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Application of triangular functions for solving Vasicek model

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Abstract. This paper introduces a numerical method for solving the vasicek model by using a stochastic operational matrix based on the triangular functions (TFs) in combination with the collocation method. The method is stated by using conversion the the vasicek model to a stochastic nonlinear system of $2m + 2$ equations and $2m + 2$ unknowns. Finally, the error analysis and some numerical examples are provided to demonstrate applicability and accuracy of this method.

Keywords: Triangular functions, Stochastic operational matrix, Vasicek model, collocation method.

2010 AMS Subject Classification: Primary 65C30, 60H35, 65C20; Secondary 60H20, 68U20.

1. Introduction

The vasicek model is a mathematical model describing the evolution of interest rates where play important role in finance. This model can be used for interest rate derivative valuation and also adapted for credit market. It is based on the ornstein-uhlenbeck process (is the first account of a bond pricing model), that incorporates stochastic interest rate and can be also seen as a stochastic investment model. The short-term interest rate process $(X(t))_{t\in\mathbf{R}^+}$ solves the equation

$$dX(t) = \alpha(\beta - X(t))dt + \sigma dB(t),$$

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where $dX(t)$ be the change in the short-term interest rate, α be the speed of mean reversion, β be the average interest rate and σ be the volatility of the short rate.
The main disadvantage is that, under vasicek model, it is theoretically possible for the interest rate to become negative. This shortcoming was fixed in the Cox-Ingersoll-Ross (CIR) model. The CIR process is a markov process with continuous paths defined by the following SDE:

$$dX(t) = \alpha(\beta - X(t))dt + \sigma\sqrt{X(t)}dB(t),$$

or

$$X(t) = X_0 + \int_0^t \alpha(\beta - X(s))ds + \int_0^t \sigma\sqrt{X(s)}dB(s), \tag{1}$$

where $B(s)$ is a standard Brownian motion (SBM) defined on a complete probability space $(\Omega, \digamma, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ with natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and $X(t)$ is unknown stochastic processes defined on same probability space.

In the last years, many methods are proposed and applied for numerical solutions of stochastic differential equations [5, 8, 9, 10], because these kinds of equations can not be solved analytically. Hence, it is importance to provide their numerical solutions.
In this work, we reduce the Eq. (1) to the stochastic nonlinear system of $2m+2$ equations and $2m+2$ unknowns without integration by using operational matrices based on the TFs in combination with the collocation technique, with several advantages in reducing computations and making convergence faster than the other methods.

The results of the paper are organized as follows: In Section 2, we state some essential preliminaries which play fundamental role in our method. In Section 3, we solve Eq. (1) by using the stochastic operational matrix based on the TFs in combination with the collocation method. In Sections 4 and 5, we provide the error analysis and some numerical examples to demonstrate the applicability and accuracy of presented method. Finally, in Section 6, we give a brief conclusion.

2. Preliminaries

The first, we state the basic properties of the SBM that play important role in solving Eq. (1). For more details see [2, 12].

Let $h(t, X),\ g(t, X) : (0, T) \times R \to R$ be measurable functions and continuous with the main properties as follows:
$\mathbf{A}_1$. There are constants $k_1,\ k_2 > 0$, such that:

$$\begin{cases} |g(t,X) - g(t,Y)| \leqslant k_1|X - Y|, & (lipschitz\ continuity), \\ |g(t,X)| < k_2(1 + |X|), & (lineargrowth). \end{cases}$$

$\mathbf{A}_2$. There are constants $k_3,\ k_4 > 0$, such that:

$$\begin{cases} |h(t,X) - h(t,Y)| < k_3|X - Y|, & (lipschitz\ continuity), \\ |h(t,X)| < k_4(1 + |X|), & (lineargrowth). \end{cases}$$

For X, $Y \in R$ and $t \in (0, T)$.

Theorem 2.1 [2, 12] Let $g(t, X(t))$ and $h(t, X(t))$ hold in conditions $\mathbf{A}_1$, $\mathbf{A}_2$ and $E \mid X_0 \mid^2 < \infty$, then, there exists a unique solution for Eq. (1).

Definition 2.2 The SBM $\{B(t),\ t \geqslant 0\}$ is the stochastic process with main properties as follows:
1. The process has independent increments for $0 \leqslant t_0 \leqslant t_1 \leqslant ... \leqslant t_n \leqslant T$.
2. $B(t+h) - B(t)$ is normally distributed with mean 0 and variance h, for all $t \geqslant 0$, $h > 0$.
3. $B(t)$ is a continuous function.

