An Algorithm for Multi-Realization of Nonlinear MIMO Systems

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Abstract

This paper presents a theoretical approach to implementation of the "Multi-realization of nonlinear MIMO systems". This method aims to find state-variable realization for a set of systems, sharing as many parameters as possible. In this paper a special nonlinear multi-realization problem, namely the multi-realization of feedback linearizable nonlinear systems is considered and an algorithm for achieving minimal stably-based multi-realization of a set of nonlinear feedback linearizable systems is introduced. An example that illustrates this algorithm is also presented.

Keywords: Feedback linearizable nonlinear systems, Multi-realization, Multiple model adaptive control.

1. Introduction

The original motivation for studying multi-realization problems comes from multiple-model adaptive control (MMAC) algorithms [1]. The implementation of "multi-controller" architecture is an important issue for MMAC applications. As argued in for example [2], because at any instant of time only one of the constituent controllers is to be applied to the plant, it is only necessary to generate one control signal at any time. Often, this means significant simplification can be achieved if all control signals are capable of being generated by a single system. In other words, rather than implementing each of the controllers in the family as a separate dynamical system, one can often achieve the same results using a single controller with adjustable parameters. Because the single controller state is, in effect, shared by the family of controllers, this implementation is termed a "state shared" multi-realization.

Most literature on system realization deals with the problem of passing from a transfer function description of a single linear time-invariant (LTI) system [3], [4], [5] to a state space description or matrix fraction description. Morse [2] introduced the concept and showed how to perform multi-realization of several linear SISO systems in the context of examining MMAC for SISO plants. Paper [6] investigates the multi-realization of several linear MIMO systems. The results are applicable to MMAC problems for linear MIMO plants.

As an extension to the linear case presented by Morse [2] for (MMAC) algorithms, Su *et. al.* were the first researchers who investigated the Multi-realization of nonlinear systems [7].

This approach to multi-realization of nonlinear MIMO systems is the basis of the method presented in this paper.

In this paper, the multi-realization technique for nonlinear MIMO systems is reviewed and an algorithm for implementing this method for nonlinear controllers is presented. Furthermore, this technique can realize "bumpless" transfer between nonlinear multivariable systems which is an effective

way to improve poor transient response of systems at time of switching. In this regard, a set of nonlinear controllers are considered and the proposed algorithm is used for switching between them.

2. Multi-realization of Nonlinear systems

The notations and symbols in this paper are standard and can be found in [8]. In the following discussion, when we use a symbol P or (P, x₀) to denote a system, it includes both input-to-state and state-to-output mappings and not just the input-output mapping. Occasionally, when only the state space equation is presented (i.e. only input-to-state mapping is considered), it implies all state variables are directly measurable and therefore observable. As a state is always associated with a system, we sometimes omit the sub-index of the state variable when referring to a particular system.

The problem of multi-realization of general nonlinear systems is more complicated and difficult than in the linear case. Here we first consider a special nonlinear multi-realization problem: the multi-realization of feedback linearizable nonlinear systems.

2.1. The multi-realization problem

Suppose that there are given a number of nonlinear systems $P_i \ (i \in \{1, 2, ..., N\})$ described by:

$$(P_i, x^0) : \begin{cases} \dot{x} = f_i(x) + g_i(x)u & x \in \Re^{n_{ix}}, u \in \Re^m \\ y = h_i(x) & y \in \Re^p \end{cases} \quad (1)$$

It is assumed that the mappings f_i, g_i and functions h_i are smooth in their arguments [8]. Further assume that, $g_i(x^0)$ has rank m and (P_i, x^0) satisfies both controllability and observability rank conditions [9] at the point x^o.

