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A weighted metric method to optimize multi-response robust problems

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Abstract

In a robust parameter design (RPD) problem, the experimenter is interested to determine the values of control factors such that responses will be robust or insensitive to variability of the noise factors. Response surface methodology (RSM) is one of the effective methods that can be employed for this purpose. Since quality of products or processes is usually evaluated through several quality characteristics or responses, more attention should be paid to multi-response parameter design to improve quality of several responses simultaneously. There are many optimization methods in multi-objective decision-making (MODM) area which could be used for this purpose. In this article, some of these optimization techniques are reviewed and a criterion is considered to determine the optimum control setting factors for multi-response RPD problems. A sensitivity analysis is performed to investigate the effect of different scenarios on the solution.

Keywords: Multi-objective decision making; Response surface methodology; Robust parameter design; Weighted metric method

1. Introduction

In response surface methodology to robust parameter design, the ultimate goal is to identify the settings of control factors (x's) which lead to an optimal solution with a minimum variation. In order to achieve this goal, we usually consider two objective functions corresponding to the two parameters of a desired quality characteristic, namely the mean response μ_y and the variance response σ_y^2. Next, settings for the control factors are determined such that the values achieved for the two objective functions are as close to their ideal values as possible. Many authors [19, 23, 24, 25, 26, 30] consider RPD problems in RSM framework. The review paper by Myers et al. [27] gives a thorough discussion of RSM. They believe that the solution of the robust parameter design problem in RSM framework would be one of the most important areas for research. Most of the published literature on robust design methodology is usually concerned with a single response. However, a common problem in product or process design is to determine the optimal parameter levels when there are multiple responses that should be considered simultaneously. There are different optimization techniques available in multi-objective decision-making area that could be considered for solving optimization problems. While dealing with several objective functions, due to possible contradiction among the objective functions, it is very unlikely to find a setting for control factors which could optimize all the objective functions simultaneously. Hence, in this situation one should search for efficient solutions. Usually there are many alternative solutions instead of a few single point solutions. The string of solutions generated as such is referred to as Pareto optimal solutions (POS). In other words, POS is the solution that one objective function cannot be improved unless at the expense of other objective functions.

Murphy et al. [24] and others believe that most statistically based multi-response objective functions could be formed by combining objective functions into a single response. The common statistical tech-

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niques for optimizing a single response have been categorized into desirability function and loss function approaches. The desirability function approach originally proposed by Harrington [17] is constructed by the geometric average of the individual desirability functions of each response. A modified version of the desirability function which allows the decision maker (DM) to place the ideal target value anywhere within the specifications is introduced by Derringer and Suich [16]. Derringer [15] added explicit weighting terms to the geometric average of the individual desirability functions. Del Castillo et al. [14] noted that since these desirability functions are non-differentiable at the target points, only direct search optimization methods are applicable. Thus, they propose a piecewise continuous desirability function in which the non-differentiable points are corrected. Latter, different forms of desirability functions are proposed by Kim and Lin [20] and Chiao and Hamada [10]. The desirability function approach does not consider the covariance structure of the responses. According to Ko et al. [21], the major advantage of the loss function compared to desirability function approach is its ability to incorporate both the covariance structure of the responses and the process economics. References on loss function approach include [1, 3, 29, 31, 32, 36, 37]. The squared-error loss function proposed by Berger [5] is commonly considered as a loss function in estimation problems. In this method, the expected value of loss function can be easily expressed by:

$$E(L) = A_0 E(Y-T)^2 = A_0[Var(Y) + (E(Y)-T)^2] \quad (1)$$

where Y and T are the actual response and the target value, respectively; and A_0 is a proportional constant representing the economic costs of the squared-error loss. To provide decision makers with flexible weighting of the off-target squared and variance components, Box and Jones [8] introduced the following general class of squared-error loss functions, where $0 \le w \le 1$,

$$E(L) = A_0[w(E(Y)-T)^2 + (1-w)Var(Y)] \quad (2)$$

Weighted metric method, henceforth referred to as L_p method, discussed in Section 3 is a general form of Equation 2. This method is used by Ardakani and Noorossana [2] to solve an RPD problem with a single response. L_p method is applicable when robust design, in the case of multiple quality characteristics, is desirable and unlike loss function methods the correlation structure of the multiple responses does not severely affect the resulting analysis.