Definition 2.3 Let $\nu = \nu(S, T)$ be the class of functions $\alpha(t, \omega) : [0, \infty) \times \Omega \longrightarrow R$ such that:
1. The function $\alpha(t, \omega)$ be $\beta \times \mathcal{F}$ measurable.
2. The function $\alpha(t, \omega)$ is $\mathcal{F}_t$-adapted.
3. $E\big[\int_S^T \alpha^2(t, \omega) dt\big] < \infty$.

Theorem 2.4 [2, 12] Let $f \in \nu(S, T)$, then

$$E\big[(\int_S^T f(t, \omega) dB(t)(\omega))^2\big] = E\big[\int_S^T f^2(t, \omega) dt\big].$$

Finally, we introduce some essential properties of the TFs that are needful for this paper. For more details see [1, 3, 4, 6, 7, 11].
1. The 1D-TF vector are defined as follows:

$$T(t) = \begin{pmatrix} T1(t) \\ T2(t) \end{pmatrix},$$

where

$$\begin{cases} T1(t) = [T_0^1(t), \ldots, T_i^1(t), \ldots, T_{m-1}^1(t)]^T, \\ T2(t) = [T_0^2(t), \ldots, T_i^2(t), \ldots, T_{m-1}^2(t)]^T, \end{cases}$$

with

$$T_i^1(t) = \begin{cases} 1 - \frac{t-ih}{h} & ih \leqslant t < (i+1)h, \\ 0 & otherwise, \end{cases}$$

and

$$T_i^2(t) = \begin{cases} \frac{t-ih}{h} & ih \leqslant t < (i+1)h, \\ 0 & otherwise, \end{cases}$$

where $h = \frac{T}{m}$.
2. Let the function $f(t) \in L^2\big((0, T)\big)$, then

$$f(t) \approx F^T . T(t),$$

where $F = [f_1, f_2]^T$, $f_1 = (f(ih))_{m\times 1}$, $f_2 = (f((i+1)h))_{m\times 1}$ and $i = 0, 1, .., m-1$.

3.

$$\int_0^t T(s)ds \approx P_T.T(t),$$

where

$$P_T = \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix},$$

with

$$P1 = \begin{pmatrix} 0 & \frac{h}{2} & \frac{h}{2} & \dots & \frac{h}{2} \\ 0 & 0 & \frac{h}{2} & \dots & \frac{h}{2} \\ 0 & 0 & 0 & \dots & \frac{h}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{m\times m},$$

and

$$P2 = \begin{pmatrix} \frac{h}{2} & \frac{h}{2} & \frac{h}{2} & \dots & \frac{h}{2} \\ 0 & \frac{h}{2} & \frac{h}{2} & \dots & \frac{h}{2} \\ 0 & 0 & \frac{h}{2} & \dots & \frac{h}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{h}{2} \end{pmatrix}_{m\times m}.$$

4.

$$\int_0^t T(s)dB(s) \approx P_S.T(t),$$

where

$$P_S = \begin{pmatrix} P1_S & P1_S \\ P2_S & P2_S \end{pmatrix},$$

with

$$P1_S = \begin{pmatrix} \alpha(0) & \beta(0) & \beta(0) & \dots & \beta(0) \\ 0 & \alpha(1) & \beta(1) & \dots & \beta(1) \\ 0 & 0 & \alpha(2) & \dots & \beta(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \beta(m-2) \\ 0 & 0 & 0 & \dots & \alpha(m-1) \end{pmatrix}_{m\times m},$$

$$P2_S = \begin{pmatrix} \gamma(0) & \rho(0) & \rho(0) & \dots & \rho(0) \\ 0 & \gamma(1) & \rho(1) & \dots & \rho(1) \\ 0 & 0 & \gamma(2) & \dots & \rho(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \rho(m-2) \\ 0 & 0 & 0 & \dots & \gamma(m-1) \end{pmatrix}_{m\times m},$$

and

$$\begin{cases} \alpha(i) = (i+1)[B((i+0.5)h) - B(ih)] - \int_{ih}^{(i+0.5)h} \frac{s}{h} dB(s), \\ \beta(i) = (i+1)[B((i+1)h) - B(ih)] - \int_{ih}^{(i+1)h} \frac{s}{h} dB(s), \\ \gamma(i) = -i[B((i+0.5)h) - B(ih)] + \int_{ih}^{(i+0.5)h} \frac{s}{h} dB(s), \\ \rho(i) = -i[B((i+1)h) - B(ih)] + \int_{ih}^{(i+1)h} \frac{s}{h} dB(s). \end{cases}$$