Find (if possible) a parameter-dependent system (the functions v_{qi} and h_{qi} can be chosen for each i):

$$(\tilde{P}_i, \xi^0) : \begin{cases} \dot{\xi} = A_0 \xi + B_0 \upsilon_{qi} & \xi \in \Re^{n_\xi}, \upsilon_{qi} \in \Re^m \\ y = h_{qi}(\xi) & y \in \Re^p \end{cases} \quad (2)$$

such that under coordinate transformation and regular state feedback $v_{qi} = \alpha_{qi}(\xi) + \beta_{qi}(\xi)u$, the system $(\tilde{P}_i, \xi_0)$ can be transformed to a decomposition form as follows:

$(\bar{P}_i, \bar{\xi}^0)$:

$$\begin{cases} \bar{\xi}_1 = f_{i1}(\bar{\xi}_1, \bar{\xi}_2) + g_{i1}(\bar{\xi}_1, \bar{\xi}_2)u & \bar{\xi}_1 \in \Re^{n_\xi - n_{ix}} \\ \bar{\xi}_2 = f_i(\bar{\xi}_2) + g_i(\bar{\xi}_2)u & \bar{\xi}_2 \in \Re^{n_{ix}} \\ y_i = h_i(\bar{\xi}_2) & y \in \Re^p \end{cases} \quad (3)$$

System $(\tilde{P}_i, \xi^0)$ is considered as a multi-realization of non-linear systems (P_i, x^0). Furthermore if we could find a stable A_0 with the smallest possible dimension, we call the state space description (2) a minimal stably based multi-realization of the set of nonlinear systems (P_i, x^0).

2.2. Theorem 1

Assume given N distinct systems $P_i \ (i \in \{1, 2, ..., N\})$ described by state variable equations of the standard type with $A_i \in R^{n_i \times n_i}$, $B_i \in R^{n_i \times m}$ and $C_i \in R^{p \times n_i}$ (with the pairs (A_i, B_i) and (C_i, A_i) are controllable and observable, and B_i has full column rank) for each system P_i such that all controllable pairs (A_i, B_i) have an identical controllability indices $d_{i1}, ..., d_{im}$ and define the ordered set $\{r_{i1}, \ r_{i2}, \ r_{i3}, \ ...\}$ to be the set of controllability indices reordered so that $r_{i1} \le r_{i2} \le \cdots \le r_{im}, \ \forall i \in \{1, 2, ..., N\}$. Then, the dimension of a minimal stably-based multi-realization (with input transformation) of the set of systems $P_i \ (i \in \{1, 2, ..., N\})$ is equal to

$$\bar{n} = \sum_{j=1}^{m} \max_{1 \le i \le N} (r_{ij}) \quad (4)$$

The proof of theorem is given in [7].

2.3. Theorem 2

The multi-realization problem is solvable if and only if the following equivalent conditions (*a*) or (*b*) hold:

(*a*) For each system P_i (described by equation (1)), there exist a neighborhood U of x^0 and m real-valued functions $\lambda_{i_1}, ..., \lambda_{i_m}$ defined on U, such that the system

$$\begin{cases} \dot{x} = f_i(x) + g_i(x)u \\ y = \lambda_i(x) \end{cases} \quad (5)$$

has some relative degree $\{r_{i1}, ..., r_{im}\}$ at x^0 and $r_{i_1} + r_{i_2} + ... + r_{i_m} = n_i$.

(b)

    i)   for   each $i \in \{1, 2, ..., N\}$ and $l \in \{1, 2, ..., n-1\}$ the distribution G_{il} has constant dimension near x^0.

    ii) the distribution $G_{i,n-1}$ ($i \in \{1, 2, ..., N\}$) has dimension n;

    iii)   for   each $i \in \{1, 2, ..., N\}$ and $l \in \{1, 2, ..., n-1\}$ the distribution G_{il} is involutive.

where

$$G_{i0} = span\{g_{i1}, ..., g_{im}\}$$

$$G_{i1} = span\{g_{i1}, ..., g_{im}, ad_{f_i} g_{i1}, ..., ad_{f_i} g_{im}\} \qquad (6)$$

$$...$$

$$G_{il} = span\{ad_{f_i}^k g_{ij} : 0 \le k \le l, 1 \le j \le m\}$$

(See theorem 5.2.3 in [8]).

    Furthermore, the smallest possible dimension of A_0 in equation (2) is equal to:

$$\bar{n} = \sum_{j=1}^{m} \max_{1 \le i \le N} (r_{ij}) \qquad (7)$$

The proof of theorem is given in [7].