Next section provides a brief review on some of the existing optimization methods that could be applied to robust design problems. In section 3, the L_p method and its application in multiple robust problems is discussed. Section 4 presents a numerical example using the proposed method. Concluding remarks are provided in the final section.

2. RSM approaches to RPD problems

This section provides an overview on two RSM approaches, namely the single and dual model approaches considered by many researchers in RPD. The paper by Robinson et al. [34] is a useful reference which reviews these approaches. In dual model approach, responses are fitted to the mean and variance separately. This approach was developed by Myers and Carter [25] for solving problems where the experimenter can identify a primary response function Y_P that is to be optimized subject to some specified value θ of a secondary response, Y_S . In other words, the problem is to optimize Y_P subject to $Y_S = \theta$. Depending on the goal, Y_P and Y_S can be characterized as μ_y or σ_y^2 . It is assumed that in a region of interest R, both responses can be estimated by fitting second-order response surface models. The sample means and variances of the responses from what is called as the outer array advocated by Taguchi are taken as the data for fitting the responses. After estimating the location and dispersion parameters, Lagrangian multipliers are used to find optimum solution. Vining and Myers [38] proposed a response surface approach to solve the dual response model with the added constraint $X'X = \rho^2$ for restricting the search area to a spherical region of radius ρ . However, since the constraints in the optimization problem all involve equalities, we cannot often find a feasible solution. To overcome this obstacle, one can replace the equality in the constraint with an inequality and apply the method proposed by Del Castillo and Montgomery [13] which uses generalized reduced gradient (GRG) algorithm to optimize the problem. A common approach to solve the above problem was to set the primary response equal to a specific value and then optimize the second response. For the case of *target is best* where one is trying to keep the mean μ at a specified target value T, Lin and Tu [23] proposed a different approach minimizing $(\hat{\mu}-T)^2+\hat{\sigma}^2$. This approach is based on the definition of mean squares error (MSE) and leads to a substantial reduction in the response variability when a small bias is allowed in the response. They demonstrated that their approach is superior to those proposed by Vining and Myers [38] and Del Castillo and Montgomery [13]. Copeland and Nelson [11] proposed a method which allows decision maker to determine maximum distance from target. They also showed that their method is as effective as the approach proposed by Lin and Tu [23]. They used the Nelder-Mead simplex procedure proposed by Nelder and Mead [28]. They also proposed objective functions for the *larger is better* and *smaller is better* scenarios.

The second approach referred to as single model was introduced by Welch et al. [39]. In this approach, both control and noise factors are considered simultaneously in a single design called a *combined array*. Useful references on the combined array and its applications include [6, 7, 25, 27, 35]. These designs typically require fewer runs than Taguchi's *crossed arrays* used in the dual model and also allow the experimenter to estimate potentially important interactions. The response model is generally given in the following form:

$$y(\mathbf{x},\mathbf{z})=\beta_0+\mathbf{x}'\boldsymbol{\beta}+\mathbf{x}'\mathbf{B}\mathbf{x}+\mathbf{z}'\boldsymbol{\gamma}+\mathbf{x}'\boldsymbol{\Delta}\mathbf{z}+\varepsilon \tag{3}$$

where $y(\mathbf{x},\mathbf{z})$, $\mathbf{x}(r_x\times 1)$, and $\mathbf{z}(r_z\times 1)$ denote the response matrix, control factors vector, and noise variables vector, respectively. The quantity β_0 is the intercept, $\boldsymbol{\beta}$ is a vector of coefficients for the linear effects in control variables, $\mathbf{B}$ contains the coefficients for the quadratic effects in control factors and the control$\times$control interactions, $\boldsymbol{\gamma}$ is a vector of coefficients for the linear effects in the noise variables, and $\boldsymbol{\Delta}$ contains the coefficients of the interaction effects between control and noise factors that is critical for the success of RPD. The experimental error, *i.e.* the error due to the inability of the model to explain the real physical phenomenon, is defined by ε. It is assumed that ε_i's in the experiment follow a normal distribution with mean zero and variance σ^2. The response surface model for the mean, assuming $E(\mathbf{z})=\mathbf{0}$, is given by:

$$\mathrm{E}\left[y(\mathbf{x},\mathbf{z})\right]=\beta_0+\mathbf{x}'\boldsymbol{\beta}+\mathbf{x}'\mathbf{B}\mathbf{x}. \tag{4}$$

The response surface model for the process variance is given by

$$\mathrm{Var}\left[y(\mathbf{x},\mathbf{z})\right]=\mathrm{Var}\left[\left(\boldsymbol{\gamma}+\boldsymbol{\Delta}'\mathbf{x}\right)'\mathbf{z}\right]+\sigma_{\varepsilon}^2, \tag{5}$$

or equivalently

$$\mathrm{Var}\left[y(\mathbf{x},\mathbf{z})\right]=\left(\boldsymbol{\gamma}+\boldsymbol{\Delta}'\mathbf{x}\right)'\boldsymbol{\Sigma}_z\left(\boldsymbol{\gamma}+\boldsymbol{\Delta}'\mathbf{x}\right)+\sigma_{\varepsilon}^2. \tag{6}$$

where $\mathrm{Var}(\mathbf{z})=\boldsymbol{\Sigma}_z$ is usually assumed to be $\sigma_z^2\mathbf{I}$. Like dual model approach, in this approach, two responses are also considered, a location response corresponding to μ_y and a dispersion response corresponding to σ_y which are necessary to formulate the standard RPD problems. Now optimization techniques should be applied to determine factor settings which lead to an optimum solution.

3. Weighted metric method (L_p method)

A brief discussion on L_p method is given in this section. L_p method which is usually discussed in MODM references such as [4, 12, 18] is among optimization techniques that combines multiple objectives into a single objective. The weighted L_p distance measure of any solution $\mathbf{x}$ from the ideal solution $f(\mathbf{x}^{\max j})$ can be minimized as follows:

$$L_p=\{\sum_{j=1}^{k}w_j\left(f_j(\mathbf{x}^{\max j})-f_j(\mathbf{x})\right)^p\}^{1/p} \tag{7}$$

where w_j is a non-negative weight assigned to each objective function by DM and p indicates the importance of each objective function deviation from its ideal value. When $p=1$ is used, the resulting problem reduces to a weighted sum of the deviations. When $p=2$ is used, a weighted Euclidean distance of any point in the objective space from the ideal point is minimized. When $p=\infty$ is considered, the largest deviation $w_j\left(f_j(\mathbf{x}^{\max j})-f_j(\mathbf{x})\right)$ is minimized, that is,

$$\underset{x}{Min}\ \left\{\underset{j}{Max}\ \ w_j\left(f_j(\mathbf{x}^{\max j})-f_j(\mathbf{x})\right)\right\} \tag{8}$$

which is equivalent to:

$$\begin{cases} Min \;\; y \\ s.t: \\ y \geq w_j \left(f_j(\mathbf{x}^{\max j}) - f_j(\mathbf{x}) \right) \quad \forall j \end{cases} \tag{9}$$

Chankong and Haimes [9] showed that when L_p method is used then all solutions corresponding to $1 \leq p \leq \infty$ and $w > 0$ are efficient solutions. On the other hand, if one considers constrained L_p problems then according to Miettinen [26] only L_∞ leads to pareto optimal solutions. These are two characteristics of the L_p method.