3. Solving NSDE by using the TFs

To find a solution for the Eq. (1) we can write it as follows:

$$X(t) = X_0 + \int_0^t p(s)ds + \int_0^t q(s)dB(s). \tag{2}$$

with

$$\begin{cases} \alpha(\beta - X(s)) = g(s, X(s)) = p(s), \\ \sigma\sqrt{X(s)} = h(s, X(s)) = q(s), \end{cases} \tag{3}$$

with substituting (3) in Eq. (2), we get

$$X(t) = X_0 + \int_0^t p(s)ds + \int_0^t q(s)dB(s). \tag{4}$$

By using properties of the TFs, we can write

$$\begin{cases} p(s) \approx P^T.T(s), \\ q(s) \approx Q^T.T(s), \end{cases} \tag{5}$$

where

$$P = (p_i)_{2m\times 1} = \big(p(0), p(h), \dots, p((m-1)h), p(h), p(2h), \dots, p(mh)\big)_{2m\times 1},$$

and

$$Q = (q_i)_{2m\times 1} = \big(q(0), q(h), \ldots, q((m-1)h), q(h), q(2h), \ldots, q(mh)\big)_{2m\times 1}.$$

With substituting (5) in (4), we get

$$X(t) \approx X_0 + \int_0^t P^T.T(s)ds + \int_0^t Q^T.T(s)dB(s), \tag{6}$$

or

$$X(t) \approx X_0 + P^T P_T T(t) + Q^T P_S T(t). \tag{7}$$

Also, by substituting (7) in (3), we obtain

$$\begin{cases} p(t) \approx g(t, X_0 + P^T P_T T(t) + Q^T P_S T(t)), \\ q(t) \approx h(t, X_0 + P^T P_T T(t) + Q^T P_S T(t)). \end{cases} \tag{8}$$

Now, with replacing $\approx$ by $=$, the relation (8) is approximated via the collocation method in $m+1$ nodes $t_j = \frac{j}{\frac{1}{T}m+1}$ $(j = 0, 1, \ldots, m)$, as follows:

$$\begin{cases} p(t_j) = g(t_j, X_0 + P^T P_T T(t_j) + Q^T P_S T(t_j)), \\ q(t_j) = h(t_j, X_0 + P^T P_T T(t_j) + Q^T P_S T(t_j)), \end{cases} \tag{9}$$

or

$$\begin{cases} P^T T(t_j) = g(t_j, X_0 + P^T P_T T(t_j) + Q^T P_S T(t_j)), \\ Q^T T(t_j) = h(t_j, X_0 + P^T P_T T(t_j) + Q^T P_S T(t_j)), \end{cases} \tag{10}$$

where Eq. (10) is the nonlinear system of $2m+2$ equations and $2m+2$ unknowns. From solving Eq. (10), we can conclude

$$X(t) = X_m(t) = X_0 + P^T P_T T(t) + Q^T P_S T(t). \tag{11}$$

4. Error analysis

Theorem 4.1 Let $f(t)$ be an arbitrary real bounded function on $(0,1)$, $|f'(t)| \leqslant M$ and $e(t) = f(t) - \hat{f}(t)$ that $\hat{f}(t)$ denotes the TFs of $f(t)$. Then,

$$||e(t)||^2 \leqslant O(h^2),$$

where $||e(t)||^2 = \int_0^1 |e(t)|^2 dt$.

Proof. By using properties of the TFs, we can write

$$|e(t)| = |f(t) - \hat{f}(t)| = |f(t) - \sum_{i=0}^{m-1} f(ih)(1 - \frac{t-ih}{h}) + f((i+1)h)(\frac{t-ih}{h})|.$$

Let $t \in (ih,\ (i+1)h)$, so, we get

$$|e(t)| = |f(t) - \hat{f}(t)| = |f(t) - f(ih)(1 - \frac{t-ih}{h}) - f((i+1)h)(\frac{t-ih}{h})| =$$
$$|f(t) - f(ih) + \big(f(ih) - f((i+1)h)\big)(\frac{t-ih}{h})| \leqslant |f(t) - f(ih)| + |f(ih) -$$
$$f((i+1)h)||\frac{t-ih}{h}| \leqslant |f(t) - f(ih)| + |f(ih) - f((i+1)h)|,$$

by using the mean value theorem, we get

$$|e(t)| \leqslant |f'(\alpha)|(t - ih) + |f'(t)h| \leqslant Mh,$$

consequently

$$||e(t)||^2 = \int_0^1 |e(t)|^2 dt \leqslant M^2h^2 \leqslant O(h^2). \qquad \blacksquare$$