3. The Multi-realization Algorithm

    In this section an algorithm for achieving stably-based multi-realization of a set of nonlinear feedback linearizable systems is presented:

    Step 1. For each nonlinear multivariable system P_i given by equation (1), the vector relative degree, can be evaluated such that:

i) $L_{g_j} L_f^k h_i(x) = 0$

for all $1 \le j \le m$, $k < r_i - 1$ and $1 \le i \le m$, and all x in a neighborhood of x^0.

ii) The $m \times m$ matrix

$$A(x) = \begin{pmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & ... & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & ... & L_{g_m} L_f^{r_2-1} h_2(x) \\ ... \\ L_{g_1} L_f^{r_m-1} h_m(x) & ... & L_{g_m} L_f^{r_m-1} h_m(x) \end{pmatrix} \qquad (8)$$

is non-singular at $x = x_0$.

    Step 2. The multi-realization problem is solvable if and only if the equivalent conditions (*a*) or (*b*), presented earlier in Theorem 1, hold. If neither of these conditions is true, real-valued functions $\lambda_{i1}(x), ..., \lambda_{im}(x)$ defined on U should be chosen using Theorem 5.2.3 in [8] such that the following system satisfies the $r_{i1} + r_{i2} + \cdots + r_{im} = n_i$ condition

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = \lambda(x) \end{cases} \qquad (9)$$

    Step 3. Evaluate the minimal multi-realization degree, $\bar{n} = \sum_{j=1}^{m} \max_{1 \le i \le N} (r_{ij})$, Using Theorem 1.

    Step 4. Construct the following stable system which is of order $\bar{n}$

$$(\tilde{P}_i, \xi_0): \quad \dot{\xi} = A_0 \xi + B_0 \upsilon_{qi} \qquad (10)$$

    Step 5. This step of the algorithm has two cases:
i) If the dimension of P_i is equal to the dimension of state-space equation (10), proceed to Step 6.
ii) If the dimension of P_i is less than the dimension of state-space equation (10), then its dimension should be augmented with a linear transformation $\eta = T_i(\xi)$ such that the system in equation (10) can be transformed as

$$(\tilde{P}_i, \eta^0): \begin{cases} \dot{\eta} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \eta + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \upsilon_{qi} \\ \\ Y = \begin{bmatrix} 0 & C_2 \end{bmatrix} \eta \end{cases} \qquad (11)$$

Then proceed to step 7.

    Step 6. Construct for the parameter dependent system (2), an invertible transformation $\bar{\xi} = \Phi_i(\xi)$ and regular state feedback $\upsilon_{qi} = \alpha_{qi}(\xi) + \beta_{qi}(\xi)$ using the state-space exact linearization approach, such that the state-space equation of $(\tilde{P}_i, \xi_0)$ can be transformed in a form of $(\bar{P}_i, \bar{\xi}_0)$. Thus, if the transformation is selected as $\bar{\xi} = \Phi_i(\xi)$, and the output function is selected as $y = h_i(\bar{\xi})$, then the multi-realization is obtained.

    Step 7. Construct for the parameter dependent system (2), an invertible transformation $\bar{\xi} = \Psi_i(\eta) = [\eta_1^T, \Psi_{i2}^T(\eta)]^T$ and regular state feedback $\upsilon_{qi} = \alpha_{qi}(\eta_2) + \beta_{qi}(\eta_2)u$ using state-space exact linearization approach, such that the state-space equations of $(\tilde{P}_i, \xi_0)$ can be transformed in a form of $(\bar{P}_i, \bar{\xi}_0)$. Thus, if the overall transformation is selected as $\bar{\xi} = \Phi_i(\xi) = \Psi_i(T_i(\xi))$, and the output

function is selected as $y = h_i(\Psi_{i2}(\eta_2)) = h_i(\overline{\xi_2})$, then the multi-realization is obtained.

4. Example

We show the multi-realization strategy of the following systems.

$$P_1 : \begin{cases} \dot{x} = \begin{bmatrix} x_2 + u_1 \\ -x_1 + \dfrac{x_1^3}{6} - x_2 + u_2 \end{bmatrix}, \\ y = [x_1, x_2]^T \end{cases}$$

$$P_2 : \begin{cases} \dot{x} = \begin{bmatrix} -3x_1 + x_1^2 x_2^2 + u_1 \\ -x_2 + x_1^3 x_3 + u_2 \\ -x_3 + x_2^2 \end{bmatrix} \\ y = [x_1, x_3]^T \end{cases} \tag{12}$$

The relative degrees associated with output channels (r_1 and r_2) in systems P_1 and P_2 are calculated according to *step 1* above and found to satisfy the condition (*a*) in *step 2*, $r_{11} + r_{12} = 1 + 1 = n_1$ and $r_{21} + r_{22} = 1 + 2 = n_2$ respectively. Both P_1 and P_2 are linearizable.