In Equation 7, it is assumed that objective functions have the same scale. If $f_j(\mathbf{x})$'s do not have the same scale then each objective function could be made scale-less using either of the following equations:

$$L_p = \left(\sum_{j=1}^{k} w_j [\frac{f_j(\mathbf{x}^{\max j}) - f_j(\mathbf{x})}{f_j(\mathbf{x}^{\max j})}]^p \right)^{1/p} \tag{10}$$

or

$$L_p = \left(\sum_{j=1}^{k} w_j [\frac{f_j(\mathbf{x}^{\max j}) - f_j(\mathbf{x})}{f_j(\mathbf{x}^{\max j}) - f_j(\mathbf{x}^{\min j})}]^p \right)^{1/p} \tag{11}$$

Since the values of each objective function and the quantity of L_p in Equation 11 are between zero and one, one could also formulate and solve the problem in fuzzy environment.

4. Numerical example

The following example comes from a case study discussed by Romano et al. [33]. The robust design experiment was conducted on the elastic element of a force transducer. This example involves a combined array design with three control and two noise variables. Control factors are the three parameters defining the element configuration, namely lozenge angle (x_1), bore diameter (x_2), and half-length of the vertical segment (x_3). Noise factors are the deviation of the lozenge angle from its nominal value (z_1) and the deviation of the bore diameter from its nominal value (z_2). The two responses y_1 and y_2 define the non-linearity and the hysteresis, respectively. The fitted response surface functions for these indicators are given by:

$$\begin{aligned} \hat{y}_1 &= 1.38 - 0.361x_1 - 0.155x_2 + 0.0771x_3 - 0.148x_1x_2 \\ &+ 0.0218x_1x_3 + 0.013x_2x_3 + 0.0481x_1^2 - 0.0588z_1 \\ &- 0.0116z_2 + 0.01x_1z_1 \end{aligned} \tag{12}$$

$$\begin{aligned} \hat{y}_2 &= 1.64 + 0.592x_1 + 0.438x_2 - 0.095x_3 + 0.301x_1x_2 \\ &- 0.143x_1x_3 + 0.201x_1^2 - 0.0844x_1x_2x_3 \\ &+ 0.0794x_1z_1 \end{aligned} \tag{13}$$

It is assumed that noise factors are uncorrelated and $\sigma_{z_1}^2 = \sigma_{z_2}^2 = 1$. In the experiment, the corresponding deterministic noise factors, i.e. z_1 and z_2, have fixed levels. The values of error variances computed by Köksoy [22] are $s_1^2 = 0.0003253$ and $s_2^2 = 0.024$. The models in equations 4 and 6 are used to generate the mean and variance responses. These responses for y_1 are as follows:

$$\begin{aligned} \mu_1 &= 1.38 - 0.361x_1 - 0.155x_2 + 0.0771x_3 - 0.148x_1x_2 \\ &+ 0.0218x_1x_3 + 0.013x_2x_3 + 0.0481x_1^2 \end{aligned} \tag{14}$$

$$\sigma_1^2 = (-0.0588 + 0.01x_1)^2 + (-0.0116)^2 + s_1^2 \tag{15}$$

The mean and variance responses for y_2 are correspondingly as follows:

$$\begin{aligned} \mu_2 &= 1.64 + 0.592x_1 + 0.438x_2 - 0.095x_3 + 0.301x_1x_2 \\ &- 0.143x_1x_3 + 0.201x_1^2 - 0.0844x_1x_2x_3 \end{aligned} \tag{16}$$

$$\sigma_1^2 = (0.0794x_1)^2 + s_2^2 \tag{17}$$

The L_p metric method is used to set control factors such that the means responses, i.e. equations 14 and 16, achieve a target value of one while the variance responses, i.e. equations 15 and 17, have minimum values. The L_p metric discussed in the previous section requires the ideal values for the mean and variance responses. The ideal values shown in Table 1 are obtained by optimizing each response individually in a search area of the form $-1 \leq x_i \leq 1, \;\; i = 1,2,3$. In other words; it is assumed that control factors can take values between -1 and 1.