Let

$$\begin{cases} p^m(t) = g(t, X_m(t)), \\ q^m(t) = h(t, X_m(t)), \end{cases} \tag{12}$$

and

$$\begin{cases} \hat{p}(t) = \hat{g}(t, X_m(t)), \\ \hat{q}(t) = \hat{h}(t, X_m(t)), \end{cases} \tag{13}$$

where $\hat{p}(t)$ and $\hat{q}(t)$ are defined by properties of the TFs. Also, let $X_m(t)$ be numerical solution of Eq. (1) defined in Eq. (11).

Theorem 4.2 Let $X_m(t)$ be the numerical solution of Eq. (1) defined in Eq. (11) and let conditions $\mathbf{A}_1$, $\mathbf{A}_2$ and $E \mid X_0 \mid^2 < \infty$ hold. Then,

$$\| X(t) - X_m(t) \|^2 \leqslant O(h^2), \qquad t \in (0,1), \tag{14}$$

where $\| X \|^2 = E[X^2]$.

Proof

$$X(t) - X_m(t) = \int_0^t (p(s) - \hat{p}(s))ds + \int_0^t (q(s) - \hat{q}(s))dB(s), \tag{15}$$

via $\big(x+y\big)^2 \leqslant 2\big(x^2+y^2\big)$ and the property of the Itô isometry for the SBM, we can write

$$\| X(t) - X_m(t) \|^2 \leqslant 2\big(\| \int_0^t (p(s) - \hat{p}(s))ds \|^2 + \| \int_0^t (q(s) - \hat{q}(s))dB(s) \|^2 \big) \leqslant 2$$
$$\big(\int_0^t \| p(s) - \hat{p}(s) \|^2 ds + \| \int_0^t (q(s) - \hat{q}(s))ds \|^2 \big) \leqslant 2\big(\int_0^t \| p(s) - \hat{p}(s) \|^2 ds +$$
$$\int_0^t \| q(s) - \hat{q}(s) \|^2 ds\big) \leqslant 2\big(2 \int_0^t \| p(s) - p^m(s) \|^2 ds + 2 \int_0^t \| p^m(s) - \hat{p}(s) \|^2 ds$$
$$+2 \int_0^t \| q(s) - q^m(s) \|^2 ds + 2 \int_0^t \| q^m(s) - \hat{q}(s) \|^2 ds\big) \leqslant 4\big(\int_0^t \| p(s) - p^m(s) \|^2$$
$$ds + \int_0^t \| p^m(s) - \hat{p}(s) \|^2 ds + \int_0^t \| q(s) - q^m(s) \|^2 ds + \int_0^t \| q^m(s) -$$
$$\hat{q}(s) \|^2 ds\big). \tag{16}$$

By using Theorem (4.1), we have

$$\begin{cases} \| p^m(s) - \hat{p}(s) \|^2 \leqslant L_1 h^2, & L_1 > 0, \\ \| q^m(s) - \hat{q}(s) \|^2 \leqslant L_2 h^2, & L_2 > 0. \end{cases} \tag{17}$$

Also, by using conditions $\mathbf{A}_1$ and $\mathbf{A}_2$, we have

$$\begin{cases} \int_0^t \| p(s) - p^m(s) \|^2 ds \leqslant L_3 \int_0^t \| X(s) - X_m(s) \|^2 ds, \\ \int_0^t \| q(s) - q^m(s) \|^2 ds \leqslant L_4 \int_0^t \| X(s) - X_m(s) \|^2 ds. \end{cases} \tag{18}$$

With substituting (17) and (18) in (16), we obtain

$$\| X(t) - X_m(t) \|^2 \leqslant 4\big(L_1 h^2 + L_3 \int_0^t \| X(s) - X_m(s) \|^2 ds + L_2 h^2 +$$
$$L_4 \int_0^t \| X(s) - X_m(s) \|^2 ds\big), \tag{19}$$

or

$$\eta(t) \leqslant m + n \int_0^t \eta(s) ds,$$

where $m = 4(L_1 h^2 + L_2 h^2)$, $n = 4(L_3 + L_4)$ and $\eta(s) = \| X(s) - X_m(s) \|^2$. Furthermore, from Gronwall inequality, we get

$$\eta(t) \leqslant m(1 + n \int_0^t \exp\big(n(t-s)\big) ds), \quad t \in (0, 1),$$