According to Theorem 2, the minimal order of the multi-realization is $\bar{n} = 3$, therefore, we construct a third order stable system $\tilde{P}_i$, with these specifications:

1) (A_0, B_0) is a controllable pair.
2) B_0 has full column rank.
3) The controllability indices, $d_j = \max\limits_{1 \le i \le N}(r_{ij})$, are increasingly ordered.
4) A_0 is stable and minimal with dimension

$$\bar{n} = \sum_{j=1}^{m} \max_{1 \le i \le N}(r_{ij}).$$

as

$$\dot{\xi} = A_0 \xi + B_0 \upsilon_{qi} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \upsilon_{qi1} \\ \upsilon_{qi2} \end{bmatrix}. \tag{13}$$

where the controllability indices are $d_1 = 1$ and $d_2 = 2$.

Since the dimension of P_1 is less than that of Equation 13, a linear transformation $\eta = T(\xi)$ is constructed to implement an observable /unobservable decomposition. To build the decomposition form, we select a new matrix C for

the purpose of pole/zero cancellation. The resulting transfer function of the system has the form:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow$$

$$C(sI - A_0)^{-1} B_0 = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{(s+2)} \end{bmatrix} \tag{14}$$

According to virtual output $y = [\xi_1 \quad \xi_2 + \xi_3]^T$, to get the decomposition form, it is easy to construct the transformation $\eta = T(\xi)$. At first the observability matrix ϕ_o is defined as

$$\phi_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \tag{15}$$

It can be seen that its rank is equal to 2 and the system is unobservable. To construct the linear transformation matrix $T(\xi)$ two independent row vectors of matrix ϕ_0 are considered as T_1 and a third arbitrary row vector $[0 \quad 0 \quad 1]$ which is independent of T_1 is selected as T_2. Therefore the transformation matrix can be shown as

$$T = \begin{bmatrix} T_2 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \tag{16}$$

$$\eta = T\xi = [\xi_3 \quad \xi_1 \quad \xi_2 + \xi_3]^T$$

and the system $(\tilde{P}_i, \xi^0)$ is transformed as follows:

$$(P_i, \eta^0) : \begin{cases} \dot{\eta} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \upsilon_{qi} \\ y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \end{cases} \tag{17}$$

$\eta_1 \in \Re^1, \eta_2 \in \Re^2$

which is an observable /unobservable decomposition form.

Using equation 5.7 from [8], system's equations can be achieved in the new coordinates as:

$$\dot{\xi}_1^1 = \dot{\xi}_1 = b_1(\xi) + a_{11}(\xi)u_1 + a_{12}(\xi)u_2$$
$$= \xi_2 + \xi_3 + u_1$$
$$\dot{\xi}_1^2 = \dot{\xi}_2 + \dot{\xi}_3 = b_2(\xi) + a_{21}(\xi)u_1 + a_{22}(\xi)u_2 \qquad (18)$$
$$= -\xi_1 + \frac{\xi_1^3}{6} - (\xi_2 + \xi_3) + u_2.$$

Without loss of generality Equation above can be transformed into the following form by adding and subtracting a ξ_3 term to and from the second equation.

$$\begin{cases} \dot{\xi}_1 = \xi_2 + \xi_3 + u_1 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = -\xi_1 + \dfrac{\xi_1^3}{6} - \xi_2 - 2\xi_3 + u_2 \end{cases} \qquad (19)$$