Since the ranges of objective functions are very different, Equation 11 is suitable to find optimum points. L_p function for this example is as follows:

$$L_P = \left\{ w_1 \left| \frac{\mu_1 - T_1}{\mu_1^{\max} - \mu_1^{\min}} \right|^p + w_2 \left(\frac{\sigma_1 - \sigma_1^{\min}}{\sigma_1^{\max} - \sigma_1^{\min}} \right)^p \right. \\ \left. + w_3 \left| \frac{\mu_2 - T_2}{\mu_2^{\max} - \mu_2^{\min}} \right|^p + w_4 \left(\frac{\sigma_2 - \sigma_2^{\min}}{\sigma_2^{\max} - \sigma_2^{\min}} \right)^p \right\}^{1/p} \quad (18)$$

The quantities of $\mu^{\max}, \mu^{\min}, \sigma^{\max}$ and $\sigma^{\min}$ are the extreme values for corresponding responses in the search area. T_1 and T_2 are determined by DM and represent the target values for μ_1 and μ_2, respectively. The quantity of w's indicate the importance of each response determined by DM. Replacing target values and ideal values in Equation 18 leads to the following optimization problem:

$$\underset{x}{Min} \left\{ w_1 \left| \frac{\mu_1 - 1}{1.1982} \right|^p + w_2 \left(\frac{\sigma_1 - 0.053304}{0.018761} \right)^p \right. \\ \left. + w_3 \left| \frac{\mu_2 - 1}{2.3601} \right|^p + w_4 \left(\frac{\sigma_2 - 0.15492}{0.01916} \right)^p \right\} \quad (19)$$

$$\textit{Subject to}: \\ -1 \le x_1, x_2, x_3 \le 1$$

In order to solve this optimization problem, DM should select desired weights (w 's) in advance. These weights determine the importance of each response from the viewpoint of DM. For instance, if the standard deviation responses are more important to the decision maker compared to the mean responses then relatively larger values are assigned to w_2 and w_4. For comparison purposes, one set of weights corresponding to twenty one different decision makers' preferences is arbitrary assigned to each objective function. A row corresponding to the indifference case, i.e. when DM has no priority among responses $(w_1 = w_2 = w_3 = w_4 = 0.25)$ is also considered for comparison purposes. Also, three different values of p, i.e. 1, 2 and ∞, are considered to optimize Equation 19. The optimization module of MATLAB software is used for finding optimum solutions. The results for the L_p metric method for the case of $p = 1$ are presented in Table 2.

Since all L_p solutions are efficient, there is no any superiority among these twenty one solutions. Each of the solutions has the best values of objective functions from the point of view of the DM. In other words, according to the definition of Pareto optimality, moving from one Pareto optimal solution to another requires a trade-off. As a result, one cannot improve any criterion without worsening the value of at least another criterion. Similar analysis was conducted for different values of p. The results for the case of p equal to two and infinitive are presented in Tables 3 and 4, respectively.

In L_p method, an increase in the value of p indicates that DM makes a more conservative decision in trading off among the objective functions. In order to determine the effect of p and w on solutions, two-way analysis of variance is used. This analysis helps to investigate the effect of p and w in the L_p method. The three levels of p, i.e. 1, 2 and ∞, and twenty one levels of weights are selected as two factors in two-way ANOVA method. μ_1, σ_1, μ_2 and σ_2 shown in Tables 2 through 4 are considered as response variables. Tables 5 to 8 present the results of two-way ANOVA and 95% confidence intervals based on pooled standard deviations for each response.

According to the ANOVA results and confidence intervals illustrated in Table 5, since the quantity of p-values for p and w are 0.276 (more than 0.05) and zero, respectively, μ_1 is not affected by the value of p but it is affected by w. Similar analyses can be concluded from Tables 7 and 8. Table 6 shows that the value of p has an effect on σ_1. Thus, we can generally conclude that solutions in L_p method are not sensitive to the value of p but, as we expected, L_p results are heavily depended on the weights allocated to the responses by DM.

Table 1. The maximum and minimum values of objective functions.

Response	Min	Max	Range
μ_1	0.6522	1.8504	1.1982
σ_1	0.053304	0.072065	0.018761
μ_2	1.0713	3.4314	2.3601
σ_2	0.15492	0.17408	0.01916

Table 2. L_p optimum solutions for $p=1$.

