mathcal{U}(x))(cl(G) \cap int(cl(U)) = \emptyset) \\ (b)\end{array}\right\} \Rightarrow \\ &\Rightarrow (\exists V \in eR(Y, f(x)))(V \subset f(cl(G))) \\ &\Rightarrow (\exists V \in eR(Y, f(x)))(V \cap f(int(cl(U))) = \emptyset) \\ &\Rightarrow f(x) \notin e\text{-}cl_\theta(f(int(cl(U)))) \\ &\Rightarrow f(x) \in X \setminus e\text{-}cl_\theta(f(int(cl(U)))). \end{aligned}$$

$(c) \Rightarrow (d)$: Let F be any regular closed set of X.

$$\left.\begin{array}{r} F \in RC(X) \Rightarrow e\text{-}cl_\theta(f(int(F))) = e\text{-}cl_\theta(f(int(cl(int(F))))) \\ (c) \end{array}\right\} \Rightarrow$$
$$\Rightarrow e\text{-}cl_\theta(f(int(F))) \subset f(cl(int(F))) = f(F).$$

$(d) \Rightarrow (e)$: Let U be any open subset of X.

$$\left.\begin{array}{r} (U \in \tau)(cl(U) \in RC(X)) \\ (d) \end{array}\right\} \Rightarrow e\text{-}cl_\theta(f(U)) \subset e\text{-}cl_\theta(f(int(cl(U)))) \subset f(cl(U)).$$

$(e) \Rightarrow (f)$: Let U be any preopen subset of X.

$$\left.\begin{array}{r} U \in PO(X) \Rightarrow (U \subset int(cl(U)))(int(cl(U)) \in \tau) \\ (e) \end{array}\right\} \Rightarrow$$
$$\Rightarrow e\text{-}cl_\theta(f(U)) \subset e\text{-}cl_\theta(f(int(cl(U)))) \subset f(cl(int(cl(U)))) \subset f(cl(U)).$$

$(f) \Rightarrow (g)$: Let U be any preopen subset of X.

$$\left.\begin{array}{r} U \in PO(X) \Rightarrow X \setminus cl(U) \in \tau \\ (f) \end{array}\right\} \Rightarrow e\text{-}cl_\theta(f(X \setminus cl(U))) \subset f(cl(X \setminus cl(U)))$$
$$\begin{aligned} &\Rightarrow e\text{-}cl_\theta(Y \setminus f(cl(U))) \subset f(X \setminus int(cl(U))) = Y \setminus f(int(cl(U))) \\ &\Rightarrow Y \setminus e\text{-}int_\theta(f(cl(U))) \subset Y \setminus f(int(cl(U))) \\ &\Rightarrow f(U) \subset f(int(cl(U))) \subset e\text{-}int_\theta(f(cl(U))). \end{aligned}$$

$(g) \Rightarrow (h)$: Straightforward.
$(h) \Rightarrow (i)$: Let $U \subset X$.

$$\begin{aligned} \left.\begin{array}{r} U \subset X \Rightarrow f(U) \subset Y \\ (h) \end{array}\right\} &\Rightarrow f^{-1}(e\text{-}cl_\theta(f(U))) \subset cl_\theta(f^{-1}(f(U))) = cl_\theta(U) \\ &\Rightarrow e\text{-}cl_\theta(f(U)) \subset f(cl_\theta(U)). \end{aligned}$$

$(i) \Rightarrow (j)$: Let $U \subset X$.

$$\left.\begin{array}{r} U \subset X \Rightarrow cl_\theta(U) \in C(X) \Rightarrow int(cl_\theta(U)) \subset X \\ (i) \end{array}\right\} \Rightarrow$$
$$\Rightarrow e\text{-}cl_\theta(f(int(cl_\theta(U)))) \subset f(cl_\theta(int(cl_\theta(U)))) = f(cl(int(cl_\theta(U)))) \subset f(cl_\theta(U)).$$

$(j) \Rightarrow (a)$: Straightforward. ■

Theorem 2.9 If X is a regular space and $f : (X, \tau) \to (Y, \sigma)$ is a bijective function, then the following statements are equivalent:
(a) f is weakly eR-open.
(b) For each θ-open set A in X, $f(A)$ is e-θ-open in Y.