Now using Equations 13 and 19, state feedback can be formulated as

$$\upsilon_{q1} = \alpha_{q1}(\xi) + \beta_{q1}(\xi)u$$
$$= \begin{bmatrix} \xi_1 + \xi_2 + \xi_3 \\ -\xi_1 + \dfrac{\xi_1^3}{6} + \xi_2 + \xi_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (20)$$

and the invertible transformation $\bar{\xi}$ and the output matrix can be stated as

$$\bar{\xi} = \Phi_1(\xi) = \Psi(\eta) = \eta = T(\xi) = \begin{bmatrix} \xi_3 & \xi_1 & \xi_2 + \xi_3 \end{bmatrix}^T \quad (21)$$
$$y = h_{q1}(\xi) = \begin{bmatrix} \xi_1 & \xi_2 + \xi_3 \end{bmatrix}^T.$$

then the implemented system can be rewritten as

$$\dot{\bar{\xi}} = \begin{bmatrix} -\bar{\xi}_2 + \dfrac{\bar{\xi}_2^3}{6} - \bar{\xi}_3 - \bar{\xi}_1 + u_2 \\ \bar{\xi}_3 + u_1 \\ -\bar{\xi}_2 + \dfrac{\bar{\xi}_2^3}{6} - \bar{\xi}_3 + u_2 \end{bmatrix} \qquad (22)$$
$$y = \begin{bmatrix} \bar{\xi}_2 & \bar{\xi}_3 \end{bmatrix}^T$$

The above equation is in the form of Equation 1 in the multi-realization problem and according to Equation 17, $\bar{\xi}_1$ is unobservable but stable.

The same procedure is applicable to system P_2, except that since this system's dimension is equal to the dimension of the state space 13, there is one less step to carry out (i.e. there is no need to construct the linear transformation $\eta = T(\xi)$).

Using state feedback and local coordinate transformation, the system's equations in the new coordinates can be written in accordance with equation 5.7 from [8] as:

$$\begin{cases} \dot{\xi}_1 = -3\xi_1 + \xi_1^2(\xi_2 + \xi_3) + u_1 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = -3(\xi_2 + \xi_3) + 2\xi_1^3\xi_2(\xi_2 + \xi_3)^{\frac{1}{2}} \\ \qquad + \xi_2 + 2(\xi_2 + \xi_3)^{\frac{1}{2}} u_2 \end{cases} \qquad (23)$$

and other system settings are as follows

$$\upsilon_{q2} = \alpha_{q2}(\xi) + \beta_{q2}(\xi)u$$
$$= \begin{bmatrix} -2\xi_1 + \xi_1^2(\xi_2 + \xi_3) \\ 2\xi_1^3\xi_2(\xi_2 + \xi_3)^{\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2(\xi_2 + \xi_3)^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (24)$$

$$\bar{\xi} = \Phi_2(\xi) = \begin{bmatrix} \xi_1 & (\xi_2 + \xi_3)^{\frac{1}{2}} & \xi_2 \end{bmatrix}^T,$$

$$y = h_{q2}(\xi) = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^T$$

then the implemented system P_2 is

$$(\overline{P_2}, \bar{\xi}_0): \begin{cases} \dot{\bar{\xi}} = \begin{bmatrix} -3\bar{\xi}_1 + \bar{\xi}_1^2\bar{\xi}_2^2 + u_1 \\ -\bar{\xi}_2 + \bar{\xi}_1^3\bar{\xi}_3 + u_2 \\ -\bar{\xi}_3 + \bar{\xi}_2^2 \end{bmatrix} \\ y = \begin{bmatrix} \bar{\xi}_1 & \bar{\xi}_3 \end{bmatrix}^T \end{cases} \qquad (25)$$

So as to ensure the transformation $\bar{\xi} = \Phi_2(\xi)$ is smooth and the feedback is "regular static state feedback" [8], it is required that $(\xi_2 + \xi_3)^{\frac{1}{2}} \neq 0$.

5. Conclusion

In this paper, an algorithm for applying the multi-realization method to a set of feedback linearizable nonlinear systems is presented. Using the proposed method a minimal stably based multi-realization is achieved. As an example, a set of feedback linearizable controllers are considered and multi-realization is used as a method of switching between the controllers when the operating conditions of the system change.

This paper does not attempt to prove that the control performance of switching between nonlinear controllers is superior to that of linear controllers. However, it does provide more options for switching controller selection and proposed stably-based multi-realization approach to implement switching between controllers which are not necessarily linear.

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