No.	w_1	w_2	w_3	w_4	x_1	x_2	x_3	μ_1	σ_1	μ_2	σ_2	L_p
1	0.1	0.1	0.1	0.7	0.038039	-1	-1	1.462	0.062231	1.3082	0.15495	0.100295
2	0.1	0.1	0.3	0.5	0.03967	-1	1	1.5916	0.062216	1.114	0.15495	0.112151
3	0.1	0.1	0.5	0.3	0.041229	-1	1	1.5913	0.062201	1.1143	0.15495	0.121457
4	0.1	0.1	0.7	0.1	0.046362	-1	1	1.5903	0.062153	1.1153	0.15496	0.130839
5	0.1	0.3	0.1	0.5	0.14505	-1	-1	1.4379	0.061228	1.3428	0.15535	0.189002
6	0.1	0.3	0.3	0.3	0.2079	-1	1	1.5614	0.060639	1.1509	0.1558	0.197105
7	0.1	0.3	0.5	0.1	0.42871	-1	1	1.526	0.058579	1.2166	0.15861	0.193396
8	0.1	0.5	0.1	0.3	0.3936	-1	-1	1.3859	0.058906	1.4409	0.15804	0.249039
9	0.1	0.5	0.3	0.1	0.95568	-1	1	1.4603	0.05371	1.4525	0.1725	0.198509
10	0.1	0.7	0.1	0.1	1	-1	-1	1.2842	0.053304	1.7846	0.17408	0.156963
11	0.3	0.1	0.1	0.5	0.12541	1	-1	1.0691	0.061412	2.3088	0.15524	0.124324
12	0.3	0.1	0.3	0.3	0.10167	-1	-1	1.4475	0.061634	1.3282	0.15513	0.20145
13	0.3	0.1	0.5	0.1	0.19295	-1	1	1.564	0.060779	1.1472	0.15568	0.216207
14	0.3	0.3	0.1	0.3	0.27082	0.92753	-1	1.0098	0.060051	2.4348	0.1564	0.19431
15	0.3	0.3	0.3	0.1	0.61367	-1	-1	1.3449	0.056862	1.5486	0.1624	0.252023
16	0.3	0.5	0.1	0.1	1	0.56736	1	1.0015	0.053304	2.5034	0.17408	0.164076
17	0.5	0.1	0.1	0.3	0.26022	1	-1	1	0.06015	2.4618	0.15629	0.11988
18	0.5	0.1	0.3	0.1	0.29931	0.90736	-1	1	0.059785	2.4562	0.15673	0.229094
19	0.5	0.3	0.1	0.1	0.71091	0.20157	-1	1	0.055963	2.4578	0.16488	0.156271
20	0.7	0.1	0.1	0.1	0.26768	0.98199	-1	1	0.060081	2.4607	0.15637	0.105582
21	0.25	0.25	0.25	0.25	0.24288	-1	-1	1.4167	0.060312	1.3785	0.15612	0.236079

Table 3. L_p optimum solutions for $p=2$.