(c) For any set B of Y and any θ-closed set A in X containing $f^{-1}(B)$, there exists an e-θ-closed set F in Y containing B such that $f^{-1}(F) \subset A$.

Proof. $(a) \Rightarrow (b)$: Let A be a θ-open set in X.

$$\left.\begin{array}{r} A \in \theta O(X) \Rightarrow Y \setminus f(A) \subset Y \\ (a)(\text{Theorem } 2.7(d)) \end{array}\right\} \Rightarrow f^{-1}(e\text{-}cl_\theta(Y \setminus f(A))) \subset cl_\theta(f^{-1}(Y \setminus f(A)))$$
$$\Rightarrow X \setminus f^{-1}(e\text{-}int_\theta(f(A))) \subset cl_\theta(X \setminus A) = X \setminus A$$
$$\Rightarrow A \subset f^{-1}(e\text{-}int_\theta(f(A)))$$
$$\Rightarrow f(A) \subset e\text{-}int_\theta(f(A)).$$

$(b) \Rightarrow (c)$: Let B be any set in Y and A be a θ-closed set in X such that $f^{-1}(B) \subset A$.

$$\left.\begin{array}{r} (B \subset Y)(A \in \theta C(X))(f^{-1}(B) \subset A) \Rightarrow (X \setminus A \in \theta O(X))(B \subset Y \setminus f(X \setminus A)) \\ (b) \end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r} (f(X \setminus A) \in e\theta O(Y))(B \subset Y \setminus f(X \setminus A)) \\ F := Y \setminus f(X \setminus A) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (F \in e\theta C(X))(B \subset Y)(f^{-1}(F) = f^{-1}(Y \setminus f(X \setminus A)) = f^{-1}(f(A)) = A.)$$

$(c) \Rightarrow (a)$: Let B be any set in Y.

$$\left.\begin{array}{r} (B \subset Y)(f^{-1}(B) \subset cl_\theta(f^{-1}(B))) \\ X \text{ is regular} \Rightarrow cl_\theta(f^{-1}(B)) \in \theta C(X) \\ (c) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists F \in e\theta C(Y))(B \subset F)(f^{-1}(F) \subset cl_\theta(f^{-1}(B)))$$
$$\Rightarrow (\exists F \in e\theta C(Y))(B \subset F)(f^{-1}(e\text{-}cl_\theta(B)) \subset f^{-1}(F) \subset cl_\theta(f^{-1}(B))).$$

Then from Theorem 2.8(h) f is weakly eR-open. ■

Theorem 2.10 If X is a regular space and $f : (X, \tau) \to (Y, \sigma)$ is a bijective function, then the following statements are equivalent:
(a) f is weakly eR-open.
(b) f is e-θ-open.
(c) For each $x \in X$ and each open set U of X containing x, there exists an e-open set V of Y containing $f(x)$ such that e-$cl(V) \subset f(U)$.

Proof. $(a) \Rightarrow (b)$: Let W be a nonempty open subset of X.

$$\left.\begin{array}{r} x \in W \in \tau \\ X \text{ is regular} \end{array}\right\} \Rightarrow (\exists U_x \in \mathcal{U}(x))(cl(U_x) \subset W)$$
$$\Rightarrow W = \cup\{U_x | x \in W\} = \cup\{cl(U_x) | x \in W\}$$
$$\Rightarrow \left.\begin{array}{r} f(W) = \cup\{f(U_x) | x \in W\} \\ f \text{ is weakly } eR\text{-open} \end{array}\right\} \Rightarrow f(W) = \cup\{f(U_x) | x \in W\}$$
$$\subset \cup\{e\text{-}int_\theta(f(cl(U_x))) | x \in W\}$$
$$\left.\begin{array}{r} \subset e\text{-}int_\theta(\cup\{f(cl(U_x)) | x \in W\}) \\ f \text{ is bijective} \end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r} f(W) \subset e\text{-}int_\theta(f(\cup\{cl(U_x) | x \in W\})) = e\text{-}int_\theta(f(W)) \\ e\text{-}int_\theta(f(W)) \subset f(W) \end{array}\right\} \Rightarrow$$
$$\Rightarrow e\text{-}int_\theta(f(W)) = f(W)$$
$$\Rightarrow f(W) \in e\theta O(Y).$$

$(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$: Straightforward. ■

Theorem 2.11 If $f : (X, \tau) \to (Y, \sigma)$ is weakly eR-open and strongly continuous, then

f is e-θ-open.