so

$$\| X(t) - X_m(t) \|^2 \leqslant O(h^2).$$

■

5. Numerical examples

In this section, we give some numerical results to illustrate our main results. The numerical results have been shown in Figures (1-4) via a comparison between numerical solution of deterministic model and numerical solution of stochastic model. Also, the numerical solution of deterministic model has been approximated using properties of the TFs. In addition, we assume $X_0 = 0.03,\ \alpha = 0.05,\ \beta = 0.3$ and $\sigma = 0.002$ in Figures (1-2) and $X_0 = 0.5,\ \alpha = 0.2,\ \beta = 0.005$ and $\sigma = 0.002$ in Figures (3-4).

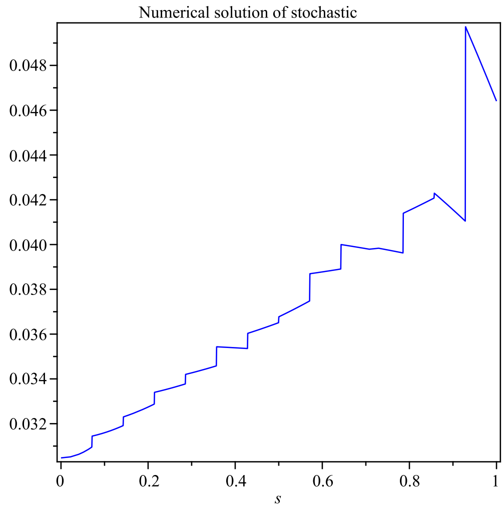

Fig.1

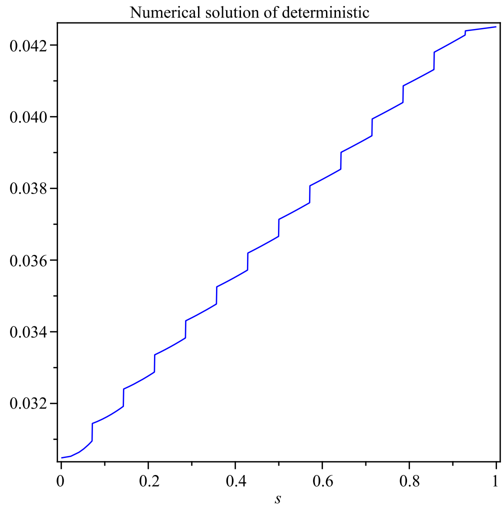

Fig.2

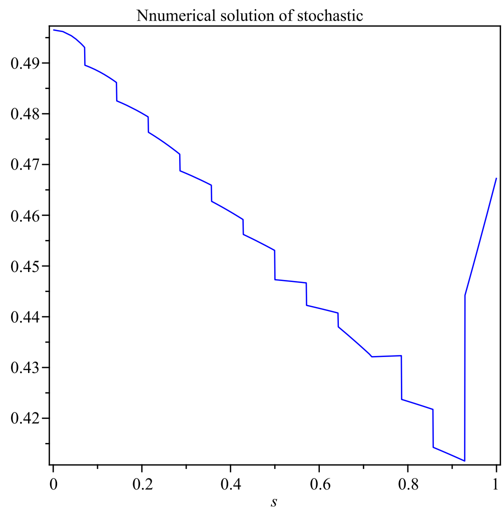

Fig.3

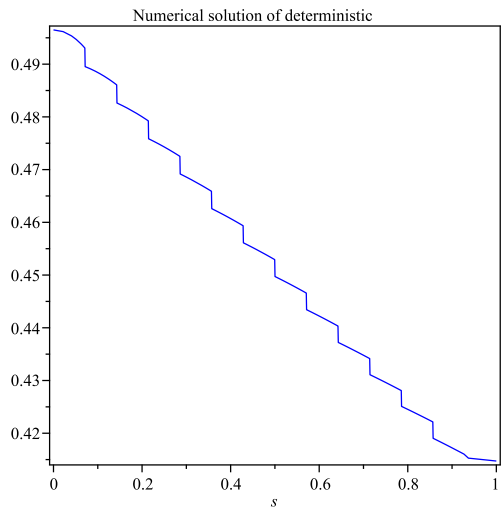

Fig.4

6. Conclusion

In this paper, we introduce the numerical method based on the TFs for solving the vasicek model. With using this method, we reduce Eq. (1) to the stochastic nonlinear system. Also, numerical simulations are provided to accuracy of presented method.

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