No.	w_1	w_2	w_3	w_4	x_1	x_2	x_3	μ_1	σ_1	μ_2	σ_2	L_p
1	0.1	0.1	0.1	0.7	0.23566	-0.70293	-1	1.358	0.06038	1.5328	0.15605	0.175168
2	0.1	0.1	0.3	0.5	0.25039	-1	-0.75486	1.4322	0.060242	1.3545	0.15619	0.188818
3	0.1	0.1	0.5	0.3	0.29137	-1	0.36069	1.5024	0.059859	1.2451	0.15664	0.193904
4	0.1	0.1	0.7	0.1	0.38516	-1	1	1.5326	0.058985	1.2021	0.15791	0.191038
5	0.1	0.3	0.1	0.5	0.35303	-0.81898	-1	1.3541	0.059284	1.5271	0.15743	0.229744
6	0.1	0.3	0.3	0.3	0.39984	-1	-0.21834	1.4416	0.058848	1.3512	0.15814	0.2343
7	0.1	0.3	0.5	0.1	0.52549	-1	1	1.5119	0.05768	1.2515	0.16044	0.220345
8	0.1	0.5	0.1	0.3	0.46532	-0.91755	-1	1.3525	0.058239	1.5248	0.15926	0.252167
9	0.1	0.5	0.3	0.1	0.59784	-1	0.44521	1.4592	0.057009	1.3523	0.16203	0.233758
10	0.1	0.7	0.1	0.1	0.65126	-1	-1	1.3384	0.056514	1.5689	0.16332	0.231301
11	0.3	0.1	0.1	0.5	0.26458	-0.00788	-1	1.2066	0.06011	1.9226	0.15634	0.200269
12	0.3	0.1	0.3	0.3	0.31276	-0.78083	-1	1.3552	0.05966	1.5287	0.1569	0.236855
13	0.3	0.1	0.5	0.1	0.39219	-1	-1	1.3862	0.058919	1.4403	0.15802	0.245243
14	0.3	0.3	0.1	0.3	0.40769	-0.21733	-1	1.2045	0.058775	1.913	0.15827	0.241634
15	0.3	0.3	0.3	0.1	0.53701	-0.97524	-1	1.3526	0.057573	1.5248	0.16068	0.255703
16	0.3	0.5	0.1	0.1	0.6054	-0.44954	-1	1.2046	0.056939	1.9137	0.16221	0.238709
17	0.5	0.1	0.1	0.3	0.30792	0.20969	-1	1.1448	0.059705	2.0777	0.15684	0.206898
18	0.5	0.1	0.3	0.1	0.41979	-0.57972	-1	1.2841	0.058662	1.7048	0.15846	0.257747
19	0.5	0.3	0.1	0.1	0.53789	-0.13224	-1	1.1437	0.057565	2.0693	0.1607	0.22868
20	0.7	0.1	0.1	0.1	0.41391	0.18199	-1	1.111	0.058717	2.1563	0.15837	0.203907
21	0.25	0.25	0.25	0.25	0.40939	-0.86981	-1	1.3531	0.058759	1.5256	0.15829	0.250958

Table 4. L_p optimum solutions for $p = \infty$.

No.	w_1	w_2	w_3	w_4	x_1	x_2	x_3	μ_1	σ_1	μ_2	σ_2	L_p
1	0.1	0.1	0.1	0.7	0.22694	-0.0063	0.00125	1.3018	0.060461	1.7671	0.15596	0.025188
2	0.1	0.1	0.3	0.5	0.21649	-1	-0.40353	1.4634	0.060559	1.3042	0.15587	0.024791
3	0.1	0.1	0.5	0.3	0.21153	-1	0.48613	1.5255	0.060605	1.207	0.15583	0.014248
4	0.1	0.1	0.7	0.1	0.23105	-1	1	1.5575	0.060423	1.1569	0.156	0.005637
5	0.1	0.3	0.1	0.5	0.40801	0	0	1.2407	0.058772	1.8893	0.15827	0.020088
6	0.1	0.3	0.3	0.3	0.48984	-0.70215	0.16658	1.3875	0.058011	1.5136	0.15973	0.03234
7	0.1	0.3	0.5	0.1	0.59741	-1	1	1.502	0.057013	1.2799	0.16202	0.037056
8	0.1	0.5	0.1	0.3	0.57709	0	0	1.1877	0.057201	2.0122	0.16155	0.015665
9	0.1	0.5	0.3	0.1	0.75903	-1	0.27041	1.4228	0.055518	1.4531	0.16623	0.035286
10	0.1	0.7	0.1	0.1	0.80642	-0.01851	0.004839	1.1257	0.055082	2.1837	0.16763	0.010491
11	0.3	0.1	0.1	0.5	0.28492	0.12608	-1	1.1712	0.05992	2.0119	0.15656	0.035265
12	0.3	0.1	0.3	0.3	0.49854	-0.7915	-1	1.3154	0.05793	1.6212	0.1599	0.024658
13	0.3	0.1	0.5	0.1	0.41587	-1	-1	1.3816	0.058699	1.4509	0.1584	0.018163
14	0.3	0.3	0.1	0.3	0.48984	0	0	1.2147	0.058011	1.9474	0.15973	0.040142
15	0.3	0.3	0.3	0.1	0.49853	-0.7915	-1	1.3154	0.05793	1.6212	0.1599	0.025992
16	0.3	0.5	0.1	0.1	0.75903	-0.0173	0.004304	1.1387	0.055518	2.1449	0.16623	0.034727
17	0.5	0.1	0.1	0.3	0.39205	0.19485	-1	1.1162	0.05892	2.1441	0.15802	0.029934
18	0.5	0.1	0.3	0.1	0.49868	-0.5008	-1	1.2451	0.057929	1.8045	0.1599	0.024652
19	0.5	0.3	0.1	0.1	0.66926	-0.18118	-1	1.1166	0.056348	2.1487	0.16378	0.046242
20	0.7	0.1	0.1	0.1	0.49899	0.14816	-1	1.088	0.057926	2.2138	0.15991	0.024636
21	0.25	0.25	0.25	0.25	0.49854	-0.7915	-1	1.3154	0.05793	1.6212	0.1599	0.061644