Proof. Let U be any open subset of X.

$$\left.\begin{array}{r} \left.\begin{array}{r} U \in \tau \\ f \text{ is weakly } eR\text{-open} \end{array}\right\} \Rightarrow f(U) \subset e\text{-}int_\theta(f(cl(U))) \\ f \text{ is strongly continuous} \end{array}\right\} \Rightarrow$$
$$\Rightarrow f(U) \subset e\text{-}int_\theta(f(cl(U))) \subset e\text{-}int_\theta(f(U)).$$ ■

The following example shows that strong continuity is not decomposition of e-θ-openness. Namely, an e-θ-open function need not be strongly continuous.

***Example* 2.12** Let $X = \{a, b\}$ and τ be the indiscrete topology for X. Then the identity function $f : (X, \tau) \to (X, \tau)$ is an e-θ-open function but it is not strongly continuous.

Theorem 2.13 If $f : (X, \tau) \to (Y, \sigma)$ is contra e-θ-closed, then f is a weakly eR-open function.

Proof. Let U be any open subset of X.

$$\left.\begin{array}{r} U \in \tau \Rightarrow cl(U) \in C(X) \\ f \text{ is contra } e\text{-}\theta\text{-closed} \end{array}\right\} \Rightarrow f(cl(U)) \in e\theta O(Y)$$
$$\Rightarrow f(U) \subset f(cl(U)) = e\text{-}int_\theta(f(cl(U))).$$ ■

Theorem 2.14 If $f : (X, \tau) \to (Y, \sigma)$ is bijective contra e-θ-open, then f is a weakly eR-open function.

Proof. Let U be any open subset of X.

$$\left.\begin{array}{r} U \in \tau \\ f \text{ is contra } e\text{-}\theta\text{-open} \end{array}\right\} \Rightarrow f(U) \in e\theta C(Y) \Rightarrow e\text{-}cl_\theta(f(U)) = f(U) \subset f(cl(U)).$$

Then from Theorem 2.8(e) f is weakly eR-open. ■

Theorem 2.15 Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective function. If $f(cl_\theta(U))$ is e-θ-closed in Y for every subset U of X, then f is weakly eR-open.

Proof. Let U be a subset of X.
$(U \subset X)(f(cl_\theta(U)) \in e\theta C(Y)) \Rightarrow e\text{-}cl_\theta(f(U)) \subset e\text{-}cl_\theta(f(cl_\theta(U))) = f(cl_\theta(U))$.
Then from Theorem 2.8(i) f is weakly eR-open. ■

Definition 2.16 A function $f : X \to Y$ is called complementary weakly eR-open (briefly c.w.eR-o) if for each open set U of X, $f(Fr(U))$ is e-θ-closed in Y, where $Fr(U)$ denotes the frontier of U.

Examples 2.17 and 2.18 show the independence of complementary weakly eR-openness and weakly eR-openness.

***Example* 2.17** Let $X = \{a, b, c, d\}$ and

$$\tau = \{\emptyset, X, \{a, d\}\} \quad \text{and} \quad \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, X\}.$$

The identity function $f : (X, \tau) \to (Y, \sigma)$ is weakly eR-open, but it is not c.w.eR-o.

***Example* 2.18** Let $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$ and $\sigma = \{\emptyset, X, \{b\}\}$. The identity function $f : (X, \tau) \to (X, \sigma)$ is c.w.eR-o., but it is not weakly eR-open.

Theorem 2.19 If $f : (X, \tau) \to (Y, \sigma)$ is bijective weakly eR-open and c.w.eR-o, then f is e-θ-open.