Table 5. Confidence intervals and two-way ANOVA for μ_1.

```
Two-way ANOVA:
Source  DF      SS         MS        F    P-value
P        2  0.01998  0.0099892    1.48  0.240
W       20  1.60730  0.0803650  11.91  0.000
Error   40  0.26991  0.0067478
Total   62  1.89719

Confidence Intervals:
P        Mean  --+---------+---------+---------+-------
1     1.32116          (------------*-----------)
2     1.33473                (------------*-----------)
inf   1.29204  (-----------*------------)
                --+---------+---------+---------+-------
                  1.260     1.290     1.320     1.350
```

Table 6. Confidence intervals and two-way ANOVA for σ_1.

```
Two-way ANOVA:
Source  DF       SS         MS         F    P-value
P        2  0.0000147  0.0000073  6.18  0.005
W       20  0.0002091  0.0000105  8.82  0.000
Error   40  0.0000474  0.0000012
Total   62  0.0002712

Confidence Intervals:
P         Mean  ---------+---------+---------+---------
1    0.0593095                   (--------*-------)
2    0.0586869          (-------*-------)
inf  0.0581289  (-------*-------)
                ---------+---------+---------+---------
                       0.05820   0.05880   0.059400
```

Table 7. Confidence intervals and two-way ANOVA for μ_2.

```
Two-way ANOVA:
Source  DF     SS       MS       F    P-value
P        2   0.1939  0.096973   2.29  0.114
W       20   8.8529  0.442644  10.46  0.000
Error   40   1.6925  0.042312
Total   62  10.7393

Confidence Intervals:
P       Mean  -+---------+---------+---------+--------
1    1.69172             (----------*-----------)
2    1.60415   (-----------*----------)
inf  1.73794                    (----------*----------)
              -+---------+---------+---------+--------
               1.520     1.600     1.680     1.760
```

Table 8. Confidence intervals and two-way ANOVA for σ_2.

```
Two-way ANOVA:
Source  DF       SS         MS       F    P-value
P        2  0.0000202  0.0000101  1.59  0.217
W       20  0.0009270  0.0000463  7.29  0.000
Error   40  0.0002542  0.0000064
Total   62  0.0012013

Confidence Intervals:
P        Mean  ----+---------+---------+---------+-----
1    0.159215        (-----------*----------)
2    0.158690  (----------*-----------)
inf  0.160063              (-----------*----------)
               ----+---------+---------+---------+-----
                  0.1580    0.1590    0.1600    0.1610
```

5. Conclusion

In this paper, L_p method was introduced as a vehicle for optimizing multiresponse problems. This method was applied to a multiresponse example to generate efficient solutions. In addition, a sensitivity analysis was conducted for different values of p and w to evaluate the performance of the proposed method. The results indicate that the optimization method heavily depends on the weight w allocated to each response by the decision maker. However, the solutions obtained by this method are robust with respect to p. One of the advantages of the proposed method is its ability and flexibility to generate efficient solutions regardless of the values of p and w.

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