Proof. Let U be an open subset in X with $x \in U$. Since f is weakly eR-open, by Theorem 2.7(i) there exists an e-θ-open set V containing $f(x) = y$ such that $V \subset f(cl(U))$. Now $Fr(U) = cl(U) \setminus U$ and thus $x \notin Fr(U)$. Hence $y \notin f(Fr(U))$ and therefore $y \in V \setminus f(Fr(U))$. Put $V_y = V \setminus f(Fr(U))$. Now V_y is an e-θ-open set since f is c.w.eR-o. Since $y \in V_y$, then $y \in f(cl(U))$. But $y \notin f(Fr(U))$ and thus $y \notin f(Fr(U)) = f(cl(U)) \setminus f(U)$ which implies that $y \in f(U)$. Therefore $f(U) = \cup\{V_y | (V_y \in e\theta O(Y))(y \in f(U))\}$. Hence f is e-θ-open. ■

Recall that a space X is said to be e-connected [3] if X is not the union of two disjoint nonempty e-open sets.

Theorem 2.20 If $f : (X, \tau) \to (Y, \sigma)$ is a bijective weakly eR-open of a space X onto an e-connected space Y, then X is connected.

Proof. Let f be a bijective weakly eR-open of a space X onto an e-connected space Y and suppose that X is not connected.

$$\left.\begin{array}{r} X \text{ is not connected} \Rightarrow (\exists U_1, U_2 \in \tau \setminus \{\emptyset\})(U_1 \cap U_2 = \emptyset)(U_1 \cup U_2 = X) \\ f \text{ is bijective weakly } eR\text{-open} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (f(U_i) \in \sigma \setminus \{\emptyset\})(\bigcap_i f(U_i) = \emptyset)(\bigcup_i f(U_i) = Y)(f(U_i) \subset e\text{-}int_\theta(f(cl(U_i))) = e\text{-}int_\theta(f(U_i)))\ (i = 1, 2)$$

$$\Rightarrow (f(U_i) \in \sigma \setminus \{\emptyset\})(\bigcap_i f(U_i) = \emptyset)(\bigcup_i f(U_i) = Y)(f(U_i) = e\text{-}int_\theta(f(U_i)))\ (i = 1, 2)$$

$$\Rightarrow (f(U_i) \in e\theta O(Y) \setminus \{\emptyset\})(\bigcap_i f(U_i) = \emptyset)(\bigcup_i f(U_i) = Y)\ (i = 1, 2)$$

Then Y is not e-connected which is a contradiction. ■

Definition 2.21 A space X is said to be hyperconnected [9] if every nonempty open subset of X is dense in X.

Theorem 2.22 If X is a hyperconnected space, then a function $f : (X, \tau) \to (Y, \sigma)$ is weakly eR-open if and only if $f(X)$ is e-θ-open in Y.

Proof. *Sufficiency:* Obvious.
Necessity: Let U be a nonempty open subset of X.

$$\left.\begin{array}{r} (U \in \tau)(X \text{ is hyperconnected}) \Rightarrow cl(U) = X \Rightarrow e\text{-}int_\theta(f(cl(U))) = e\text{-}int_\theta(f(X)) \\ f \text{ is weakly } eR\text{-open} \end{array}\right\} \Rightarrow$$

$$\Rightarrow f(U) \subset f(X) = e\text{-}int_\theta(f(X)) = e\text{-}int_\theta(f(cl(U))).$$ ■

Theorem 2.23 Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective weakly eR-open function. Then the following properties hold:
(a) If F is θ-closed in X, then $f(F)$ is e-θ-closed in Y.
(b) If F is θ-open in X, then $f(F)$ is e-θ-open in Y.

Proof. (a) Let $F \in \theta C(X)$.

$$\left.\begin{array}{r} F \in \theta C(X) \Rightarrow F = cl_\theta(F) \\ \text{Theorem 2.8}(i) \end{array}\right\} \Rightarrow \left.\begin{array}{r} e\text{-}cl_\theta(f(F)) \subset f(cl_\theta(F)) = f(F) \\ f(F) \subset e\text{-}cl_\theta(f(F)) \end{array}\right\} \Rightarrow$$

$$\Rightarrow f(F) = e\text{-}cl_\theta(f(F))$$
$$\Rightarrow f(F) \in e\theta C(Y).$$

(b) Similarly proved. ■

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